Homework Set 3 – Solutions
DUE: Thursday May 19

1. ΛCDM Fluctuation Power Spectrum

(a) Explain why the ΛCDM power spectrum $P(k)$ is maximum near a specific wave number $k_{\rm max}$ and why it asymptotically approaches $P_k \propto k^{-3}\log^2(k)$ at $k \gg k_{\rm max}$. (A simple derivation of the $\log^2 k$ factor is sketched in my lectures at the 1984 Varenna School* – see especially page 91.) What is the physical origin of the scale corresponding to $k_{\rm max}$?

1). The power spectrum of fluctuations has a turnover/maximum at the wavenumber corresponding to the size of the horizon at the time of matter-radiation equality, i.e. the size of the last fluctuation to come inside the horizon before the transition to matter-dominated growth; because it was the last to come into the horizon, this fluctuation was suppressed the least. The spectrum tends towards $P \propto k^{-3}$ at high k because at higher wavenumbers the fluctuations enter the horizon earlier and earlier in the radiation-dominated era; the primordial k^1 spectrum is correspondingly suppressed by an additional factor of k^{-4}. The $\log^2 k$ factor derives from the contribution of the decaying mode during the radiation dominated era, which acts more strongly on smaller fluctuations that spend more time within the horizon during this era.

(b) Carry out the calculation outlined in slide 5 of the Week 6 lecture slides** and derive the equation for the linear growth of density fluctuations $\ddot{\delta} + 2H\dot{\delta} = 4\pi G\rho_m\delta$, and verify that the growing mode grows as scale factor a in the matter dominated era and as a^2 in the radiation dominated era.

Applying the 00 component of Einstein's equations $\ddot{R} = \frac{-4\pi G}{3}(\rho + 3p)R$

to both the fluctuation and background in the figure in my Week 6, page 5, notes

one finds $$\ddot{R}(1+a) + 2\dot{R}\dot{a} + R\ddot{a} = -(4\pi G/3)\rho R(1+a+\delta),$$

or $$\ddot{\delta} + 2(\dot{R}/R)\dot{\delta} = 4\pi G\rho\delta.$$

Substituting $(\dot{R}/R) = \frac{2}{3}t^{-1}$, valid for a flat ($k = 0$) matter-dominated universe, and trying $\delta = t^\alpha$, one finds $(\alpha+1)(\alpha - \frac{2}{3}) = 0$. The general solution of (2.52) is thus

$$\delta = At^{2/3} + Bt^{-1}. \tag{2.53}$$

Notice that the amplitude of the fluctuation in the growing mode has the same rate of growth as the scale factor R in the matter-dominated universe.

An analogous calculation for a radiation-dominated universe gives

$$\delta = At + Bt^{-1}. \tag{2.54}$$

This time the growing mode for the amplitude grows as the square of the scale factor (i.e., $\delta \propto R^2$) in the radiation-dominated universe. The solution (2.54) is actually relevant only on scales larger than the horizon, since once the fluctuations come within the horizon, the radiation and baryons start to oscillate and the neutrinos freely stream away. (This is from my Varenna Lectures.)

Top-Hat Model for Galaxy formation

Assume that a proto-galaxy is a sphere of uniform density $\rho_p(t)$, whose time evolution can be described by a bound-closed Friedmann model (i.e. a "mini-universe" with $k = 1$ and $\Lambda = 0$). Assume that this sphere is embedded in a background universe which is Einstein-deSitter (i.e. $k = 0$, $\Lambda = 0$) of mean density $\rho(t)$. (At early times, the E-dS model is always a good approximation, since the dark energy contribution only became important at fairly low redshift.) We wish to determine the way the density contrast ρ_p/ρ evolves in time. Following is a guide, step by step.

2. From a small density perturbation until maximum expansion

(a) The Friedmann equation of an Einstein-deSitter model in the matter era is

$$\dot{a}^2 = \frac{2a^*}{a} - k, \quad a^* \equiv \frac{4\pi}{3} G\rho_0 a_0^3 \,,$$

where ρ_0 and a_0 are the values of the universal density and expansion factor today. Write the implicit solution for the universal expansion factor $a(t)$ in terms of the mass constant a^* and the conformal time η [defined by $d\eta \equiv dt/a(t)$], namely write the expressions for $a(\eta)$ and $t(\eta)$. Do the same for the perturbation, where you denote the corresponding quantities as a_p, a_p^*, η_p, etc.

(b) Relate the solutions inside the perturbation and in the background by demanding that the physical time t is the same in both. Use this to relate η to η_p, and then to express a in terms of η_p (rather than η). Recall that we defined $a^* \propto \rho_0 a_0^3 = \rho a^3$ (and a_p^* in analogy), and show that

$$\frac{\rho_p}{\rho} = \frac{9(\eta_p - \sin\eta_p)^2}{2(1 - \cos\eta_p)^3}.$$

(a) To find the expansion factor in terms of conformal time:

$$\begin{aligned}
\frac{da}{dt} &= \sqrt{\frac{2a^*}{a} - k} \\
d\eta &= \frac{dt}{a} \\
\eta &= \int \frac{dt}{da}\frac{da}{a} \\
&= \int \frac{1}{a}\frac{da}{\sqrt{\frac{2a^*}{a} - k}}
\end{aligned}$$

For the external flat universe ($k = 0$), the equation for conformal time is simple:

$$\begin{aligned}
\eta &= \int \frac{1}{a}\frac{da}{\sqrt{\frac{2a^*}{a}}} \\
&= \int \frac{da}{\sqrt{2a^* a}} \\
&= \sqrt{\frac{2a}{a^*}} \\
a &= \frac{a^*\eta}{2}
\end{aligned}$$

For the perturbation ($k = 1$) the situation is more complicated:

$$\begin{aligned}
\eta_p &= \int \frac{1}{a_p}\frac{da_p}{\sqrt{\frac{2a_p^*}{a_p} - 1}} \\
&= -\arctan\left(\frac{\sqrt{\frac{2a_p^*}{a_p} - 1}(a_p - a_p^*)}{a_p - 2a_p^*}\right) + C
\end{aligned}$$

The boundary conditions imply that $C = \pi/2$, so the full equation for a_p is:

$$\begin{aligned}
-\tan\eta_p - \pi/2 = \cot\eta_p &= \frac{\sqrt{\frac{2a_p^*}{a_p} - 1}(a_p - a_p^*)}{a_p - 2a_p^*} \\
\cot^2\eta_p &= \left(\frac{2a_p^*}{a_p} - 1\right)\left(\frac{a_p - a_p^*}{a_p - 2a_p^*}\right)^2 \\
a_p\cot^2\eta_p &= -\frac{(a_p - a_p^*)^2}{a_p - 2a_p^*} \\
a_p^2 - 2a_p^* a_p + \frac{a_p^{*2}}{\cot^2\eta_p + 1} &= 0 \\
a_p &= a_p^*(1 - \cos\eta_p)
\end{aligned}$$

– Elizabeth Lovegrove

(b) Time can be found from conformal time and a:

$$t = \int a\, d\eta$$
$$t = \frac{a^*}{6}\eta^3$$
$$t_p = a_p^*(\eta_p - \sin\eta_p)$$

Since physical time must be the same in both universes:

$$\frac{a^*}{6}\eta^3 = a_p^*(\eta_p - \sin\eta_p)$$
$$\eta = \left(\frac{6a_p^*}{a^*}(\eta_p - \sin\eta_p)\right)^{1/3}$$
$$\frac{\rho_p}{\rho} = \frac{a_p^*}{a^*}\frac{a^3}{a_p^3}$$
$$= \frac{a_p^*}{a^*}\left(\frac{a^*}{2}\left(\frac{6a_p^*}{a^*}(\eta_p - \sin\eta_p)\right)^{2/3}\right)^3 \frac{1}{(a_p^*(1-\cos\eta_p)^3}$$
$$= \frac{9}{2}\frac{(\eta_p - \sin\eta_p)^2}{(1-\cos\eta_p)^3}$$

– Elizabeth Lovegrove

(c) Define the density perturbation by

$$\frac{\delta\rho}{\rho} \equiv \frac{\rho_p - \rho}{\rho},$$

and use Taylor expansions to show that in the linear regime, when the perturbation is small, $\delta\rho/\rho \ll 1$, namely at early times, $\eta_p \ll 1$, the perturbation growth rate is

$$\frac{\delta\rho}{\rho} \propto t^{2/3}.$$

Compare to what we obtained using linear perturbation analysis.

(c) Taylor-expanding this equation produces the limit (dropping all constant factors):

$$\frac{\delta\rho_p}{\rho} = \frac{(\eta_p - \eta_p + \eta_p^3 - \eta_p^5)^2}{(1 - 1 + \eta_p^2 - \eta_p^4)^3} - 1$$
$$= \frac{\eta_p^6(1-\eta_p^2)^2}{\eta_p^6(1-\eta^2)^3} - 1$$
$$= \frac{1}{1-\eta_p^2} - 1$$
$$\approx \eta_p^2 + 1 - 1$$
$$t \propto \eta^3 \Rightarrow \frac{\delta\rho_p}{\rho} \propto t^{2/3}$$

– Elizabeth Lovegrove

(d) Show that at maximum expansion, when the perturbation turns around, the density contrast is

$$\frac{\rho_p}{\rho} = \frac{9\pi^2}{16} \simeq 5.5$$

Note that this is true for any spherical perturbation, no matter when it reaches its maximum expansion.

(d) The perturbation scale factor a_p reaches its maximum when $\eta_p = \pi$, so the density at maximum expansion is:

$$\frac{\rho_p}{\rho} = \frac{9}{2}\frac{(\pi - \sin\pi)^2}{(1-\cos\pi)^3}$$
$$= \frac{9}{2}\frac{\pi^2}{8} = \frac{9\pi^2}{16}$$

3. Dark-matter collapse

(a) Let the mass inside the perturbation be M, and its radius at maximum expansion be R_{max}. Assume that the kinetic energy at maximum expansion is zero (namely no non-radial motions). Assume that the collapse ends in virial equilibrium, where the kinetic energy equals - half the potential energy:

$$V^2 = \frac{GM}{R_{vir}}.$$

Use energy conversation during the collapse of dark matter (as was done in class) to show that

$$R_{vir} = \frac{1}{2}R_{max}.$$

What is the corresponding growth of density inside the halo between maximum expansion and virialization?

(b) What is the density contrast in the virialized halo relative to the background cosmological density at the time of virialization? In addition to the two factors already computed above, we have to include the decrease of the cosmological density between the time of maximum expansion (t_{max}) and the time of virialization (t_{vir}). Take this time to be roughly the time of collapse of the closed "mini-universe", namely when $\eta_p = 2\pi$. Show that the density contrast at virialization is

$$\frac{\rho_p}{\rho} \simeq 176.$$

3). At maximum expansion the kinetic energy is zero, so total energy $E = W_m$, where W_m is the potential energy at maximum. At virialization:

$$
\begin{aligned}
E &= \frac{1}{2}W = W_m \\
\frac{GM^2}{2R_{vir}} &= \frac{GM^2}{R_m} \\
R_{vir} &= \frac{1}{2}R_m
\end{aligned}
$$

The density of the perturbation grows as:

$$
\begin{aligned}
\rho'_p &= \frac{3M}{4\pi R_{vir}^3} \\
&= \frac{3M}{4\pi(1/2R_m)^3} \\
&= 8\rho_p
\end{aligned}
$$

In a flat FRW universe, density goes as $\rho = 1/(6\pi G t^2)$. At maximum extension $t = \pi a_p^*$, and at virialization $t = 2\pi a_p^*$, so $\rho' = \rho/4$.

$$
\begin{aligned}
\frac{\rho'_p}{\rho} &= \frac{8\rho_p}{\rho/4} = \frac{32\rho_p}{\rho} \\
&\approx 177
\end{aligned}
$$

– Elizabeth Lovegrove

4. The epoch of galaxy formation

(a) Let the observed mean density in a galactic halo be ρ_{vir}, when the cosmological density today is ρ_0. Based on the above computation, what is the epoch of formation (namely virialization) of this halo? Express it in terms of redshift z_{vir} (recall $1 + z = a_0/a$), and alternatively in terms of time t_{vir}/t_0.

(b) Express ρ_0 in terms of Ω_m and the Hubble constant h (where $H_0 \equiv 100h\,\mathrm{km\,s^{-1}Mpc^{-1}}$). Show that

$$
(1+z)_{vir} \simeq 6 \left(\frac{\rho_{vir}}{10^{-24}\, g\, cm^{-3}} \right)^{1/3} (\Omega_m h^2)^{-1/3}.
$$

(c) A halo is observed to have a flat rotation curve with velocity $V = 220\,\mathrm{km\,s^{-1}}$ and a virial radius of $R = 100h^{-1}$ kpc. What can we say about its formation epoch?

(d) The gas loses energy by radiation and by dissipation during the collapse. By observing the density of the gas (and stellar) component today, what can we say about the epoch of galaxy formation?

(a) For virialized halos, $\rho_P/\rho = (9/2)(2\pi)^2 = 178$. After virialization ρ_P is (approximately) constant while $\rho = \rho_0 a^{-3}$. So to determine a_{vir} set $\rho_P/\rho = (\rho_P/\rho_0)a_{vir}^{-3}$, $a_{vir} = (178\rho_0/\rho_{vir})^{1/3} = (1+z_{vir})^{-1}$. For simplicity assuming an Einstein-de Sitter universe, $t_{vir} = a_{vir}^{3/2} t_0$.

(b) For E-dS, $\rho_0 = \rho_c = 3H^2/(8\pi G) = 1.36 \times 10^{11} h_{70}^2 M_\odot$ Mpc^{-3}. More generally, $\rho_0 = \Omega_m \rho_c$. Using the latter,

$$(1+z_{vir}) = (\rho_{vir})^{1/3}(178\Omega_m\rho_c)^{-1/3} = 6.7(\rho_{vir}/10^{-24}\text{g cm}^{-3})^{1/3}(\Omega_m h^2)^{-1/3} \quad .$$

(c) The circular velocity V satisfies $V^2 = GM/R$, so plugging in the values $V = 220$ km/s and $R = 100h^{-1}$ kpc, $\rho_{vir} = (RV^2/G)\ 3/(4\pi R^3) = 1.8 \times 10^{-26} h^2$ g cm^{-3}. Then using the result from (b), $(1+z_{vir}) = 6(0.018)^{1/3}\Omega_m^{-1/3} = 1.37\Omega_m^{1/3}$. For $\Omega_m = 0.27$, this is $(1+z_{vir}) = 2.43$.

(d) Because of dissipative energy loss, the gas and stars will have much higher central density than the dark matter. If one were to use that density in the above calculation, one would get a higher formation redshift than that of the halo, which would be misleading. On the other hand, the redshift calculated in part (c) is that of the entire halo. But dark matter halos don't have constant density; instead, the central region has much higher density – which indicates that it formed at higher redshift.