Lecture

Hierarchical Clustering

Press Schechter: Halo Distribution
Extended PS: Merging Tree
Biasing: Galaxies/Subhalos in Halos
HOD: Halo Occupation Distribution
Press Schechter Formalism \( n(M,a) \)

Gaussian random field \( P(\delta) = (2\pi\sigma^2)^{-1/2} \exp(-\delta^2 / 2\sigma^2) \)

random spheres of mass \( M \)

linear-extrapolated \( \delta_{\text{rms}} \) at \( a \): \( \sigma(M,a) = \sigma_0(M) D(a) \)

fraction of spheres with \( \delta > \delta_c = 1.68 \):

\[
F(M,a) = \int_{\delta_c}^{\infty} d\delta [2\pi\sigma^2(M,a)]^{-1/2} \exp[-\delta^2 / 2\sigma^2(M,a)]
\]

\[
= (2\pi)^{-1/2} \int_{\delta_c/\sigma(M,a)}^{\infty} dx \exp(-x^2 / 2)
\]

\[
\nu_c \equiv \frac{\delta_c}{D(a) \sigma_0(M)}
\]

PS ansaz: \( F \) is the mass fraction in halos \( > M \) (at \( a \))

derivative of \( F \) with respect to \( M \):

\[
n(M,a) dM = - \left( \frac{2}{\pi} \right)^{1/2} \frac{\bar{\rho}}{M} \nu_c \frac{d \ln \sigma_0}{d \ln M} \exp \left( - \frac{\nu_c^2}{2} \right) \frac{dM}{M}
\]

Mo & White 2002
Press Schechter Formalism cont.

\[ n(M,a)dM = -\left(\frac{2}{\pi}\right)^{1/2} \frac{\bar{\rho}}{M} \nu_c \frac{d \ln \sigma_0}{d \ln M} \exp\left(-\frac{\nu_c^2}{2}\right) \frac{dM}{M} \]

Example: \[ P_k \propto k^n \rightarrow \sigma_0(M) \propto M^{-\alpha} \rightarrow \nu_c = (M/M_*)^\alpha \]
\[ \alpha = (3+n)/6 \quad \frac{d \ln \sigma_0}{d \ln M} = \alpha \]

\[ n(M) \propto \alpha \tilde{M}^{\alpha-2} \exp(-\tilde{M}^{2\alpha}/2) \quad \tilde{M} = M/M_* \]

self-similar evolution, scaled with \( M_* \)

\[ \nu_c \equiv \frac{\delta_c}{D(a) \sigma_0(M)} \quad M_*(a) \text{ defined by } \sigma(M_*,a) \equiv \delta_c \]

\[ \sigma^2(R) = (2\pi)^{-1} \int_0^\infty dk k^2 P(k) \tilde{W}^2(kR) \]

Top Hat
\[ W_R(x) = \Theta(x/R) \quad \tilde{W}_R(k) = 3[\sin(kR) - kR \cos(kR)]/(kR)^3 \]

approximate
\[ M_*(a) = M_0 D(a)^{1/\alpha} \sim 10^{13} M_0 a^5 \]
in a flat universe
\[ D(a) = a g(a)/g(1) \]
\[ g(a) \approx \frac{5}{2} \Omega_0(a) \left( \Omega_0(a)^{4/7} - \Omega_\Lambda(a) + \frac{1+\Omega_0(a)/2}{1+\Omega_\Lambda(a)/70} \right)^{-1} \]
\[ \Omega_0(a) = \frac{\Omega_m a^{-3}}{\Omega_\Lambda + \Omega_m a^{-3}} \]
Better fit using ellipsoidal collapse (Sheth & Tormen 2002)

\[ F(> M, a) \approx 0.4(1 + 0.4 / \nu^{0.4}) \text{erfc}(0.85 \nu / 2^{1/2}) \]

1\(\sigma\), 2\(\sigma\), 3\(\sigma\) 22%, 4.7%, 0.54%

Comparison of PS to N-body simulations
Press-Schechter in $\Lambda$CDM

$$\Omega_m = 0.3, \Omega_\Lambda = 0.7, h = 0.7, \sigma_8 = 0.9$$

$$\log M_* \approx 13.13 - 1.3z \quad (z \leq 2)$$
Press-Schechter by Excersion Set: \( n(M,a) \)

Linear \( \delta(x) \) at some fiducial \( a \) when \( D(a)=1 \)

Top-hat smoothing in k-space:

\[
\delta_s(x; k_c) = \int_{k<k_c} d^3k \delta(k) e^{-ikx}
\]

At a fixed point \( x \). As \( k_c \) varies, \( \delta_s \) executes a random walk:

\[
\Delta \delta_s = \delta_s(x; k_c + \Delta k_c) - \delta_s(x; k_c) = \int_{k<k_c+\Delta k_c} d^3k \delta(k) e^{-ikx}
\]

\[
\Delta \sigma_0^2 = \sigma_0^2(k_c + \Delta k_c) - \sigma_0^2(k_c)
\]

\[
\sigma_0^2(k_c) = \int d^3k P(k)
\]

\( \Delta \delta_s \) is a Gaussian random variable, independent of \( \delta_s \): Markov random walk.

PS ansatz: mass element initially at \( x \) belongs to halo of mass \( M \) at \( a \) if the random walk first crosses \( \delta_c/D(a) \) at \( \sigma_0^2(M) \)

Fraction of mass in halos \( \gg M \) = fraction of trajectories \( \delta_s(x; k_c) \) which first cross \( \delta_c/D \) at \( k_c<k(M) \)

Solution:

\[
n(M, a) dM = -\left( \frac{2}{\pi} \right)^{1/2} \frac{\rho_c}{M} \frac{d \ln \sigma_0^2}{d \ln M} \exp \left( -\frac{\nu_c}{2} \right) \frac{dM}{M} \quad \nu_c \equiv \frac{\delta_c}{D(a) \sigma_0(M)}
\]

Markov: Past of \( x \) (right) independent of its future (left), so the history of halos of mass \( M \) is superclusters and in voids are statistically identical, i.e. their galaxy populations should be identical.
Proof:

Fraction of mass in halos $\geq M = $

fraction of trajectories $\delta_s(x; k_c)$ which first cross $\delta_c/D$ at $k_c < k(M)$

For a given $a$ and $k_c = K_c$, $\delta_s$ is Gaussian:

$$P(\delta_s) = [2\pi D(a) \sigma_0(K_c)]^{-1/2} \exp(-\delta_s^2 / 2D^2 \sigma_0^2)$$

#(points $\delta_s < \delta_c$ for all $k_c < K_c$) =

#($\delta_s < \delta_c$ for $k_c = K_c$)

- #($\delta_s < \delta_c$ for $k_c = K_c$ but $\delta_s > \delta_c$ at some $k_c < K_c$)

$$P_{\text{first}}(\delta_s) = \frac{1}{(2\pi \sigma^2)^{1/2}} \left[ \exp\left(-\frac{\delta_s^2}{2\sigma^2}\right) - \exp\left(-\frac{(2\delta_c - \delta_s)^2}{2\sigma^2}\right) \right]$$

$$F(> K_c) = \int_{-\infty}^{\delta_c} P(\delta_s) d\delta_s = \int_{-\infty}^{\delta_c/\sigma} \frac{dx}{(2\pi)^{1/2}} e^{-x^2/2} - \int_{\delta_c/\sigma}^{\infty} \frac{dx}{(2\pi)^{1/2}} e^{-x^2/2}$$

Differentiate with respect to $k_c$ (or $M$):

$$n(M, a) dM = \left(\frac{2}{\pi}\right)^{1/2} \frac{\bar{\rho}}{M} \nu_c d\ln \sigma_0 \exp\left(-\frac{\nu_c}{2}\right) \frac{dM}{M} \nu_c = \frac{\delta_c}{D(a) \sigma_0(M)}$$

$$n(M) \propto \alpha \tilde{M}^{\alpha - 2} \exp\left(-\tilde{M}^{2\alpha} / 2\right) \tilde{M} \equiv M / M_*$$
Mass versus Light Distribution

- Halo mass
- Bright-end problem
- Faint-end problem
- Galaxy stellar mass
- 40% of baryons

Graph showing the distribution of mass versus light, with markers for halo mass, galaxy stellar mass, and bright-end problem.
Conditional Merger Tree: Extended Press-Schechter
Merger Tree: conditional probability
Extended Press-Schechter (EPS): Merger Tree

Given that a mass element belongs to halo $M_1$ at $z_1$, what is the probability that it belonged to halo $M_2 (<M_1)$ at $z_2 (>z_1)$?

Equivalent:

Given that $\delta_s(x;k_c)$ first crossed $\delta_c/D(a_1)$ at $k_c=k(M_1)$, what is the probability that it first crossed $\delta_c/D(a_2)$ at $k_c=k(M_2)$.

The same problem as before but with the origin shifted:

$$n(M_2|z_2,M_1,z_1)\,dM_2 = \left(\frac{2}{\pi}\right)^{1/2} \frac{M_1}{M_2} \frac{\delta_c(D_2^{-1}-D_1^{-1})}{\left(\sigma_2^2-\sigma_1^2\right)^{1/2}} \frac{d\ln(\sigma_2^2-\sigma_1^2)^{1/2}}{d\ln M_2} \exp\left(\frac{\delta_c^2(D_2^{-1}-D_1^{-1})^2}{2\left(\sigma_2^2-\sigma_1^2\right)}\right) \frac{dM_2}{M_2}$$

• # of bright E galaxies in a cluster: $M=10^{15}$ today, how many $10^{12}$ progenitors at $z=2$?
• descendents of LGBs: massive halos at $z=3$ have $n=10^{-2} \text{Mpc}^{-1}$, what mass halos do they inhibit today?
• When did the most massive progenitor include half its current mass?
• How often do two $10^{12}$ halos merge?
• Infall rate of spirals into clusters: How often does a $10^{15}$ halo accrete a $10^{12}$ halo?
Formation of galaxies in a cluster

GIF
Orbits that lead to Mergers

Binney & Tremaine
Galaxy/Halo Biasing

Examples:
- cluster clustering
- bright galaxies (LBG)
- clustering of different galaxy types
Biasing: Subhalos in Host Halos (from EPS)

Host halo: a sphere of radius $R$ today, mass $M$.

$$\delta = \frac{M}{(4\pi / 3)\bar{\rho}R^3} - 1$$

comoving initial radius

$$R_0 = R(1+\delta)^{1/3}$$

linear-extrapolated to today

$$\delta_0(\delta; \Omega, \Lambda)$$

Subhalos: average # of subs $(m, z)$ in host $(R, \delta)$, using EPS

$$N_{\text{subs}}(m, z | R_0, \delta_0)$$

Average over all $\delta$'s at fixed $R$

$$\bar{N}_{\text{halos}} = n_{\text{halos}}(m, z) (4\pi / 3)R^3$$

$$\delta_{\text{subs}} = N_{\text{subs}} / \bar{N}_{\text{halos}} - 1$$

Obtain from EPS for small subs and proto-host-halo

$$m \ll M, D\delta \ll \delta_c$$

$$\nu \equiv \frac{\delta_c}{D\sigma_0(m)}$$

$$\nu = 1 \text{ for } m = M_*(z)$$

Linear biasing factor: $b \approx 1$ for $m \approx M_*(z)$

Mo & White

Peak biasing in a Gaussian field

$$P(\delta > \nu\sigma) \propto \exp\left(-\frac{(\nu\sigma + \epsilon)^2}{2\sigma^2}\right)$$

Kaiser 1984, Bardeen et al. 86

$$\xi_{\text{subs}}(r) = (\nu / \sigma)^2 \xi_{\text{mass}}(r)$$
Elliptical galaxies in the local universe: biased with respect to the dark matter

ACDM CR : E and S0 galaxies
Credits: Mathis, Lemson, Springel, Kauffmann, White and Dekel.
Massive Ellipticals in Clusters

SDSS
Kauffmann et al. 04
Nonlinear Stochastic Biasing

\[ b(\delta)\delta = \langle g \mid \delta \rangle = \int dg \, P(g \mid \delta) \, g \]

\[ \epsilon = g - \langle g \mid \delta \rangle \]

mean biasing

\[ \hat{b} = \langle (b(\delta)\delta^2) \rangle / \sigma^2 \]

“linear” biasing

\[ \tilde{b}^2 = \langle (b(\delta)\delta^2)^2 \rangle / \sigma^2 \]

nonlinearity

biasing scatter

\[ \sigma_b^2 = \langle \epsilon^2 \rangle / \sigma^2 \]
Correlation Function and HOD
Galaxy type correlated with large-scale structure

Semi-Analytic Modeling
Power Spectrum

![Power Spectrum Graph]

- Intergalactic hydrogen clumping
- Gravitational lensing
- Cluster abundance
- Cosmic microwave background
- SDSS galaxy clustering

Density fluctuations vs. Scale (millions of lightyears)
$\Lambda\text{CDM Power Spectrum}$

$$P(k) \propto k T^2(k)$$

$$T(k) = \frac{\ln(1 + 2.34q)}{2.34q} \left(1 + 3.89q + (16.1q)^2 + (5.46q)^3 + (6.71q)^4\right)^{-1/4}$$

$$q = \frac{k}{\Omega_m h^2 \text{Mpc}^{-1}}$$

normalization:

$$\sigma_8 \equiv \sigma_{\text{tophat}} (R = 8h^{-1}\text{Mpc})$$
**Correlation Function**

\[
\xi(r) = \langle \delta(\vec{x}) \delta(\vec{x} + \vec{r}) \rangle_{\vec{x}} \tag{1}
\]

\[
\rightarrow \xi(r) = \left\langle \sum_{k} \sum_{k'} \tilde{\delta}_k \tilde{\delta}_{k'} e^{i(k-k') \cdot \vec{r}} e^{-i k \cdot r} \right\rangle
\]

\(\delta\) real, can replace by complex conjugate \(\bar{\delta}_{k'}(-k') = \bar{\delta}_{k'}^*(k')\)

all cross terms \(k \neq k'\) vanish on average because of periodic boundary conditions

isotropy \(\langle |\delta_k|^2(k) \rangle = |\delta_k|^2(k)\)

angular integration \(\int_0^\pi \cos(k r \cos \theta) \sin \theta d\theta = (kr)^{-1} \int_0^{kr} \cos y dy\)

\[
\xi(r) = \frac{V}{2\pi} \int \left\langle |\delta_k|^2 \right\rangle e^{-ik \cdot r} d^3 k
\]

**Example**

\[
P(k) \propto k^n \rightarrow \xi(r) \propto r^{-(n+3)} \int_{kr=0}^{\infty} dx \ x^{n+1} \sin x
\]

**Alternative interpretation:**

Construct a realization: select \(\rho(x_1)\) and \(\rho(x_2)\) from an ensemble.

Place a galaxy at volume \(\delta V\) with probability \(\delta P = \rho(x) \delta V\)

\[
\delta = (\rho - \langle \rho \rangle) / \langle \rho \rangle \ \rightarrow \ \langle \rho(x) \rho(x+r) \rangle = \langle \rho \rangle^2 [1 + \xi(r)]
\]

Excess probability over Poisson

\[
1 + \xi(r) = \frac{\# \text{ pairs } (r)}{\# \text{ Poisson pairs } (r)}
\]
Galaxy Correlation Function

Crude Description: power law

\[ \xi(r) = \left( \frac{r}{r_0} \right)^{-\gamma} \]

\[ r_0 \approx 5 \ h^{-1} \text{Mpc} \quad \gamma \approx 1.8 \]

Zehavi et al. 2004
SDSS
Measured Correlation Functions

\[ \xi(r_p, \pi) \]

\[ \omega_p(r_p) = 2 \int_0^\infty d\pi \xi(r_p, \pi) \]

\[ \omega_p(r_p) = 2 \int_0^\infty dy \xi[(r_p^2 + y^2)^{1/2}] \]

\[ = 2 \int_{r_p}^\infty r \, dr \xi(r) (r^2 - r_p^2)^{-1/2} \]

\[ \xi(r) = -\pi^{-1} \int_r^\infty \omega_p(r_p) (r_p^2 - r^2)^{-1/2} \, dr_p \]

\[ v_{\text{obs}} = cz = Hr + v_{\text{pec}} \]
Galaxy Correlation Function

Zehavi et al. 04 SDSS
Biasing: Luminosity

\[ \frac{b}{b_\star} = \frac{\omega_p(L)}{\omega_p(L_\star)} \text{ at } r_p = 2.7 h^{-1} \text{Mpc} \]

Zehavi et al. 04  SDSS
Luminosity Dependence of Galaxy Clustering

Correlation length (Mpc/h)

2dF Survey

Luminosity
Biasing: color

![Graph showing biasing color]
Luminosity function: Early vs Late type

$M_{*_{\text{crit}}} \sim 3 \times 10^{10} M_\odot$

SDSS
Baldry et al. 04
HOD model of Clustering

HOD = Halo Occupation Distribution

Galaxies $m > m_{\text{min}}$ in a halo $M$: conditional probability $P(N|M)$

Correlation Function

$$\xi(r) = 1 + \xi_{1h}(r) + \xi_{2h}(r)$$

$$1 + \xi_{1h}(r) = \frac{1}{2\pi r^2 n_g} \int_0^\infty \frac{1}{2R(M)} \frac{dn}{dM} dM \left[ \frac{1}{2} \left\langle N(N-1) \right\rangle_M f \left( \frac{r}{2R(M)} \right) \right]$$

$\#$ Poisson pairs $n(M)$ av. $\#$ pairs in halo $\#$ pairs $(r)$ from universal $\rho(r)$

$$\xi_{2h}(r) = \left\langle n(M) \left\langle N \right\rangle_M \xi_{\text{halos } M}(r) \right\rangle$$
Dark-Matter Halo Occupation Distribution

$M \sim M_\star(t) \rightarrow \text{group}$ at $z=0 \sim 10^{13}M_\odot$ at $z=1 \sim 10^{12}M_\odot$

$M \ll M_\star(t) \rightarrow \text{early formation, satellites decay by dynamical friction}$

$$\frac{m_{\text{sat}}}{M_{\text{halo}}} < (0.01 - 0.1) \left( \frac{M_{\text{halo}}}{M_\star(0)} \right)^{0.3}$$

$\text{Kravtsov et al. 04, N-body simulations}$

$n=5.86\times10^{-2}$ (h$^3$ Mpc$^{-3}$)
HOD from Correlation Function

Zehavi et al. 04   SDSS
Biasing: Luminosity

Zehavi et al. 04, SDSS
Biasing: Luminosity
Biasing: color

Graphs showing the relationship between various parameters, including $w_p$, $M_r$, and $N$ for different color categories (blue, red).