

### Homework Set 1

DUE: Thursday April 19

1. Cosmological constant. (a) Calculate the value of the cosmological constant  $\Lambda$ , and also calculate  $\Omega_\Lambda = \Lambda/3H_0^2$ , the corresponding vacuum energy density in units of critical density  $\rho_c$ . Assume that  $\Omega_m = 0.3$  and that the curvature vanishes ( $k = 0$ ). Express  $\Lambda$  and  $\rho_\Lambda \equiv \Omega_\Lambda \rho_c$  in several different units, including natural units ( $\hbar = c = 1$ ; take  $\rho_\Lambda \equiv \Omega_\Lambda \rho_c$  to have units of  $eV^4$ ).

(b) The acceleration due to the cosmological constant equals  $\Lambda r/3$ . Numerically compare that to the sun's gravitational acceleration  $GM_\odot/r^2$  in order to find the distance  $r$  at which they are equal.

2. For a flat universe with  $\Omega_{m,0} < 1$  and positive cosmological constant  $\Omega_{\Lambda,0} = 1 - \Omega_{m,0}$ , the density contributions of the matter and cosmological constant are equal when the scale factor has the value  $a_{m\Lambda} = (\Omega_{m,0}/\Omega_{\Lambda,0})^{1/3}$ . This equals 0.75 for the Benchmark Model:  $\Omega_{m,0} = 0.3$ ,  $\Omega_{\Lambda,0} = 0.7$ . Show that for this case the Friedmann equation can be integrated to give the expression

$$H_0 t = \frac{2}{3\sqrt{1 - \Omega_{m,0}}} \ln[y^{3/2} + \sqrt{1 + y^3}],$$

where  $y \equiv a/a_{m\Lambda}$ . Show that for  $a \ll a_{m\Lambda}$ , this reduces to

$$a(t) \approx \left(\frac{3}{2}\sqrt{\Omega_{m,0}}H_0 t\right)^{2/3},$$

and for  $a \gg a_{m\Lambda}$ , it reduces to

$$a(t) \approx a_{m\Lambda} \exp(\sqrt{1 - \Omega_{m,0}}H_0 t).$$

Show finally that the age of the universe today in this case is

$$t_0 = \frac{2}{3H_0\sqrt{1 - \Omega_{m,0}}} \ln \left[ \frac{\sqrt{1 - \Omega_{m,0}} + 1}{\sqrt{\Omega_{m,0}}} \right],$$

and that for the Benchmark Model this is  $t_0 = 0.964H_0^{-1}$ .

3. Popularizations of cosmology often talk about the “size of the universe” at some earlier time, but they usually mean by this the size that the present-epoch horizon was at that time. (For example, in the *Cosmic Voyage* IMAX film, Rocky Kolb says that everything that we can presently see was once only as big as a marble that he holds in his hand.) To

clarify this, make a log-log plot in which the horizontal axis is time since the Big Bang, from  $10^{-30}$  s to  $10^{20}$  s, and the vertical axis is length, and plot curves representing both (a) the size in the past of the present-epoch horizon, and (b) the size of the current horizon (i.e., the distance that light has travelled since the Big Bang), as a function of the time since the Big Bang. (These curves cross at  $t_0$ .) Use any cosmology that you like, for example Einstein-de Sitter, but specify which one you are using. Briefly discuss some implications of your plot.

4. Geometry. (a) Show that if  $k = 0$  and the scale factor  $a$  grows as  $t^{2/3}$ , the apparent angular sizes of distant objects of the same linear size have a minimum at  $z = 1.25$ . (b) Under the same assumptions, suppose that a galaxy is observed at  $z = 1.25$ . For what fraction of the Hubble time has its light been travelling toward us? (*Be careful to answer what this question asks; don't just calculate something similar.*)

5. Consider a galaxy of physical (visible) size 5 kpc. What angle would this galaxy subtend if situated at redshift 0.1? 1? 5? Do the calculation in a flat universe, first with zero cosmological constant, and then in the Benchmark Model with  $\Omega_{m,0} = 0.3$ . (*You are welcome to use Ned Wright's Javascript cosmology calculator – see below. Extra Credit: Check one or two of the Javascript results for the Benchmark Model by direct integration.*)

6. The astronomical convention is that the relationship between apparent magnitude  $m$  and absolute magnitude  $M$  is

$$m - M = 5 \log\left(\frac{d_L}{10pc}\right) + K$$

where  $d_L$  is the luminosity distance and  $K$  is a correction for the redshifting of the spectrum of the source (Dodelson eq. 2.81). Plot  $m - M$  as a function of redshift for a flat matter-dominated universe (this can be done analytically) and for the Benchmark Model (you need to evaluate numerically a 1D integral). Neglect the  $K$  correction. Extra Credit: Compare with a plot showing high-redshift supernova data, for example Dodelson's Fig. 1.7.

7. Suppose astronomers measure the age of a galaxy at redshift  $z = 2.5$ . How old would this galaxy have to be (at the time the light from it was emitted) in order to rule out the hypothesis that  $\Omega_M = 1$  with negligible vacuum and radiation energy density. Use  $H_0 = 70$  km/sec/Mpc. (This is Weinberg, *Cosmology*, page 569, problem 4.)

---

Note: Ned Wright's Cosmology web page, with many useful links, is

<http://www.astro.ucla.edu/~wright/cosmolog.htm> ;

his Javascript distance calculator is at

<http://www.astro.ucla.edu/~wright/CosmoCalc.html> .