Lecture

Non-linear Growth of Structure

Spherical Collapse,

Virial Theorem,

Zel'dovich Approximation,

N-body Simulations

Filamentary Structure: Zel'dovich Approximation

Approximate the displacement from initial position

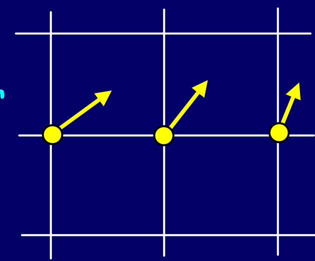
$$x(q,t) = q + D(t)\,\psi(q), \quad \psi = -\nabla\phi$$

Velocity & acceleration along displacement → trajectories straight lines

$$\dot{x} = \dot{D}\psi, \quad \ddot{x} = \ddot{D}\psi \propto \dot{x}$$

as in linear

central force → potential flow



In physical coordinates

$$r = ax, \quad v = \dot{r} = \dot{a}x + a\dot{x} = Hr + v_{pec}$$

Density (Lagrangian):

continuity $\rho(x,t)\,d^3x = \rho_q d^3q$

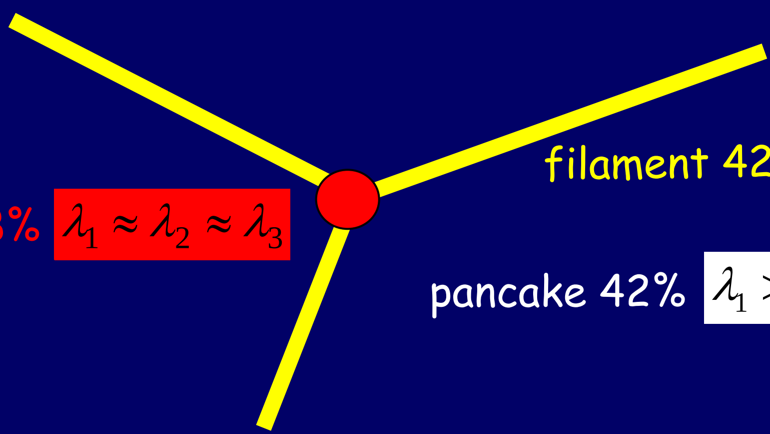
$$\rightarrow \rho(x,t) = \frac{\rho_q}{\|\partial\vec{x}/\partial\vec{q}\|} = \frac{\rho_q}{(1-D(t)\lambda_1)(1-D(t)\lambda_2)(1-D(t)\lambda_3)}$$

$$\lambda_i \equiv \frac{\partial^2\phi}{\partial^2 q_i}, \quad \lambda_1 \geq \lambda_2 \geq \lambda_3$$

Jacobian

→ caustics

deformation tensor eigenvalues



filament 42% $\lambda_1 \approx \lambda_2 >> \lambda_3$

cluster 8% $\lambda_1 \approx \lambda_2 \approx \lambda_3$

pancake 42% $\lambda_1 >> \lambda_2 \approx \lambda_3$

Zel'dovich Approximation cont'd

$$\rho(x,t) = \frac{\rho_q}{(1-D(t)\lambda_1)(1-D(t)\lambda_2)(1-D(t)\lambda_3)}$$

$$\lambda_i \equiv \frac{\partial^2 \phi}{\partial^2 q_i}, \quad \lambda_1 \ge \lambda_2 \ge \lambda_3$$

$$\delta = \frac{\rho}{\rho_q} - 1 = -D(\lambda_1 + \lambda_2 + \lambda_3) + D^2(\lambda_1\lambda_2 + \ldots) + D^3(\lambda_1\lambda_2\lambda_3) + \ldots$$

linear $$\delta = -D(\lambda_1 + \lambda_2 + \lambda_3) = -D\,\nabla\cdot\psi = -D\,\nabla\cdot\frac{\dot{x}}{\dot{D}} = -\frac{D}{\dot{D}}\nabla\cdot v = -\frac{1}{Hf(\Omega)}\nabla\cdot v$$

→ D is the growing mode of GI obeying $\ddot{D} + 2H\dot{D} = 4\pi G\rho D$

Error:

plug density in Poisson eq. $\delta_{Poisson} \propto \nabla^2 \phi_{grav} \propto -\nabla\psi = -(\lambda_1 + \lambda_2 + \lambda_3) \propto \delta_{linear}$

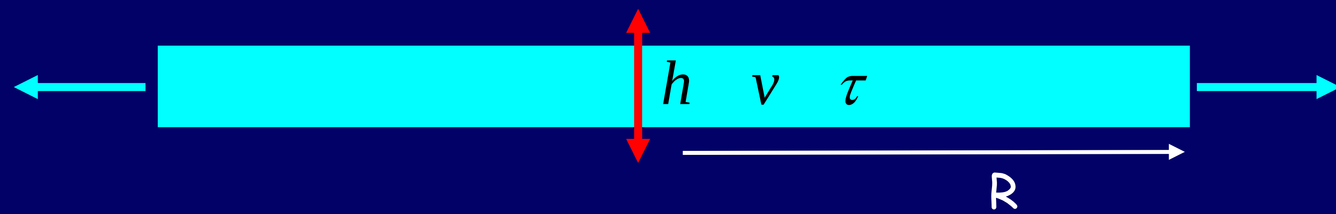
→ error is 2nd +3rd terms $$\frac{\Delta\rho}{\rho} = -(D\lambda_1)^2\left(\frac{\lambda_2}{\lambda_1} + \frac{\lambda_3}{\lambda_1} + \frac{\lambda_2\lambda_3}{\lambda_1^2}\right) + 2(D\lambda_1)^3\frac{\lambda_2\lambda_3}{\lambda_1^2}$$

error small in linear regime $D\lambda_1 << 1$

or pancakes $\lambda_1 >> \lambda_2, \lambda_3$

error big in spherical collapse $\lambda_1 \sim \lambda_2 \sim \lambda_3$

Non-dissipative Pancakes: why flat?



oscilation time << expansion time $\tau << H^{-1}$

adiabatic invariant $const. \sim \int_0^\tau \dot{x}^2 dt \sim v^2 \tau \sim h\, v$

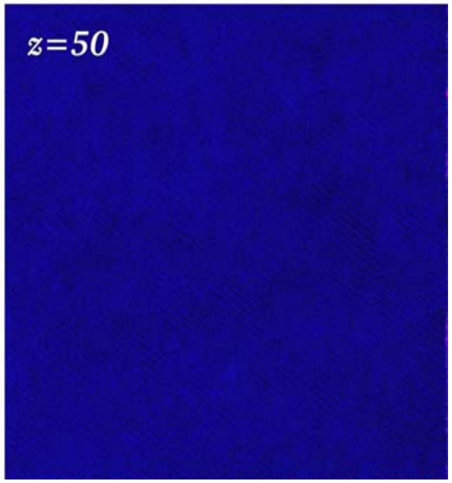
$$v \sim f\tau \propto f v^{-2} \rightarrow v \propto f^{1/3} \propto R^{-2/3}$$

$$f \propto \Sigma \propto R^{-2}$$

$$h \propto v^{-1} \propto R^{2/3}$$

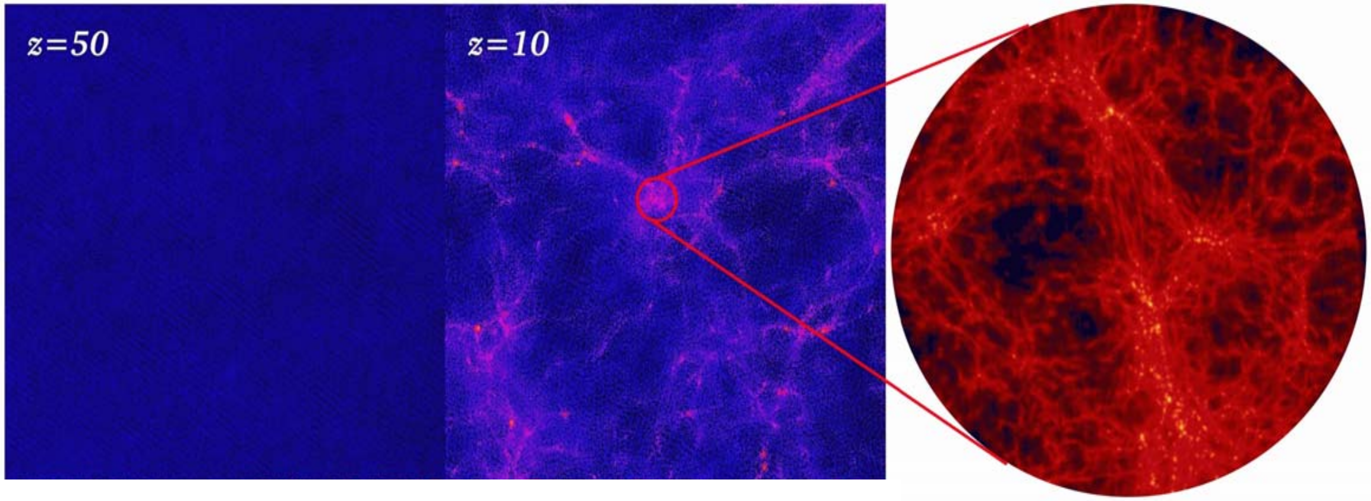
$$\frac{h}{R} \propto R^{-1/3} \propto a^{-1/3}$$

pancake becomes flatter in time



N-body simulation

ΛCDM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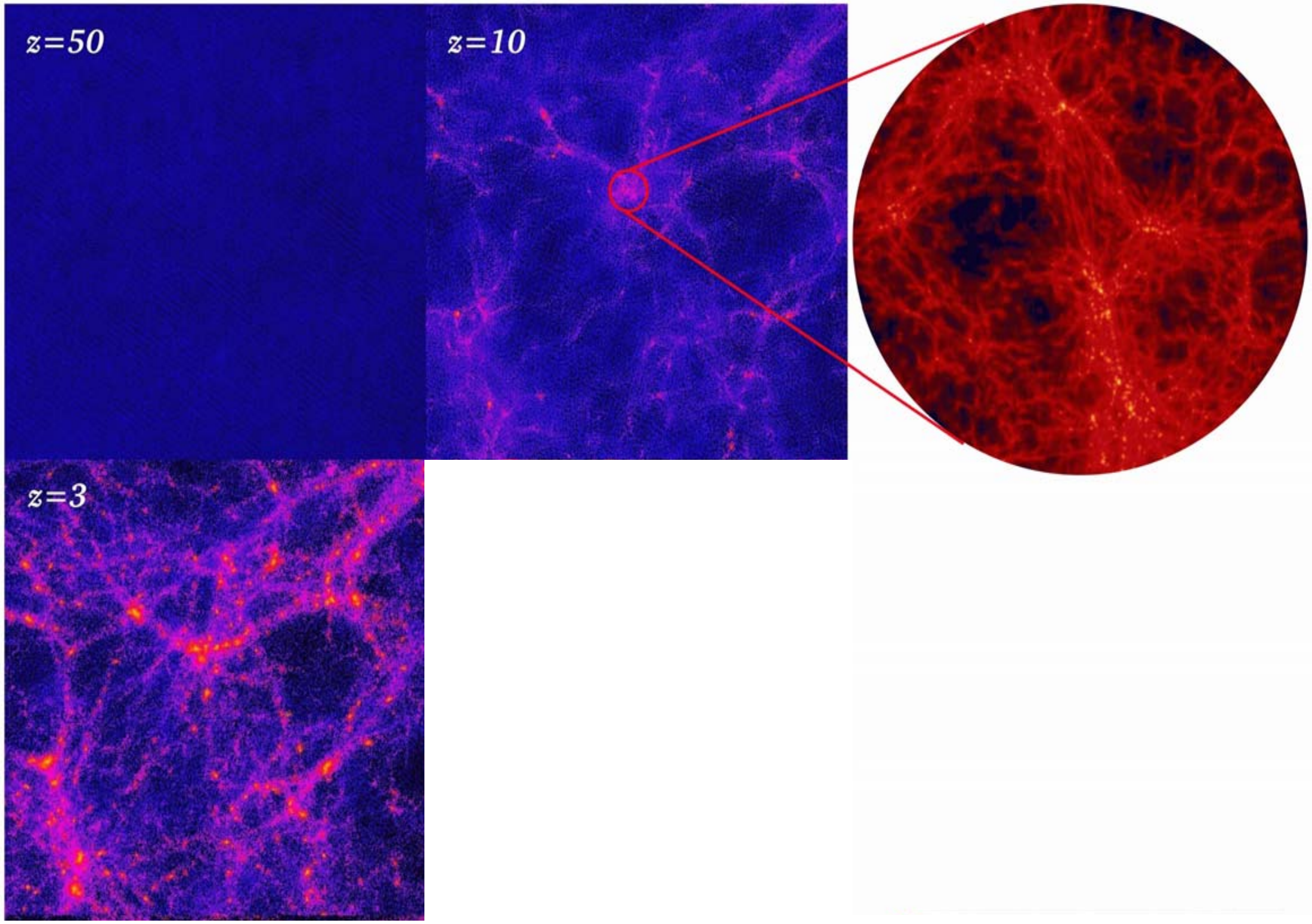
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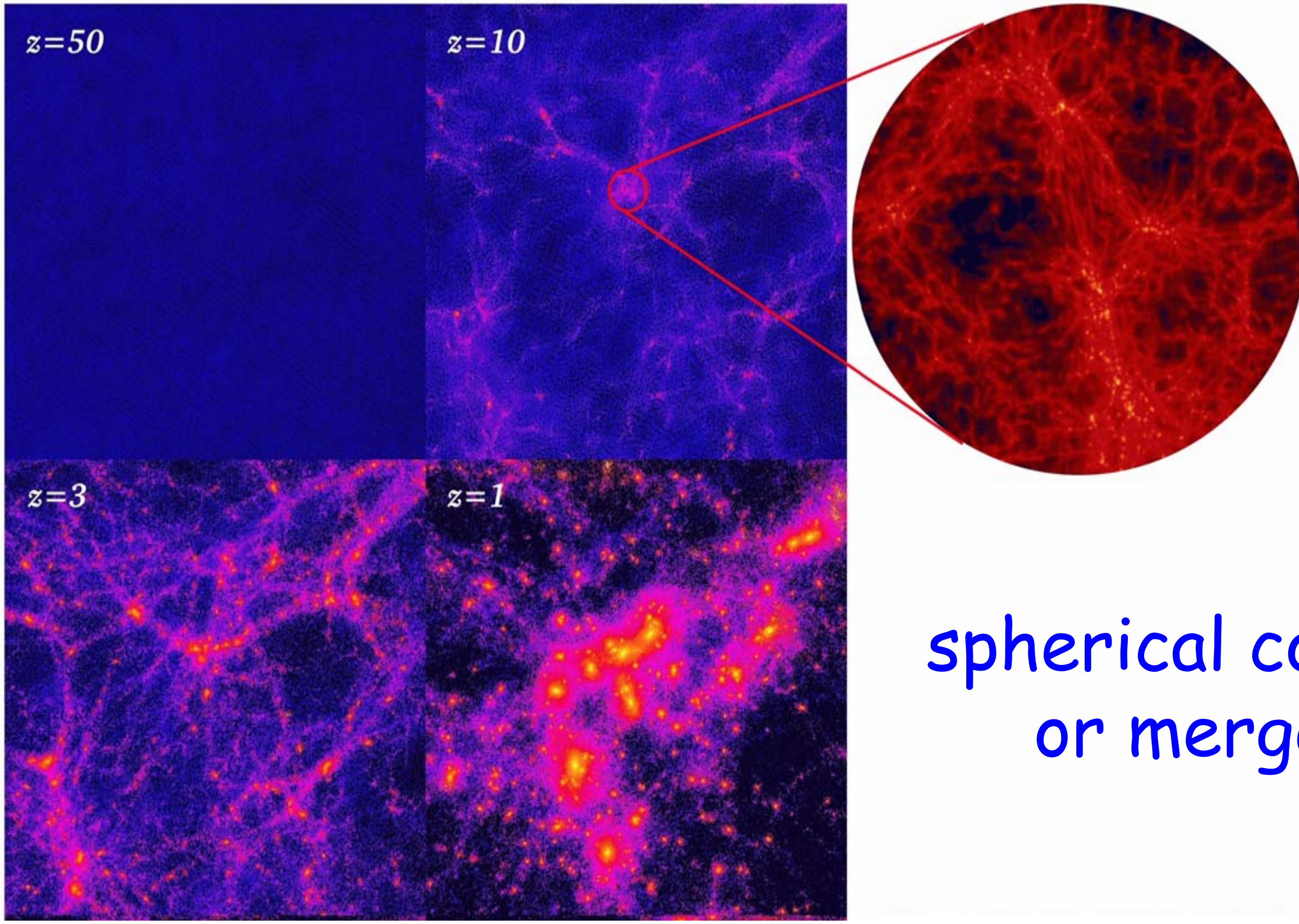
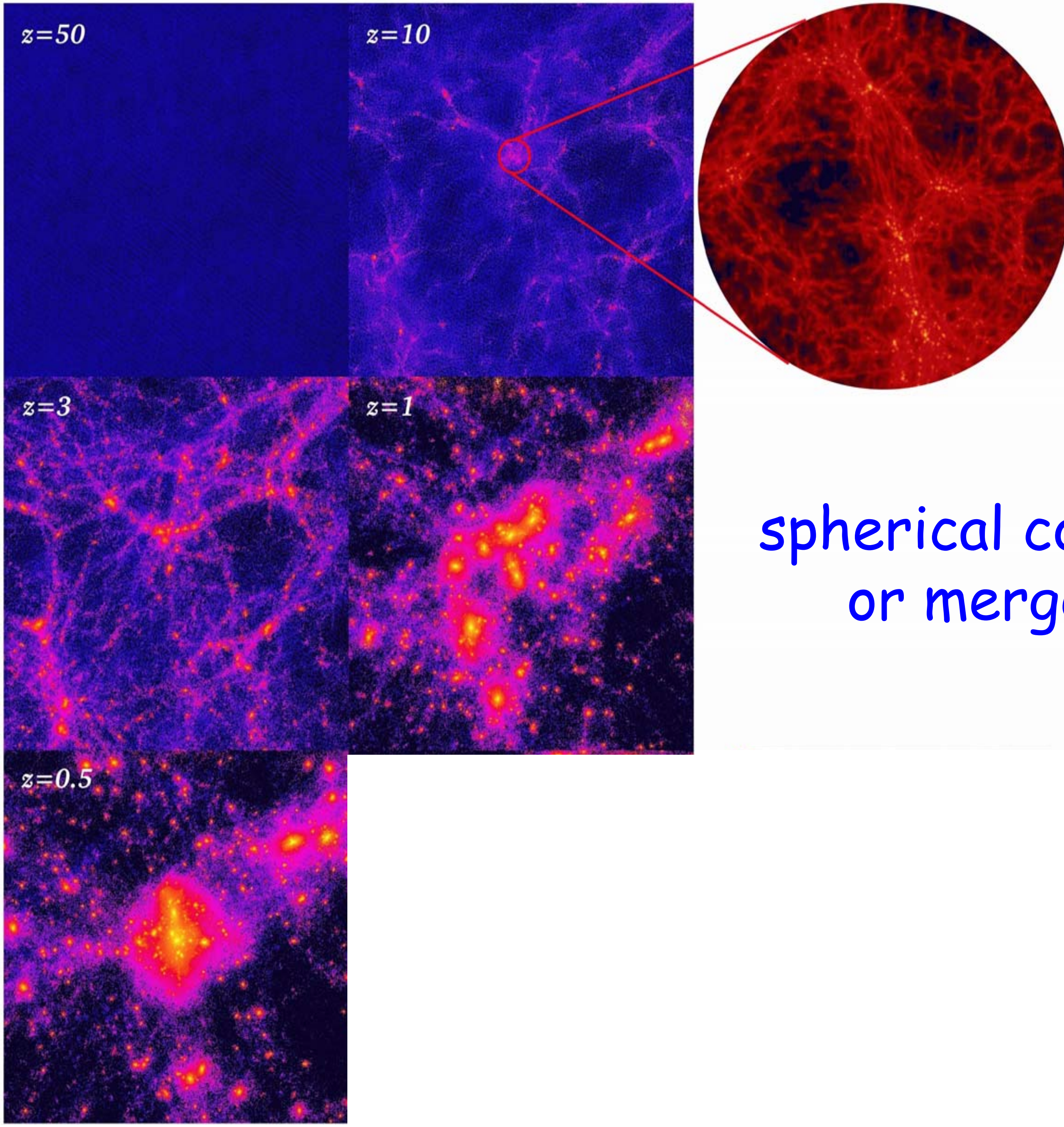


spherical collapse
or mergers



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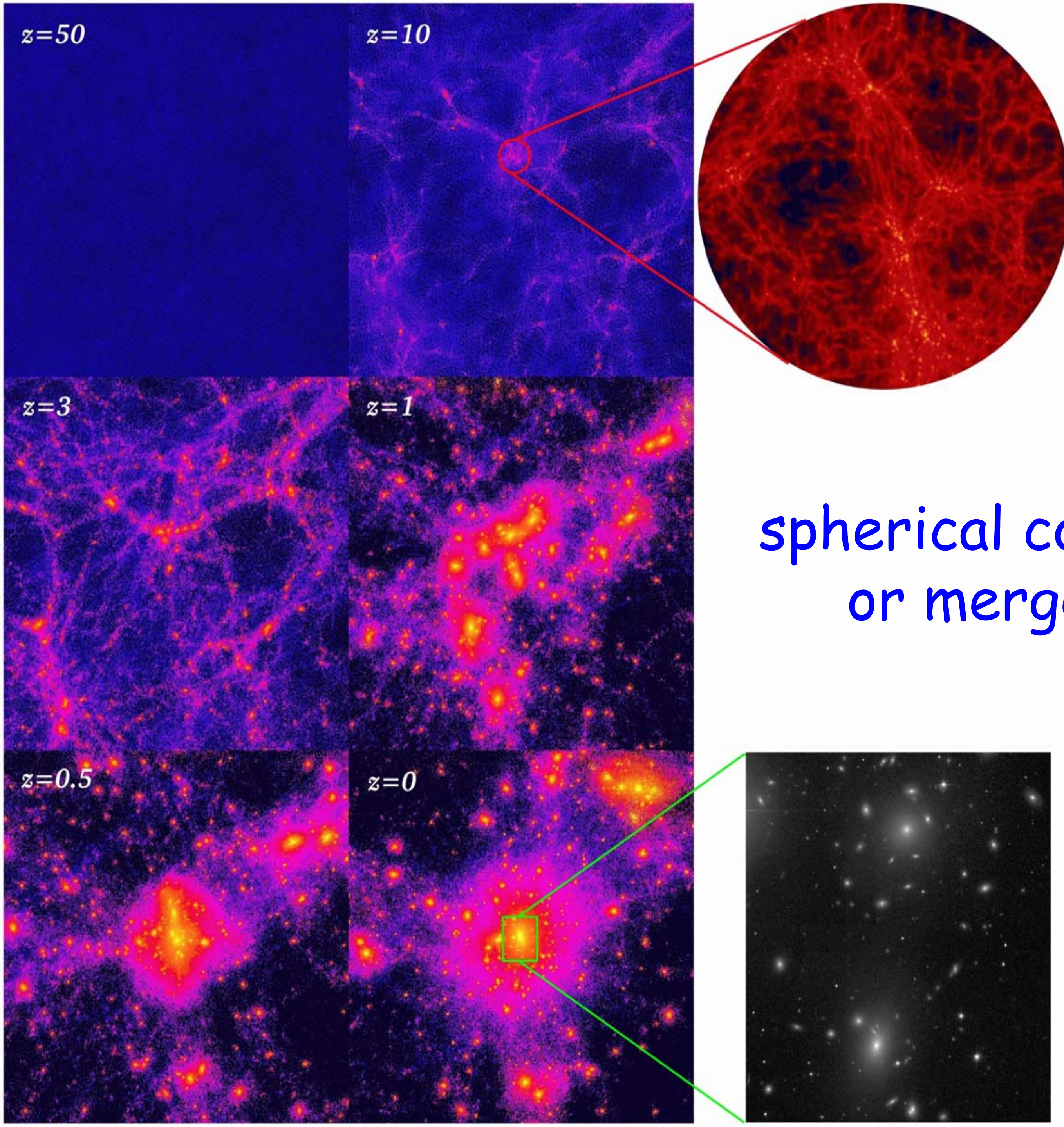
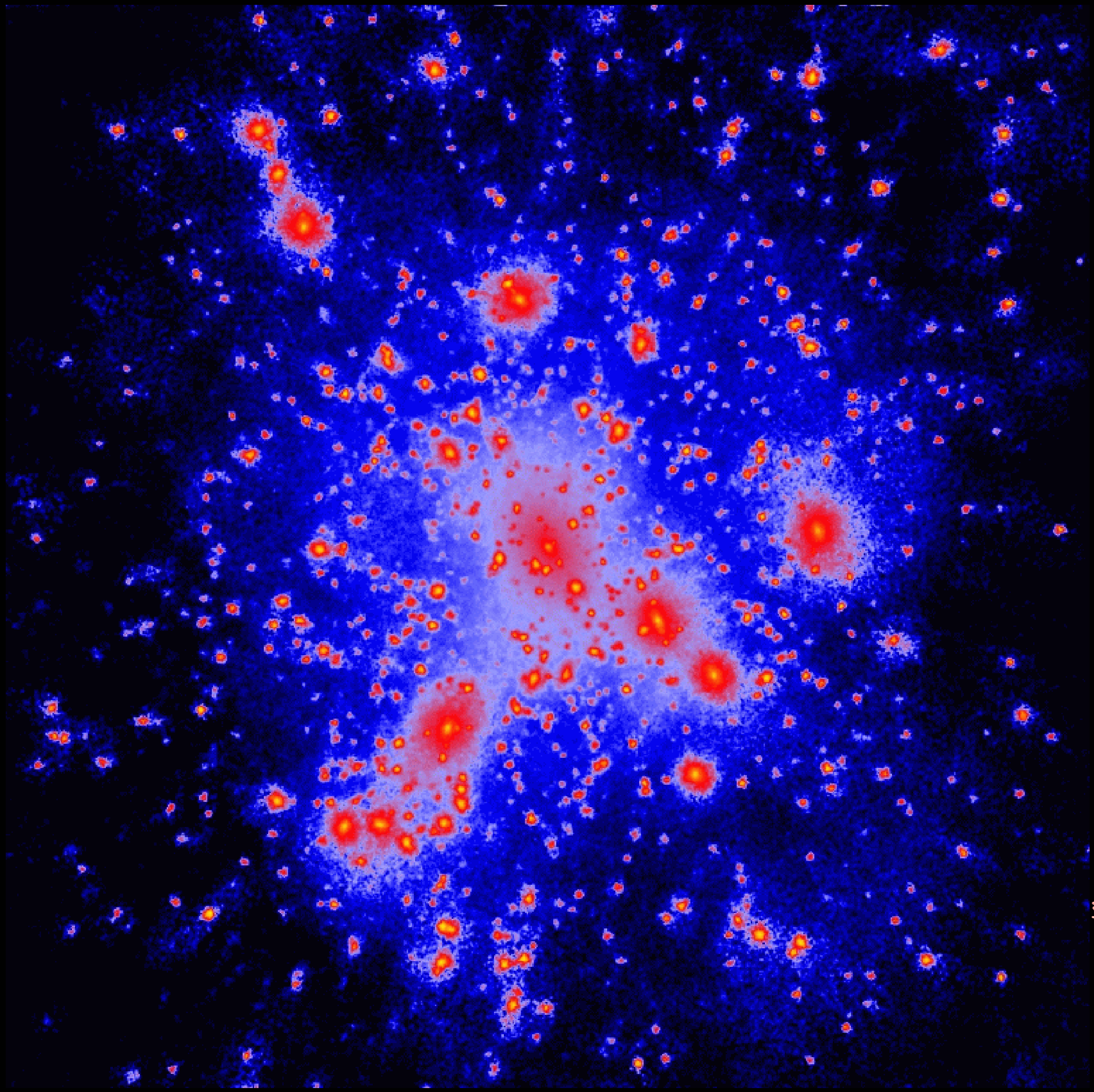


spherical collapse
or mergers

N-body simulation of Halo Formation



Top-Hat Model (Λ=0, matter era)

a bound sphere (k=1) in EdS universe (k=0)

$$\dot{a}^2 = \frac{2a^*}{a} - k \qquad a^* \equiv (4\pi/3)G\rho a^3 = const.$$

conformal time

$$d\eta \equiv \frac{dt}{a(t)}$$

$$a = (a^*/2)\eta^2 \qquad t = (a^*/6)\eta^3$$

$$a_p = a_p^*(1-\cos\eta_p) \qquad t = a_p^*(\eta_p - \sin\eta_p)$$

$$t_p = t \rightarrow \eta^3(\eta_p) = \frac{6a_p^*}{a^*}(\eta_p - \sin\eta_p) \rightarrow a(\eta_p) = \frac{1}{2}\left(\frac{6a_p^*}{a^*}(\eta_p - \sin\eta_p)\right)^{2/3}$$

overdensity:

$$a^* \propto \rho a^3 \qquad a_p^* \propto \rho_p a_p^3 \rightarrow \frac{\rho_p}{\rho} = \frac{a_p^*}{a^*}\left(\frac{a}{a_p}\right)^3 = \frac{9(\eta_p - \sin\eta_p)^2}{2(1-\cos\eta_p)^3}$$

linear perturbation $\delta\rho/\rho << 1 \quad \eta_p << 1$

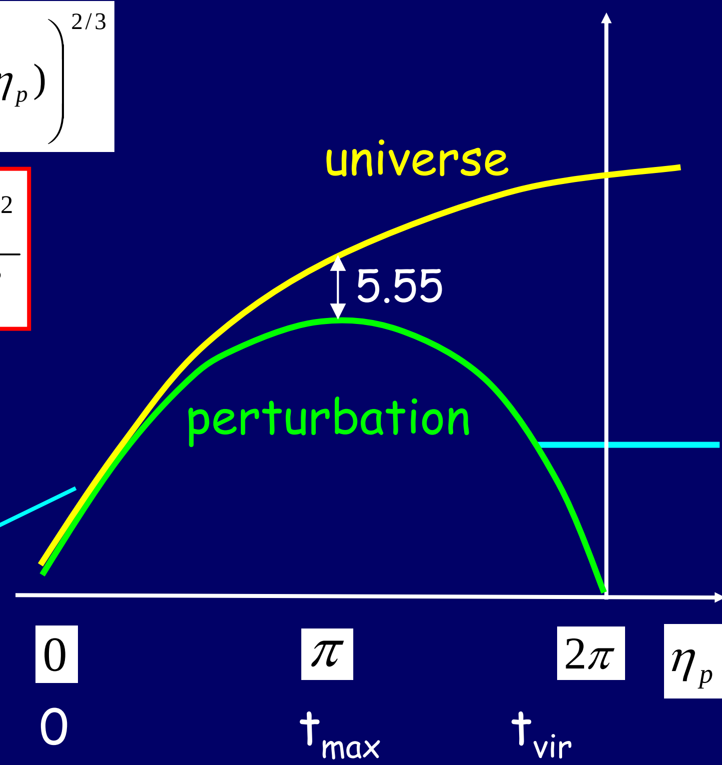
Taylor $\cos\eta \approx 1 - \frac{1}{2}\eta^2 + \frac{1}{24}\eta^4 \quad \sin\eta \approx \eta - \frac{1}{6}\eta^3 + \frac{1}{120}\eta^5$

$$\frac{\delta\rho}{\rho} \equiv \frac{\rho_p - \rho}{\rho} \approx 0.15\eta_p^2 \propto a \propto t^{2/3}$$

$$\delta \propto a$$

turnaround $\eta_p = \pi$

$$\frac{\rho_p}{\rho} = \frac{9\pi^2}{16} \approx 5.55$$



linear equivalent to collapse

$$\delta_{2\pi} = \delta(\eta_p << 1)\left(\frac{t(\eta_p = 2\pi)}{t(\eta_p << 1)}\right)^{2/3} = 0.15\eta_p^2\left(\frac{2\pi}{\eta_p^3/6}\right)^{2/3} \approx 1.68 \equiv \delta_c$$

- **Collapse to Virial Equilibrium**

$$E_{max} \simeq -\frac{GM^2}{R_{max}} \quad (E_k \simeq 0) \qquad E_{vir} \simeq \frac{1}{2} E_{grav} \simeq -\frac{1}{2}\frac{GM^2}{R_{vir}}$$

$$\text{E conserved} \quad \rightarrow \quad \boxed{\frac{R_{vir}}{R_{max}} \simeq \frac{1}{2}} \quad \rightarrow \quad \frac{\rho_{vir}}{\rho_{max}} \simeq 8$$

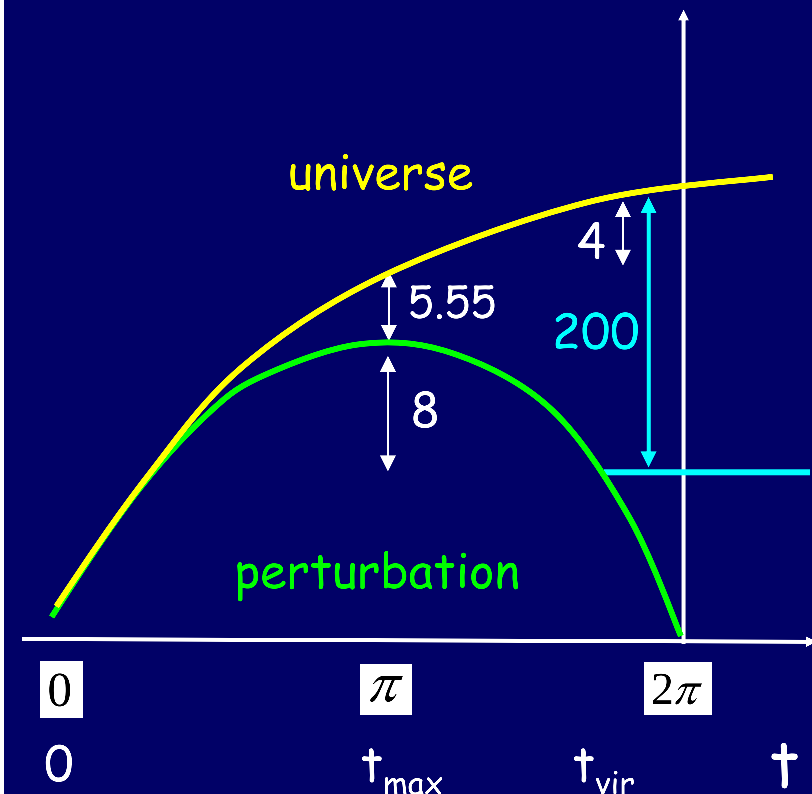
- **Virial density:**

$$\frac{\rho_{vir}}{\rho_{univ}} \simeq 5.55 \times 8 \times \left(\frac{a_{vir}}{a_{max}}\right)^3$$

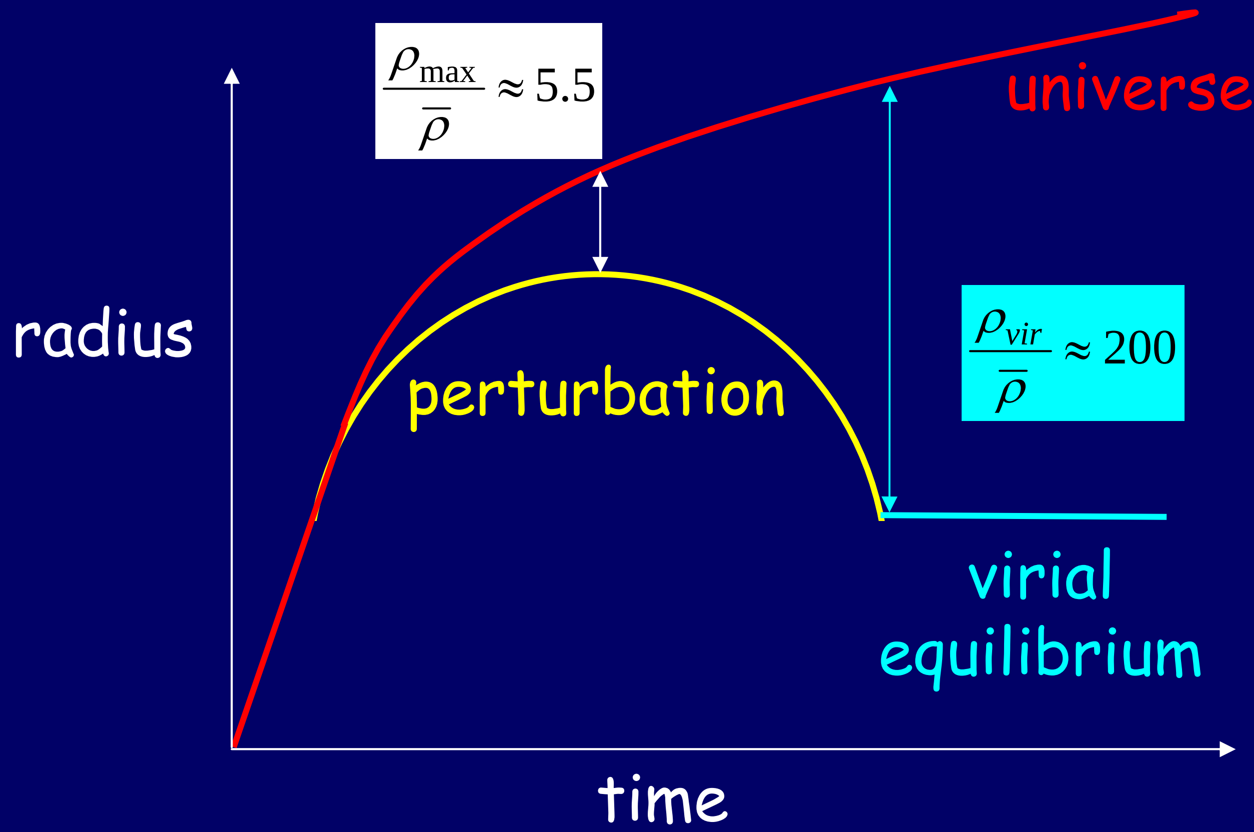
Assume virialization at collapse, $\eta_p \simeq 2\pi$,

$$\frac{t_{col}}{t_{max}} \simeq \frac{2\pi}{\pi} = 2 \quad \rightarrow \frac{a_{vir}}{a_{max}} = \left(\frac{t_{col}}{t_{max}}\right)^{2/3} \simeq 2^{2/3}$$

$$\rightarrow \boxed{\frac{\rho_{vir}}{\rho_{univ}} \simeq 5.55 \times 8 \times 4 \simeq 178 \sim 200}$$



Spherical Collapse



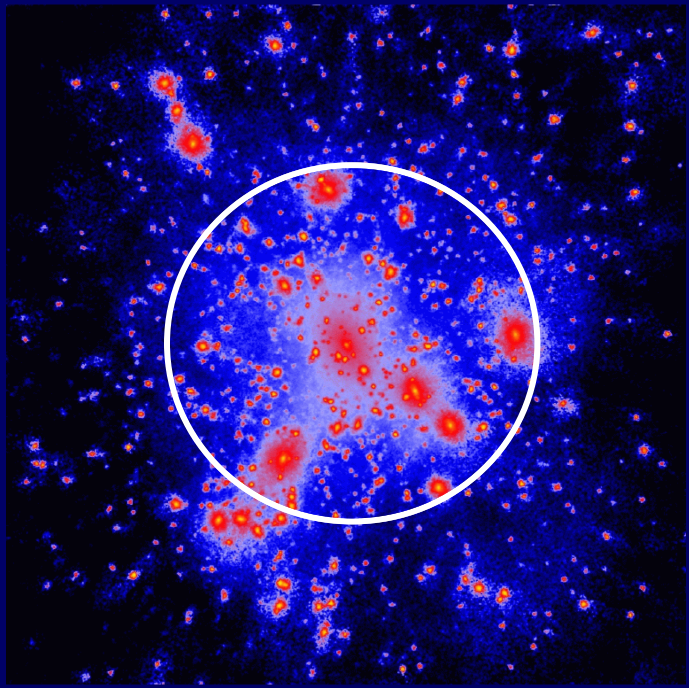
virial equilibrium: $$E = -\frac{1}{2}\frac{GM}{R_{vir}} = -\frac{GM}{R_{\max}}$$

Virial Scaling Relations

Virial equilibrium: $V^2 = \frac{GM}{R}$

Spherical collapse: $\frac{M}{(4\pi/3)R^3} = \Delta\rho_u = \Delta\rho_{u0}a^{-3}$ $\Delta \approx 200$

$\rightarrow M \propto V^3 a^{3/2} \propto R^3 a^{-3}$



Weak dependence on time of formation:

$D(a)\delta_0(M) \approx 1 \quad \rightarrow \quad a \propto M^{\alpha} \quad \alpha = (n+3)/6 \approx 0.1 - 0.2$

$M \propto V^4 \quad \text{for } n = -2$

Practical formulae:

$\rho_u \approx 2.76 \times 10^{-30}\, g\, cm^{-3}\, \Omega_{m0.3} h_{0.7}^2 a^{-3}$

$M_{11} \approx V_{100}^3 A^{-3/2} \approx R_{Mpc}^3 A^{-3}$

$A \equiv a\, (\Delta_{200}\Omega_{m0.3}h_{0.7}^2)^{-1/3}$

$\Delta(a) \approx [18\pi^2 - 82\Omega_\Lambda(a) - 39\Omega_\Lambda(a)^2]/\Omega_m(a) \quad \Delta(a << 1) \approx 178 \quad \Delta_0 \approx 340$

$\Omega_m(a) = \frac{\Omega_m a^{-3}}{\Omega_\Lambda + \Omega_m a^{-3}} \quad \Omega_m(a) + \Omega_\Lambda(a) = 1$