Power spectra and the growth of perturbations

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1 Perturbations and power spectra

1.1 Density fluctuations and power

We start with a field of density perturbations

$$\delta(\vec{x}) \equiv \frac{\rho(\vec{x}) - \langle\rho\rangle}{\langle\rho\rangle}$$

and its k-space counterpart

$$\delta(\vec{k}) = \frac{1}{V}\int \delta(\vec{x}) e^{i\vec{k}\cdot\vec{x}} d^3x$$

(where V is the volume). This is a homogeneous and isotropic field of gaussian perturbations such that $\delta(\vec{k}) = \delta(k)$ (isotropy) and

$$\begin{aligned}\langle\delta(\vec{x})\rangle &= \langle\delta(k)\rangle = 0 \\ \langle\delta(\vec{x})^2\rangle &= \sigma^2 = const \\ \langle|\delta(k)|^2\rangle &\equiv P(k),\end{aligned} \tag{1}$$

where the angle brackets denote the **ensemble average** of a large number of realizations of the density field. In general we will work in k-space, because the k modes are uncorrelated and evolve independently of one another (see the Appendix), as opposed to the case real space, where the evolution of $\delta(\vec{x}_1)$ is strongly dependent on the density fluctuations at nearby points $\{\vec{x}_2, \vec{x}_3, \ldots\}$. The power spectrum is usually parameterized as a power law, $P(k) \propto k^n$, so often, in fact, that n is generally considered one of the fundamental parameters of cosmology.

The real- and fourier-space fluctuations may be related using Parseval's theorem:

$$\frac{1}{V}\int |\delta(\vec{x})|^2\, d^3x = \frac{V}{(2\pi)^3}\int |\delta(k)|^2\, d^3k.$$

Averaging both sides, and using the fact that σ^2 is a constant, we obtain

$$\sigma^2 = \frac{V}{(2\pi)^3}\int P(k) d^3k.$$

The angular integral is trivial, and we convert to an integral over $d\ln k$ to get

$$\sigma^2 = \frac{V}{2\pi^2}\int k^3 P(k) d\ln k,$$

or, in differential form,

$$\frac{d\sigma^2}{d\ln k} = \frac{V}{2\pi^2} k^3 P(k) \equiv \Delta^2(k). \tag{2}$$

$\Delta^2(k) \propto k^3 P(k)$ is often a more convenient quantity to consider than $P(k)$ for several reasons. First, it is dimensionless (unlike the total power per k-mode, which is given by $VP(k)$ and has units of volume). Also, since $\Delta^2(k) = d\sigma^2/d\ln k$, it tells us the contribution to the density variance from each logarithmic bin in k—that is, $\Delta^2(k)$ tells us about the variance as a function of **scale**, or order of magnitude in size. Thus, if $\Delta(k_0) = 1$ at some k_0, this means that there exist order unity density fluctuations on scales of order $1/k_0$. This means that fluctuations on this scale have reached the regime of nonlinear evolution and that linear theory no longer applies on this scale.

1.2 Potential fluctuations and scale invariance

The density fluctuation field $\delta(\vec{x})$ leads to fluctuations $\phi(\vec{x})$ in the gravitational potential, according to the Poisson equation

$$\nabla^2\phi(\vec{x}) = 4\pi\delta(\vec{x}),$$

or, in k-space,

$$4\pi\delta(k) = k^2\phi(k). \tag{3}$$

(Careful, though: this equation is technically only valid on scales much smaller than the horizon. There are velocity terms that become important on larger scales in an expanding space.) These fluctuations are also gaussian, homogeneous, and isotropic, so relations analagous to those in Equations (1) hold for ϕ, σ_ϕ^2 and $P_\phi(k)$.

Slow-roll inflation predicts that the variance in ϕ is **scale-invariant**; that is,

$$\frac{d\sigma_\phi^2}{d\ln k} \propto \Delta_\phi^2(k) = const.$$

Thus, using Equation (3), the density power spectrum is

$$P(k) = \langle|\delta(k)|^2\rangle \propto k^4\langle|\phi(k)|^2\rangle = k^4 P_\phi(k) \propto k\Delta_\phi^2(k) \propto k. \tag{4}$$

That is, the spectral index is $n = 1$. This is the famous Harrison-Zeldovich scale-invariant power spectrum $P(k) \propto k$.

2 Growth of perturbations and the transfer function

2.1 The growth of linear perturbations

Note: the "derivation" given below (which follows Bartelmann's conference proceedings) is very simplified and glosses over many details in arriving at its main conclusion. A more rigorous, complete discussion may be found, e.g., *in section 15.2 of Peacock's* Cosmological Physics.

Recall that, in a perfectly homogeneous, flat universe, the Friedmann equation reads

$$H(a) = \frac{8\pi G}{3}\rho_c(a),$$

where ρ_c is the critical density. If we add in a spherically symmetric perturbation with density ρ_1 slightly larger than ρ_c, this region will evolve like a curved universe, according to

$$H_1(a) + \frac{kc^2}{a_1^2} = \frac{8\pi G}{3}\rho_1(a).$$

If the overdensity δ is sufficiently small, then $H_1 \approx H$ and $a_1 \approx a$, and we obtain

$$\delta = \frac{\rho_1 - \rho_c}{\rho_c} \approx \frac{3kc^2}{8\pi G}\frac{1}{\rho_c a^2} \propto \frac{1}{\rho_c a^2}.$$

Now, we know that ρ_c evolves as a^{-4} in the radiation-dominated phase and as a^{-3} in the matter domination phase. Therefore,

$$\delta \propto \begin{cases} a^2 & \text{before } a_{eq} \\ a & \text{after } a_{eq}, \end{cases} \tag{5}$$

where a_{eq} is the epoch of matter-radiation equality. Note that formation of new structures would effectively cease in a Λ-dominated phase.

This computation is entirely independent of scale, meaning, for example, that the power spectrum evolves as

$$P(k,a) = D(a)P(k,a_i),$$

where $D(a)$ is the linear growth factor. That is, the normalization of the power spectrum grows with time, but the growth is scale-independent—the shape of the power spectrum remains the same as it was at the initial time a_i.

2.2 The supression of growth

The discussion above is valid for all perturbations of wavelength $\lambda > d_H(a_{eq})$, where $d_H(a_{eq})$ is the comoving size of the particle horizon at matter-radiation equality. Perturbations smaller than this enter the horizon during the radiation-dominated phase of the universe. During this phase, the dark matter contributes negligibly to the gravitational potential, so the dynamics of the dark matter are strongly coupled to the dynamics of the radiation-dominated plasma. Since sub-horizon perturbations in the plasma will stop growing and oscillate as sound waves, the growth of dark matter perturbations will be strongly suppressed in this epoch (as it happens, they grow logarithmically).

Another way to see that the growth of structure must be suppressed during the radiation-dominated epoch is to compare the timescales for expansion and collapse. The expansion timescale is proportional to the dominant component of the universe:

$$t_{exp} \sim 1/\sqrt{G\rho_R}$$

during radiation-domination. The timescale for the collapse of dark-matter haloes, on the other hand is

$$t_c \sim 1/\sqrt{G\rho_M} > t_{exp}.$$

The expansion is thus much faster than the collapse, so the growth of perturbations will be suppressed until the universe becomes matter-dominated.

Radiation may only affect the physics on sub-horizon scales, however, so perturbations with $\lambda > d_H(a_{eq})$ grow unencumbered. Smaller perturbations also grow as usual until they enter the horizon at epoch a_{enter}. They begin growing again at a_{eq}, by which time δ has been suppressed by the factor $f_{sup} = (a_{enter}/a_{eq})^2$ (since super-horizon perturbations are growing as a^2 in this epoch). The physical size of the horizon at epoch a is $c/H(a)$, so the *comoving* size is

$$d_H = \frac{c}{aH(a)} \approx \frac{c}{H_0}\frac{1}{a\sqrt{a^{-4}}} \propto a$$

during radiation-domination. Thus, a_{enter} is given by $\lambda = d_H(a_{enter}) \propto a_{enter}$, and we obtain (supposing for the moment that sub-horizon perturbations are completely frozen until a_eq)

$$f_{sub} = \left(\frac{\lambda}{d_H(a_{eq})}\right)^2 \propto \left(\frac{k_0}{k}\right)^2, \tag{6}$$

where k_0 is given by the horizon scale at a_{eq}:

$$k_0 \propto \frac{1}{d_H(a_{eq})} \propto \frac{1}{a_{eq}} = 3.0 \times 10^{-5}\Omega_M h^2, \tag{7}$$

where we have used the requirement $\Omega_R a_{eq}^{-4} = \Omega_M a_{eq}^{-3}$ and our knowledge of Ω_R/h^2 to get the final equality.

2.3 The transfer function and constraining Γ

The above calculation tells us that the power spectrum after a_{eq} is related to the primordial power spectrum $P_i(k)$ by

$$P(k) = P_i(k)T^2(k)$$

Where the **transfer function** $T(k)$ is given by

$$T^2(k) \propto \begin{cases} 1 & \text{for } k \ll k_0 \\ k^{-4} & \text{for } k \gg k_0. \end{cases} \tag{8}$$

Then, for scale-invariant initial fluctuations,

$$P(k) \propto \begin{cases} k & \text{for } k \ll k_0 \\ k^{-3} & \text{for } k \gg k_0. \end{cases} \tag{9}$$

Since matter-radiation equality actually lasts a rather long time, there is a slow, smooth transition between the two regimes in k. Hence, the power spectrum has a broad peak at $k_0 \propto \Omega_M h^2$. Since cosmological distances are generally measured in terms of Mpc$/h$, if we can measure k_0 in terms of $(\text{Mpc}/h)^{-1}$ we have simultaneously measured the quantity $\Omega_M h$.

Of course, things are not nearly as simple as this. For example, perturbations grow logarithmically during the radiation-dominated epoch—they are not entirely frozen—so the supression of power is not quite as strong as advertised. Also, in computing $T(k)$ above, we have ignored entirely the effect of acoustic oscillations in the baryon fluid before recombination. These oscillations will imprint a series of oscillations on the power spectrum, exactly as they do for the CMB spectrum. The correct power spectrum can be computed using an Einsten-Boltzmann computer code. Alternatively (and often more usefully) there exist various fitting functions for $T(k)$, one of which (due to Eisenstein and Hu, `astro-ph/9710252`) is accurate to $\sim 1\%$. These depend only on numerical factors and the scaled variable $k/(\Omega_M h^2)$. Hence, measuring $T(k)$ is still tantamount to measuring $\Omega_M h$. This parameter combination is so ubiquitous in large scale structure that it has been given a name, the **shape parameter**

$$\Gamma \equiv \Omega_M h$$

(be careful here: this is the most common definition of Γ, but many authors define it to include baryonic effects as well).

2.4 Other parameter constraints

There are two other obvious cosmological parameters that can be constrained by fitting the power spectrum. One is obviously the spectral index n of the primordial power spectrum. The other is the normalization parameter σ_8, defined as the rms fluctuation of the density field, smoothed with a spherical top-hat filter of radius 8 Mpc. This is given by

$$\sigma_8 = \frac{V}{(2\pi)^3} \int P(k) \left| f(k) \right|^2 k d^3k,$$

where $f(k) = 3y^{-3}(\sin y - y \cos y)$ is the Fourier transform of the top-hat filter. The σ_8 parameter is conventionally used to set the overall normalization of the power spectrum.

2.5 The nonlinear regime

So far we have only discussed the growth of linear perturbations. However, when $\Delta^2(k) \sim 1$, perturbations of wavelength $1/k$ become nonlinear and collapse. This means that power on large scales in the nonlinear is transferred to smaller scales, creating a suppression of power at large scales, an inflection point where the power spectrum is kicked outward, and an enhancement of power on small scales. The power spectrum eventually falls off again on the smallest scales, since gravitational collapse ceases when dark matter haloes become virialized. These processes also tend to smear out and erase any baryonic oscillations present in the power spectrum on nonlinear scales. Accurate fitting functions exist for fitting the nonlinear power spectrum.

3 Measuring the power spectrum

There are many ways to measure the power spectrum, each focussing on a particular range in k. The most commonly used methods up to the present time are the cluster abundance, the CMB, and galaxy redshift surveys. The last method probes P(k) on intermediate scales; it will be discussed in detail later in the course, so we will neglect it here. The cluster abundance can be used to fix the normalization σ_8, using the Press-Schechter formalism for predicting cluster abundances from a given power spectrum. This will also be discussed laterin the course. CMB data can be used to constrain P(k) on large scales by a complicated mapping from the usual ℓ-space used for CMB maps to the more physically meaningful k-space of the power spectrum.

Other methods for measuring the power spectrum have become increasingly popular in recent years. These include weak lensing surveys, in which one can reconstruct the cosmological dark matter density field in three dimensions. Thus far, such surveys have probed small to intermediate scales. Also, studies of the Lyman alpha forest, which probe the density field of intergalactic gas in overdense regions, thus measuring the power spectrum on very small (and highly nonlinear) scales. All of these methods are reviewed in Tegmark and Zaldarriaga (`astro-ph/0207047`).

4 Appendix: Independence of the k-modes

To show that k-modes are uncorrelated, we start by defining the **correlation function** (also more properly called the autocorrelation function), $\langle\delta_1\delta_2\rangle$, where δ_1 and δ_2 are the perturbation field (in real space or k-space or whatever space we like) evaluated at two different points. In real space, homogeneity means that this is translationally invariant, viz.,

$$\langle\delta(\vec{x})\delta(\vec{x}')\rangle = \langle\delta(\vec{x}+\vec{y})\delta(\vec{x}'+\vec{y})\rangle \equiv \xi(\vec{x}-\vec{x}').$$

That is, the correlation function depends only on $\vec{x}-\vec{x}'$ The k-space correlation function is then given by taking two fourier transforms:

$$\langle\delta(\vec{k})\delta(\vec{k}')\rangle \propto \int d^3x d^3x' e^{i\vec{k}\cdot\vec{x}} e^{i\vec{k}'\cdot\vec{x}'} \langle\delta(\vec{x})\delta(\vec{x}')\rangle = \int d^3x d^3x' e^{i\vec{k}\cdot\vec{x}} e^{i\vec{k}'\cdot\vec{x}'} \xi(\vec{x}-\vec{x}').$$

Changing variables to $\vec{r}=\vec{x}-\vec{x}'$ gives

$$\begin{aligned} \langle\delta(\vec{k})\delta(\vec{k}')\rangle &\propto \int d^3r d^3x e^{i(\vec{k}+\vec{k}')\cdot\vec{x}} e^{i\vec{k}\cdot\vec{r}} \xi(\vec{r}) \\ &\propto \delta(\vec{k}+\vec{k}')P(k), \end{aligned} \tag{10}$$

where $\delta(\vec{k}-\vec{k}')$ in the last line is a delta function (not a fluctuation), and we have used the fact that $P(k) \propto \int \exp(i\vec{k}\cdot\vec{r})\xi(\vec{r})d^3r$ in the last line. Thus, the correlation function in k-space is zero unless we are correlating a given mode with itself. Different modes are uncorrelated.