Homework Set 2 - Solutions

DUE: Tuesday May 6

1. **Power Spectrum of Perturbations.** Suppose that the power spectrum of density perturbations $\delta \equiv (\delta\rho/\rho)$, with index n, is $P_k \equiv |\delta_k|^2 \propto k^n$.

a) Derive this expression for the rms density fluctuations on a comoving scale λ:

$$\left(\frac{\delta\rho}{\rho}\right)_\lambda \sim k^{3/2}|\delta_k| \propto \lambda^p$$

and find the index p in terms of n.

b) Find the index m for the rms mass fluctuations on a comoving scale λ:

$$(\delta M/M)_\lambda \propto M^m.$$

What is the criterion on n such that structures grow in a top-down manner (large masses collapse first)? In a bottom-up manner? If the primordial spectrum has $n = 1$ and the CDM transfer function goes as $T(k) \sim k^{-2}$ due to the horizon entry effect, what m does this correspond to and what is the physical interpretation of the mass fluctuation spectrum?

c) The dynamics of the perturbations are driven by the fluctuations in the gravitational potential ϕ. Find the index f for the rms potential fluctuations

$$\delta\phi_\lambda \propto \lambda^f.$$

What is the physical significance of the scale invariant power spectrum index $n = 1$? Also explain the difference between a process being scale free and being scale invariant. Which, if either, is Newtonian gravity?

1. Power Spectrum of Perturbations - Solution

a) The root mean square fluctuation amplitude $\delta\rho/\rho$ is the Fourier transform of the power spectrum, so $(\delta\rho/\rho)^2_\lambda \sim \int d^3k\, P_k \sim \int k^2 dk\, P_k \sim k^3|\delta_k|^2$. In a logarithmic interval dk/k, a good broadband measure of the fluctuations on a scale $\lambda \sim k^{-1}$, one therefore has $(\delta\rho/\rho)_\lambda \sim k^{3/2}|\delta_k|$. If $P_k = |\delta_k|^2 \sim k^n$, then $(\delta\rho/\rho)_\lambda \sim k^{(3+n)/2} \sim \lambda^{-(3+n)/2}$, so $p = -(3+n)/2$.

b) Since $M \sim \rho\lambda^3$, for comoving scales $(\delta\rho/\rho) = (\delta M/M)$. Thus we need only convert λ to M: $\lambda \sim M^{1/3}$. Therefore $(\delta M/M) \sim M^{-(3+n)/6} \sim M^m$, or $m = -(3+n)/6$. If $n > -3$ then small mass scales have larger fluctuations and so collapse first, leading to the **bottom-up** picture of structure formation. If $n < -3$ one has the **top-down** picture. Note that at short wavelengths, the transfer function for CDM would change a primordial index $n = 1$ to $n = -3$ since $P_k \sim T_k^2$. This gives **mass invariant** fluctuations $m = 0$ – all scales collapse together. (Actually, the $\log k$ in T_k for CDM means that smaller scales (larger k) collapse a little earlier on average.)

c) From Poisson's equation $k^2(\delta\phi)_\lambda \sim (\delta\rho/\rho)_\lambda$ so $(\delta\phi)_\lambda \sim \lambda^2\lambda^{-(3+n)/2} \sim \lambda^{(1-n)/2}$. Thus $f = (1-n)/2$. The $n = 1$ power spectrum is called scale invariant because the gravitational potential fluctuations are independent of scale. Note that any power law is scale free (no unique scale) but only $n = 1$ gives scale *invariant* behavior, i.e. the same amplitude as each scale enters the horizon. Newtonian gravity is scale free but the gravitational force does depend on distance, so it is not invariant (also, Newtonian perturbation growth has a characteristic length scale $c/\sqrt{G\rho}$ since the gravitational growth time scale is $1/\sqrt{G\rho}$).

2. **Growth of Density Perturbations.** For matter density perturbations with wavelengths much greater than the Jeans length, the time evolution is given by

$$\ddot{\delta} + 2H\dot{\delta} - (3/2)\Omega_m(t)\,H^2\,\delta = 0.$$

a) Rewrite this equation with the dependent variable being the scale factor a. Write any derivatives of a or H in terms of H and the deceleration parameter q. **b)** Consider the case of a flat matter universe where, on the scales considered, only a constant fraction Ω_{cl} clumps to form structure. (Possible realizations are a cold + hot dark matter universe or a dark + baryonic matter universe.) In this case the Ω_m in the source term of the evolution equation is replaced by Ω_{cl}. Solve the equation to find the behavior of the growing mode: $\delta \propto a^m$. Interpret. Check the limits $\Omega_{cl} = 0$ and 1. **c)** Consider the case of an open universe at a time dominated by the curvature. Write the evolution equation, substituting in for q and $\Omega_m(t)$, keeping only the leading order for the coefficient of each term as a gets large. What happens to the source term for the growth as the universe expands? Try a solution $\delta \propto a^m$. Find the dominant mode and give the physical interpretation (include explanation of the roles of both the drag and source terms).

2. Growth of Density Perturbations - Solution

a) Note that

$$\frac{d}{dt} = aH\frac{d}{da}$$
$$\frac{d^2}{dt^2} = a^2H^2\frac{d^2}{da^2} + \frac{d(aH)}{dt}\frac{d}{da} = a^2H^2\frac{d^2}{da^2} + \ddot{a}\frac{d}{da} = a^2H^2\frac{d^2}{da^2} - aH^2q\frac{d}{da} \quad .$$

With a prime denoting d/da and dividing by a^2H^2,

$$\delta'' + (2-q)a^{-1}\delta' - (3/2)\Omega_m(a)a^{-2}\delta = 0,$$

where the deceleration parameter $q = -\ddot{a}a/\dot{a}^2$.

b) Since the universe is flat and matter dominated then $q = 1/2$. Taking $\delta \sim a^m$ gives a characteristic equation: $m(m-1) + (3/2)m - (3/2)\Omega_{cl} = 0$. The two solutions are

$$m = \frac{1}{4}\left[-1 \pm \sqrt{1 + 24\Omega_{cl}}\right]$$

with the plus sign giving the growing mode. This indicates that the clustered component doesn't grow as fast as the usual solution $\delta \sim a$. When $\Omega_{cl} = 0$ then $m = 0$, i.e. $\delta =$ constant; there is no growth since there is no clumping. When $\Omega_{cl} = 1$ then we return to the $\delta \sim a$ solution as expected.

c) In a curvature dominated universe $q = 0$ so the equation becomes

$$\delta'' + 2a^{-1}\delta' - (3/2)\Omega_m(a)\, a^{-2}\delta = 0.$$

Looking at the source term, we see that

$$\Omega_m(a) = \frac{\Omega_{m0}\, a^{-3}}{\Omega_{m0}\, a^{-3} + (1 - \Omega_{m0})\, a^{-2}} = \left[1 + \frac{1 - \Omega_{m0}}{\Omega_{m0}} a\right]^{-1} \approx \frac{\Omega_{m0}}{1 - \Omega_{m0}} a^{-1}.$$

Therefore the source term dies off as a gets large. Trying $\delta \sim a^m$ gives a characteristic equation: $m(m-1) + 2m = 0$. This has no growing mode; the dominant mode is $\delta =$ constant. Curvature domination shuts off growth because of a combination of a weak source term and a large Hubble drag term (large $2 - q$ from small deceleration q).

3. **Spherical Collapse.** The evolution of a spherically symmetric, overdense perturbation in an $\Omega = 1$ universe can be solved analytically up to the point of singular collapse. As a consequence of Birkhoff's theorem (in a spherically symmetric universe, only the interior mass matters), the perturbation follows the equations of a $k = +1$ Friedmann universe, for which we have a parametric solution.

a) Perturbation overdensity The solutions – unperturbed for the background universe (barred quantities) and perturbed for the overdense region – for the evolution of the size of a sphere and the time are given by

$$\bar{r} = r_0\,(a/a_0) = r_0\,(\eta/\eta_0)^2 \qquad \bar{t} = t_0\,(a/a_0)^{3/2} = t_0\,(\eta/\eta_0)^3$$
$$r = A\,(1 - \cos\theta), \qquad t = B\,(\theta - \sin\theta),$$

where η is the conformal time and θ is the development angle.

At early times the density perturbation must be small ($\rho \to \bar{\rho}$) so the Friedmann equations for the universe and the perturbation region look the same. Enforce this by matching x, $\dot{x}$, $\ddot{x}$ for $x = r, \bar{r}$ with the respective time variables in order to find $r(\theta)$ and $t(\theta)$, i.e. A and B. *Hint:* Remember the definition of the conformal time parameters η and θ.

The age of the universe is unique so t and $\bar{t}$ must be equal. Use this to derive $\eta(\theta)$.

Use mass conservation to express the overdensity $\rho/\bar{\rho}$ first in terms of $r/\bar{r}$ and then as a function of θ.

Verify that turnaround occurs at $\theta = \pi$ and $\rho/\bar{\rho} = 5.55$ and that virialization occurs at $\rho/\bar{\rho} = 178$. Use that the radius is half the turnaround radius (implying $V = -2K$), but use the time corresponding to $\theta = 2\pi$. (Although $V = -2K$ at $\theta = 3\pi/2$, virialization requires $\langle V \rangle = -2\langle K \rangle$, which obtains roughly at $\theta = 2\pi$).

b) Linear regime Show that the density contrast

$$\frac{\delta\rho}{\rho} \equiv \frac{\rho - \bar{\rho}}{\bar{\rho}} \propto a$$

when $\theta \ll 1$. Show that the dimensionless velocity perturbation for $\theta \ll 1$ is

$$\delta_v \equiv \frac{v - Hr}{Hr} = -\frac{1}{3}\left(\frac{\delta\rho}{\rho}\right),$$

where $v = dr/dt$ is the perturbation's expansion velocity and H is the Hubble parameter of the background universe.

c) Astrophysical application Suppose that we observe a galaxy with rotation speed σ at radius R. If we attribute this rotation speed to the mass of a (spherical) dark halo and assume the spherical collapse model in an $\Omega = 1$ universe gives an accurate description of the formation of this halo, what is the expression for the redshift of virialization z_v? What is the value of z_v if $\sigma = 180$ km s^{-1}, $R = 30$ kpc and $H_0 = 60$ km s^{-1} Mpc^{-1}?

3. Spherical Collapse - Solution

For more on Spherical Collapse in GR, see `http://physics.ucsc.edu/~joel/Phys224/SphericalCollapse.pdf`.

a) The definitions of the conformal parameters $d\eta \equiv d\bar{t}/a$ and $d\theta \equiv dt/a_p$ imply $\eta_0 = 3t_0/a_0$ and $B = Aa_0/r_0$. At early times matching $a = a_p$ ($\bar{r} \sim a$, $r \sim a_p$) implies that matching the first derivatives with respect to times $\tau = (t, \bar{t})$, $\dot{x} = dx/d\tau = a^{-1}dx/dc$, matches them with respect to conformal times $c = (\theta, \eta)$ also. I.e. $dr/dt = d\bar{r}/d\bar{t}$ implies $dr/d\theta = d\bar{r}/d\eta$. Similarly $\ddot{x} = a^{-1}(d/dc)(a^{-1}dx/dc) = a^{-2}d^2x/dc^2 - a^{-3}(da/dc)(dx/dc)$ so already matching x, $\dot{x}$ implies $d^2r/d\theta^2 = d^2\bar{r}/d\eta^2$.

Thus the matching conditions are

$$\begin{aligned} \frac{d\bar{r}}{d\eta} &= 2r_0\eta_0^{-2}\eta &= \frac{dr}{d\theta} &= A\sin\theta \to A\theta \\ \frac{d^2\bar{r}}{d\eta^2} &= 2r_0\eta_0^{-2} &= \frac{d^2r}{d\theta^2} &= A\cos\theta \to A \end{aligned}$$

with solution $A = 2r_0\eta_0^{-2}$, $B = 2a_0\eta_0^{-2}$, and $\theta \to \eta$ at early times.

Setting $t = \bar{t}$, valid at all times, gives

$$\eta^3 = 6(\theta - \sin\theta).$$

Mass conservation says $\rho\, r^3 = \bar{\rho}\,\bar{r}^3$ so

$$\begin{aligned} \frac{\rho}{\bar{\rho}} &= \left(\frac{\bar{r}}{r}\right)^3 \\ &= \frac{1}{8}\eta^6(1-\cos\theta)^{-3} \\ &= \frac{9}{2}(\theta - \sin\theta)^2(1-\cos\theta)^{-3}. \end{aligned} \tag{1}$$

At turnaround, r is maximum hence $\theta = \pi$. Plugging this in gives an overdensity $\rho/\bar{\rho} = 9\pi^2/16 = 5.55$.

Taking virialization to be at a time given by $\theta = 2\pi$ means substituting that into $(\theta - \sin\theta)^2$, which comes from η^6, but using the half radius criterion means $1 - \cos\theta = (1 - \cos\pi)/2 = 1$. Alternately one could work out $\rho/\bar{\rho}$ at virialization relative to at turnaround: the density ρ has increased by 8 since r has decreased by 2, while $\bar{\rho} \sim a^{-3} \sim \bar{t}^{-2}$ has decreased by 4 since $\bar{t} \sim \eta^3$ has doubled ($\theta = \pi$ became 2π). Either way gives

$$\left(\frac{\rho}{\bar{\rho}}\right)_v = 18\pi^2 = 178.$$

b) Expanding equation (1) for the overdensity to $\mathrm{O}(\theta^5)$ gives

$$\frac{\delta\rho}{\rho} \equiv \frac{\rho - \bar{\rho}}{\bar{\rho}} \to \frac{3}{20}\theta^2 \to \frac{3}{20}\eta^2.$$

Since $\eta^2 \sim a$ then $\delta\rho/\rho \propto a$.

The velocity is

$$\begin{aligned} v \equiv \frac{dr}{dt} &= \frac{dr}{d\theta}\bigg/\frac{dt}{d\theta} \\ &= \frac{r_0}{a_0}\frac{\sin\theta}{1-\cos\theta}. \end{aligned}$$

The Hubble parameter is

$$H = \frac{2}{3\bar{t}} = \frac{2}{3t_0}\left(\frac{\eta}{\eta_0}\right)^{-3}.$$

One can then calculate δ_v by expanding to $\mathrm{O}(\theta^5)$. A faster way is to calculate v/r to $\mathrm{O}(\theta^{-3})$ since the next higher order term is missing. Either way, the answer is

$$\delta_v \equiv \frac{v - Hr}{Hr} = -\frac{1}{20}\theta^2 = -\frac{1}{3}\frac{\delta\rho}{\rho},$$

which is not surprising since in the linear regime ρr^3 =const implies $\delta\rho/\rho = -3(\delta r/r) = -3(v - Hr)/(Hr)$.

c) Virialization occurs at a fixed overdensity, $\rho/\bar{\rho} = 178$, as we derived in b). The background density scales as $(1+z)^3$ so the physical density of the galaxy today (constant since virialization) is

$$\rho = 178\bar{\rho}(z_v) = 178\,(1+z_v)^3 \cdot \frac{3H_0^2}{8\pi G}.$$

The density is related to the mass M by $\rho = M/(4\pi R^3/3)$ and the mass is related to the circular velocity by

$$\sigma^2 = \frac{GM}{R}.$$

Putting all this together one finds

$$(1+z_v)^3 = \frac{2}{178}\left(\frac{\sigma}{H_0 R}\right)^2.$$

Substituting the given values yields $z_v = 3.8$.