Physics 5I LECTURE 6 November 4, 2011

- More on Special Relativity

  Resolution of the Twin Paradox

  Rotations and Lorentz transformations

  Successive Lorentz transformations

  The invariant interval $c^2t^2 - x^2$ and the Light Cone

  Resolution of the Pole and Barn paradox

  Special Relativity with 4-vectors

- Next Friday Nov 11 - Veterans Day holiday

  Friday Nov 18 - Midterm Exam* (page of notes ok)

  Friday Nov 25 - Thanksgiving Vacation

  Friday Dec 2 - General Relativity and Black Holes

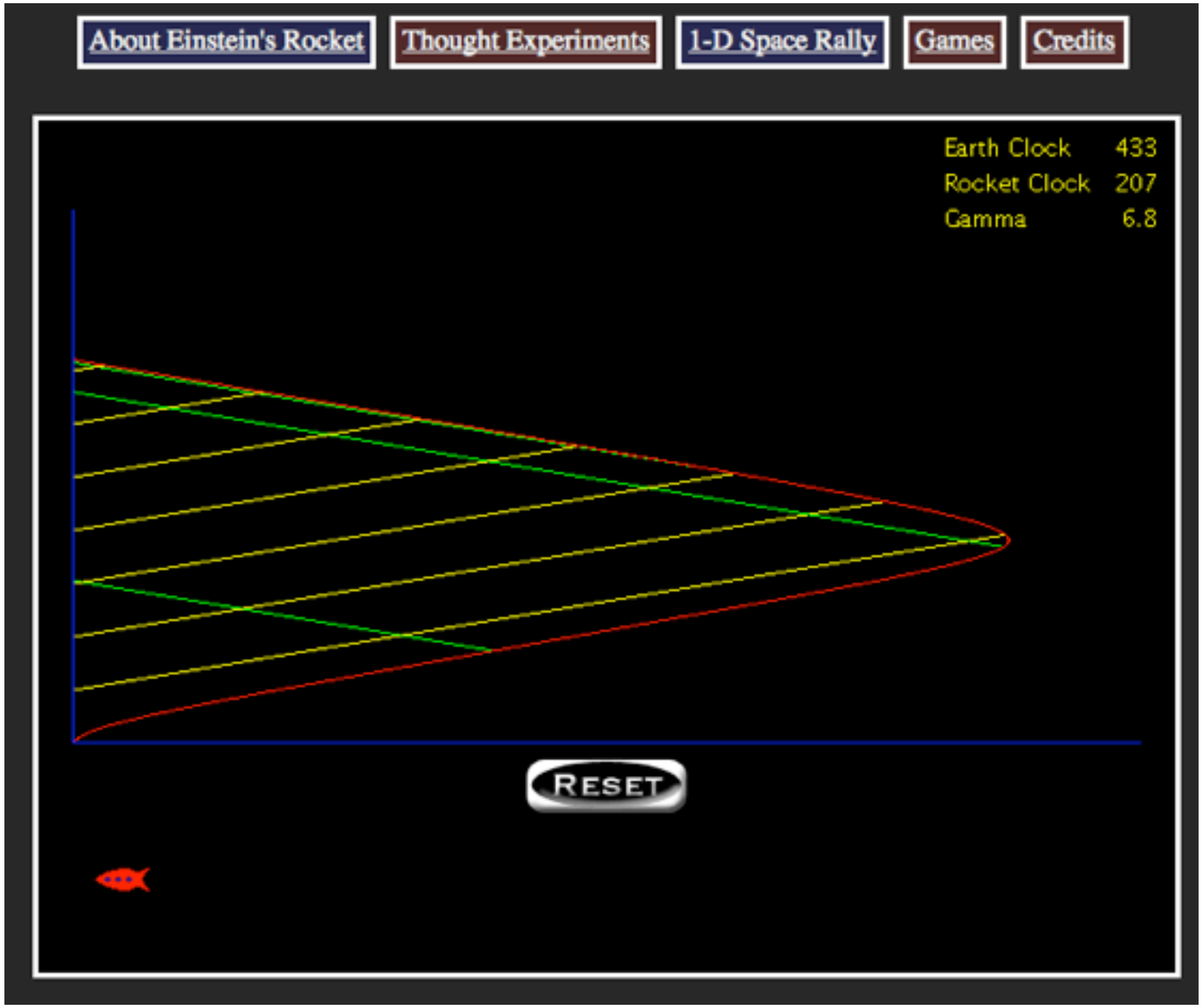
*Note: No Final Exam for Physics 5I

The "twin paradox" of Special Relativity

If Albert stays home and his twin sister Berta travels at high speed to a nearby star and then returns home, Albert will be much older than Berta when he meets her at her return.

But how can this be true? Can't Berta say that, from her point of view, it is Albert who traveled at high speed, so Albert should actually be younger?

To clarify why more time elapses on Albert's clock than on Berta's, we can use the Einstein's Rocket "1-D Space Rally".



http://physics.ucsc.edu/~snof/er.html

Rotations in 2D

Recall that a rotation by angle θ in the x-y plane is given by

$$x = x' c_\theta - y' s_\theta \qquad y = x' s_\theta + y' c_\theta.$$

where $c_\theta = \cos\theta$, $s_\theta = \sin\theta$. The reverse rotation corresponds to $\theta \to -\theta$:

$$x' = x\, c_\theta + y\, s_\theta \qquad y' = -x\, s_\theta + y\, c_\theta.$$

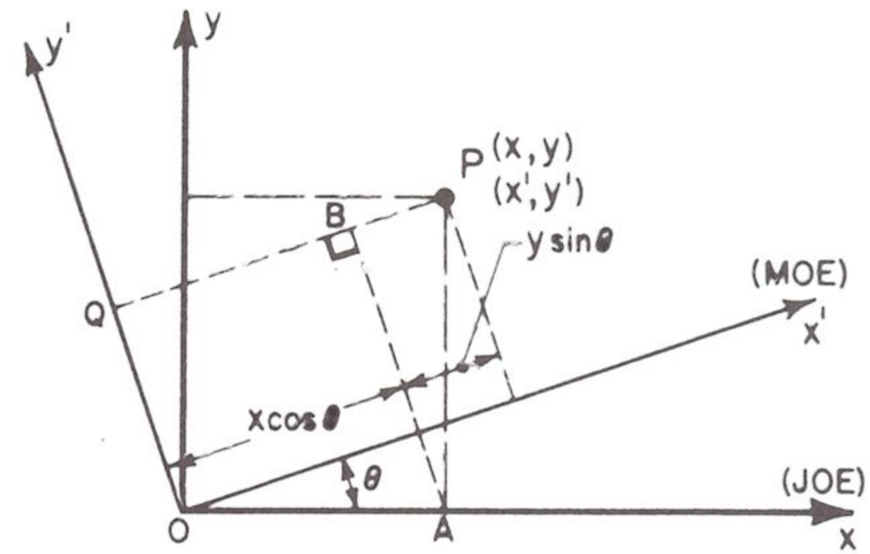


Fig. 11–2. Two coordinate systems having different angular orientations.

From the *Feynman Lectures on Physics*, vol. 1.

These transformations preserve the lengths of the vectors ($x^2 + y^2 = x'^2 + y'^2$) since $\sin^2\theta + \cos^2\theta = 1$.

Successive rotations by angles θ_1 and θ_2 correspond to rotation through angle $\theta_3 = \theta_1 + \theta_2$.

Lorentz transformation

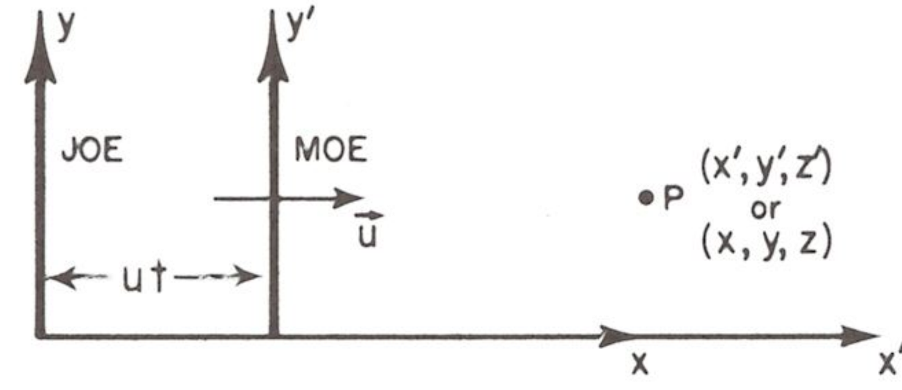


Fig. 15–1. Two coordinate systems in uniform relative motion along their x-axes.

The Lorentz transformation between inertial reference frame (x,t) and the inertial frame (x',t') moving at speed v in the x-direction when the origins of the reference frames coincide at $t = t' = 0$ is

$$x = x' C_\theta + ct' S_\theta \qquad ct = x' S_\theta + ct' C_\theta.$$

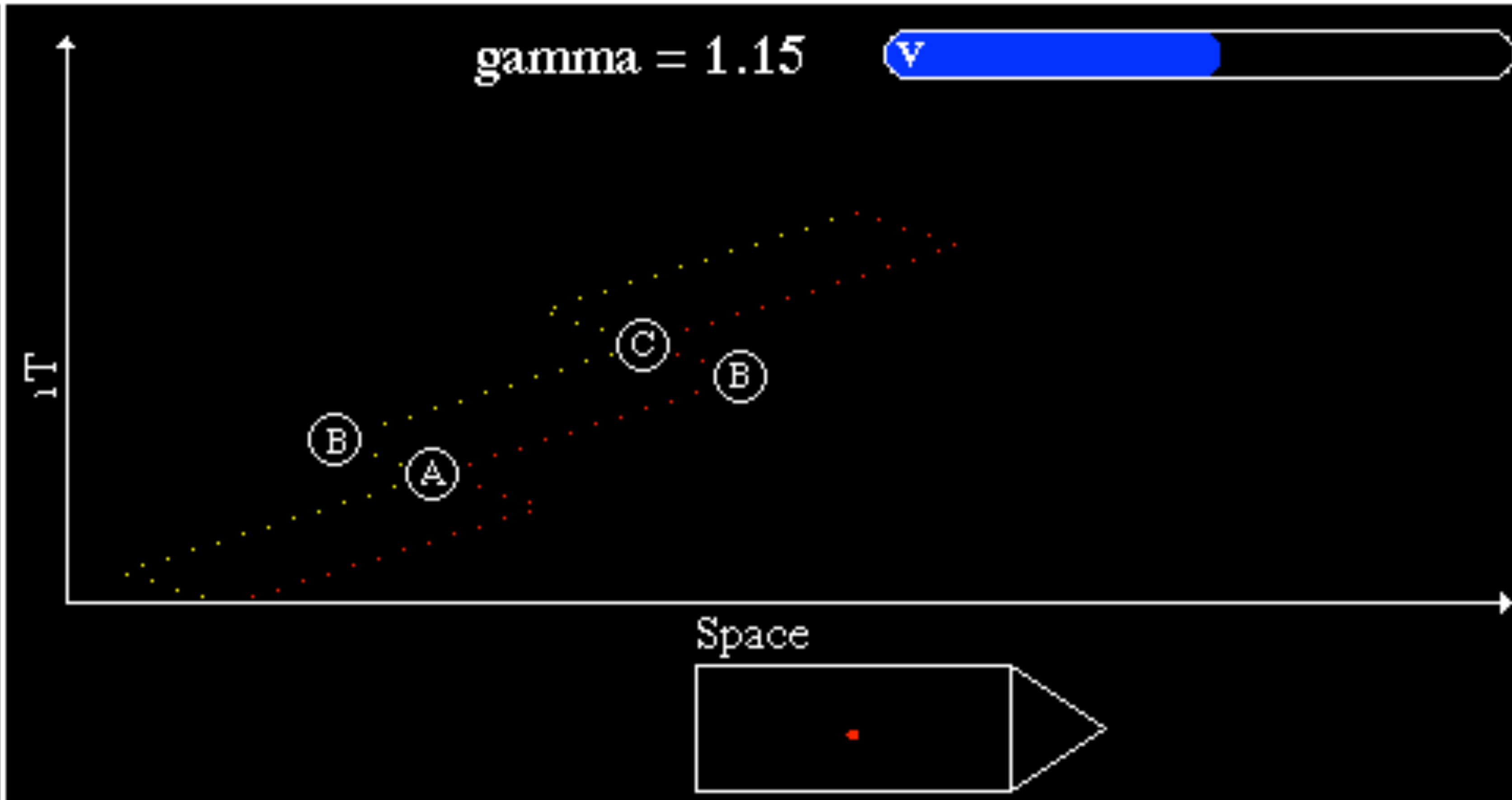
where $\gamma = (1-v^2/c^2)^{-1/2} = \cosh\theta = C_\theta$, $\gamma v/c = \sinh\theta = S_\theta$, and $\theta = \tanh^{-1} v/c$ since $\tanh\theta = \sinh\theta / \cosh\theta = v/c$. The reverse transformation is again $\theta \to -\theta$:

$$x' = x C_\theta - ct S_\theta \qquad ct' = - x S_\theta + ct C_\theta.$$

Here $\cosh\theta = (e^\theta + e^{-\theta})/2$ and $\sinh\theta = (e^\theta - e^{-\theta})/2$ are the hyperbolic functions, and $\tanh\theta = \sinh\theta / \cosh\theta$. With the correspondence above between θ and v/c, it follows that $v/c = \tanh\theta$, so $\theta = \tanh^{-1} v/c$.

Successive Lorentz transformations: The reason the θ approach is useful is that the product of two Lorentz transformations that correspond to θ_1 and θ_2 is $\theta_3 = \theta_1 + \theta_2$, which greatly simplifies things. It turns out that Lorentz transformations form a group that is a generalization of the group of rotations.

Lorentz transformations preserve the space-time interval ($c^2t^2 - x^2 = c^2t'^2 - x'^2$) because $\cosh^2\theta - \sinh^2\theta = 1$.



The light pulses start from and return to the center of the rocket simultaneously (A and C), but now they strike the end mirrors at different times (B).

http://physics.ucsc.edu/~snof/er.html

The invariant interval $c^2t^2 - x^2$ and the Light Cone

Lorentz transformations preserve the space-time interval ($c^2t^2 - x^2 = c^2t'^2 - x'^2$), as they must since that guarantees that the speed of light is c in both frames.

This means that relatively moving observers will agree as to whether the space-time interval between two events is negative (space-like ①), positive (time-like ② or ③), or zero (light-like). If two events are time-like or light-like separated, the earlier one can affect the latter one, but if they space-like separated then they can have no effect on each other. The Light Cone separates events into these three different classes.

Before relativity, people thought "Now" had an invariant meaning. But we have already seen (in the Einstein's Rocket thought-experiment concerning the flash sent from the center to the ends of the rocket) that events that one observer says are simultaneous will not be so according to an observer in relative motion. That's also the explanation of the pole-and-barn paradox. The pole is entirely inside the barn in its rest frame, but in the rest frame of the pole the barn is much shorter and both doors are simultaneously open.

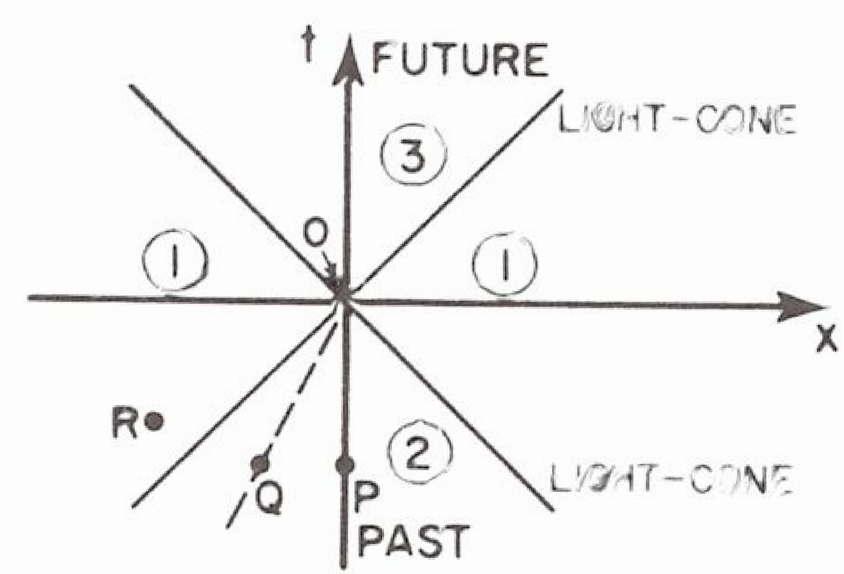


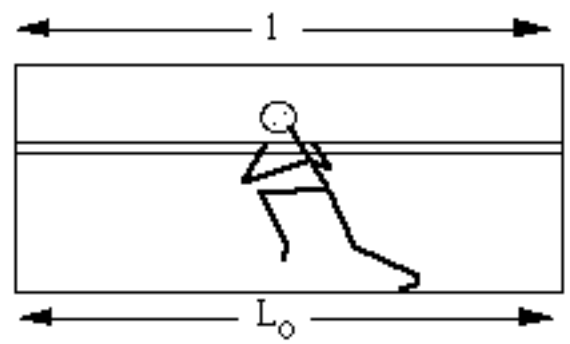
Fig. 17–3. The space-time region surrounding a point at the origin.

Resolution of the pole and barn paradox

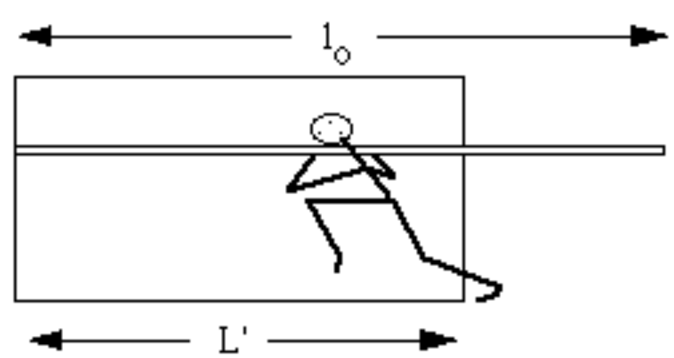
Pole vaulter Emma travels at speed v and holds a pole. If Emma measures the length of the pole, which is stationary in her frame, she measures length L_o, which is called the proper length. ('Proper' here means characteristic of its own frame, not correct or superior.) According to spectator Eric, Emma is moving so fast that he sees her pole to have a relativistic length contraction. He measures its length as $L < L_o$.

Eric organises this experiment: he builds a barn which is just as long as Emma's moving pole, as he measures it. The barn has a door at either end. Emma, with her contracted pole, will run into the barn and he will shut both doors just *when* her pole fits inside. For an instant at least, Emma's pole will be entirely inside the barn and he will have proved that her pole has shrunk.

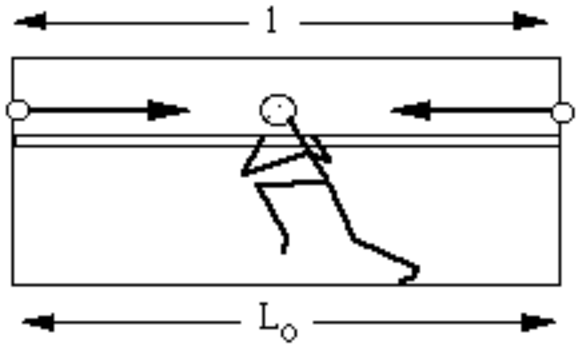
The experiment is run and Eric thinks that it is conclusive (first diagram).



Emma differs, however. Here is how she saw it:



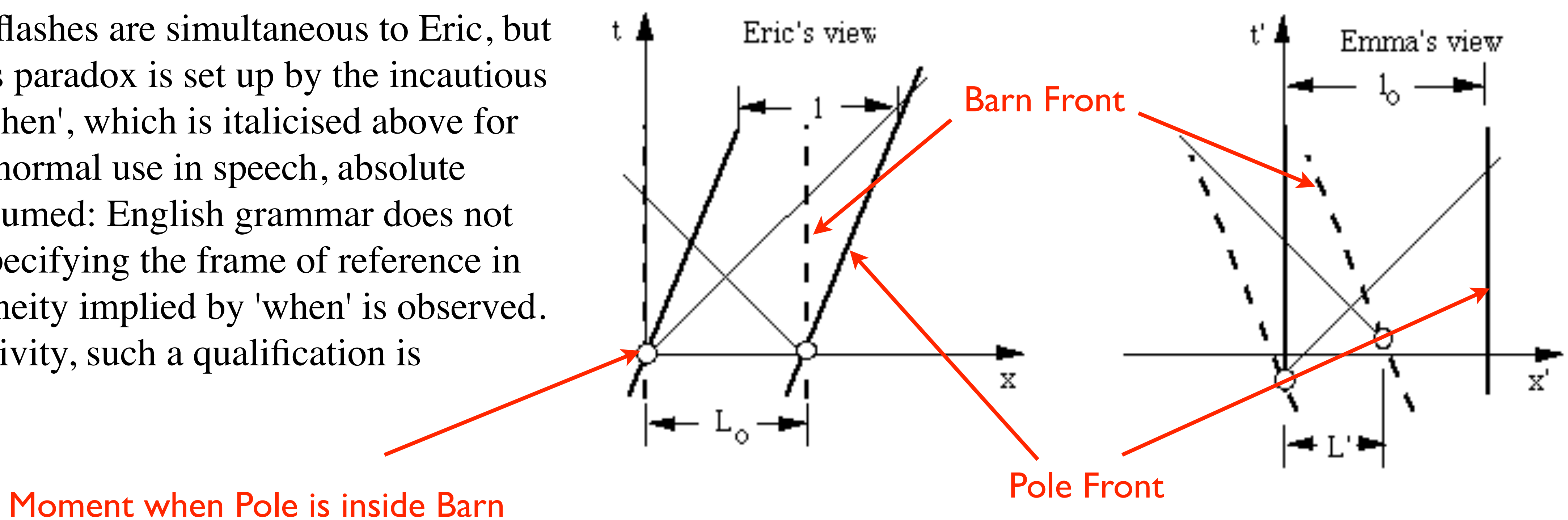
"You cheated," she says "you closed the back door *when* my pole had already poked through the front door! My pole was always longer than your barn." Their disagreement is now about the timing of the closing of the doors. Did two distant events happen simultaneously or not?



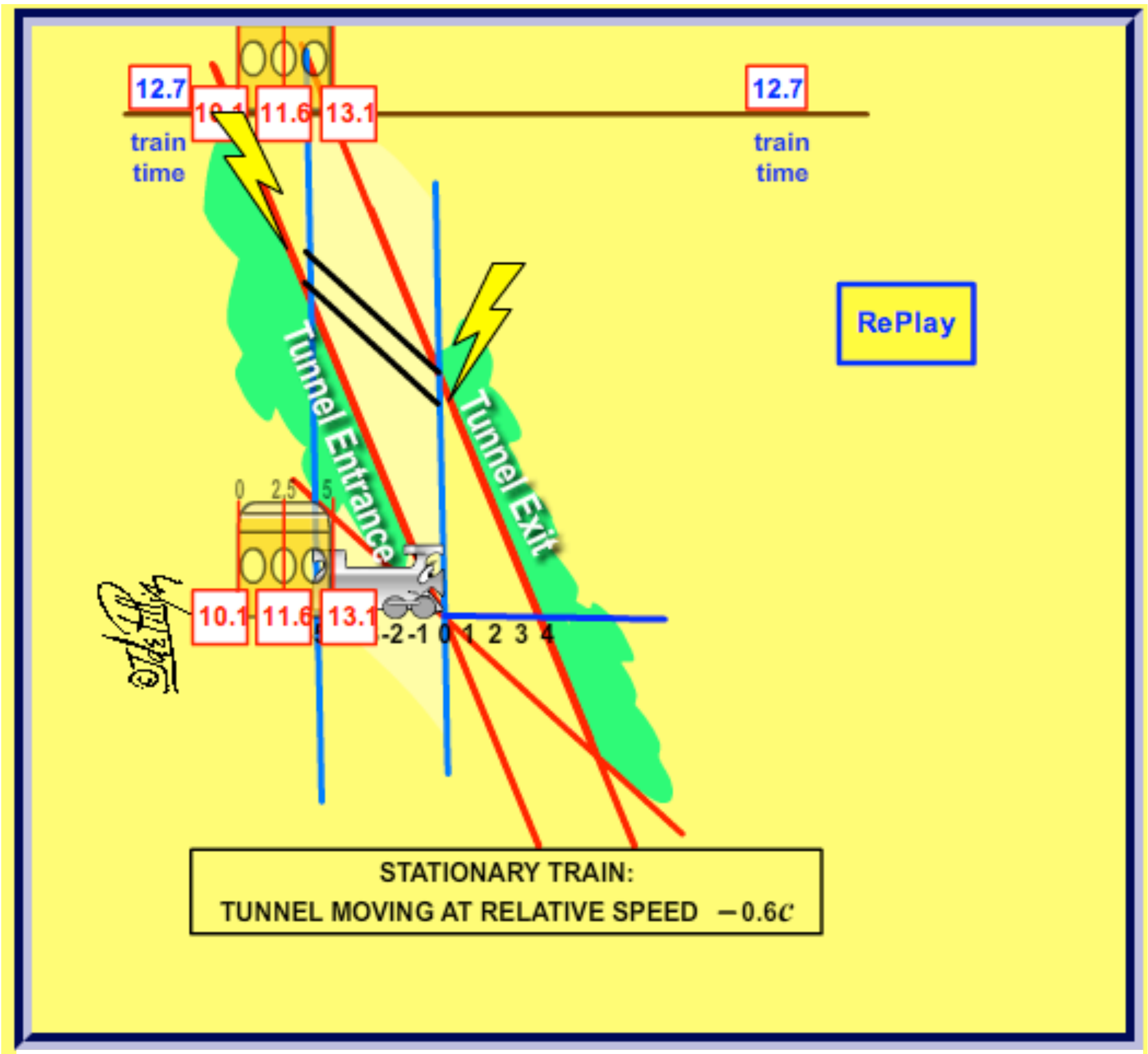
So they try again, and this time Eric sets off a flash bulb simultaneously *when* the doors are closed. Here is his view of events, with the flash bulbs represented by white circles. It just so happens that, according to Eric, Emma was at the midpoint of the shed *when* the flash bulbs went off.

Now Eric expects that Emma will receive the flash from the front before the one from behind, because he sees her moving to the left. No problem for him. Emma also receives the light from the left before she gets the light from the right - and this is the basis for her charge of cheating. For her, light travels at the speed c, she receives the light from the left first. Hence her claim that Eric fired this flash first. Simultaneity is relative: one consequence of the theory of relativity is that two observers may disagree on whether or not two events are simultaneous. (Indeed, if the two events are a long distance D apart, but close together Δt in time (if $D/\Delta t > c$), they can even disagree upon which happened first.)
Their respective points of view are shown below in what are called space-time diagrams. This means that we plot time on the vertical axis as a function of the position of events. Because the two characters disagree over time and length, we must give them two separate diagrams (below) and, to convert between the (x,t) and the (x',t') frames, we need to use the Lorentz transform equations. In this figure, the bold lines are the ends of Emma's pole, the dashed bold lines are the ends of Eric's shed, and the fine lines are the flashes of light.

Note that the two flashes are simultaneous to Eric, but not to Emma. This paradox is set up by the incautious use of the word 'when', which is italicised above for that reason. In its normal use in speech, absolute simultaneity is assumed: English grammar does not require a clause specifying the frame of reference in which the simultaneity implied by 'when' is observed. In discussing relativity, such a qualification is required.



Animation of the train and tunnel paradox



The Train and Tunnel have equal lengths d=5. When the train and the tunnel have constant speed 0.6 c relative to each other. The train's nose (at its x=0, t=0) synchronizes with the left entrance of the tunnel (at its x=0, t=0).

When the Tunnel is Stationary...
From the point of view of the (stationary) tunnel with at-rest length d = 5 , the train's length is observed to be 0.8 d = 4.

The moving train's clocks run at 80% the rate of the tunnel's clocks. (*Check the reading of the moving train's nose clock against the stationary tunnel clocks* .)

When the Train is Stationary...
From the point of view of the (stationary) train with at-rest length d = 5, the tunnel's length is observed to be 0.8 d = 4.

The moving tunnel's clocks run at 80% the rate of the train's clocks. (*Check the reading of the moving tunnel's entrance clock against the stationary train clocks*.)

http://math.ucr.edu/~jdp/Relativity/Main_Train_Tunnel.html

This animation is based on a section of the (*forthcoming*) book, ***SPECIAL RELATIVITY ILLUSTRATED,*** by John de Pillis.

Special Relativity with 4-Vectors

An quantity that transforms the same way as (ct,x) is called a 4-vector. It turns out that the combination (γ, γv/c) = γ(1,v/c) where v is the velocity, is a 4-vector, called the velocity 4-vector. Its invariant length-squared is $\gamma^2 (1-v^2/c^2) = 1$.

Multiply the rest mass of a particle m by its velocity 4-vector and you get its momentum 4-vector:

$$P = (E, pc) = mc^2\,\gamma(1, v/c)$$

Its invariant length-squared is $m^2c^4\,\gamma^2 (1-v^2/c^2) = m^2c^4 = E^2 - p^2c^2$.

For a particle of mass m, this says that $E^2 = p^2c^2 + m^2c^4$.

For the special case of a massless particle like the photon, this says that $E^2 = p^2c^2$ or $E = |p|c$. The momentum carried by a photon of energy E is $p = E$/c.

As students become more familiar with formulas like these, it's convenient to stop writing the speed of light c and just understand that powers of c are included as needed to get the right units. Then the energy-momentum-mass relation becomes $E^2 = p^2 + m^2$. We often measure mass in energy units, for example we say that the mass m_e of the electron is 0.511 MeV, even though what we really mean is that m_ec^2 = 0.511 MeV. And of course we measure distances in time units: light-years.