Homework 2 Solutions

1. (15 points) (a) Find the radius a_0 of the $n = 1$ orbit, and in terms of a_0 find the radii r_n of the orbits for the higher values of n.

Answer Applying Newton's law $F = ma$ sets the Coulomb force equal to the centripetal force:

$$\frac{e^2}{4\pi\epsilon_0 r^2} = m_e \frac{v^2}{r} \ .$$

Bohr's angular momentum quantization assumption is $m_e r v = n\hbar$. Combining these equations,

$$r = \frac{n\hbar}{m_e v} = \frac{e^2}{4\pi\epsilon_0 v^2 m_e} \quad ,$$

which implies that in the n^{th} orbit

$$v_n = \frac{e^2}{4\pi\epsilon_0 n\hbar} = \frac{\alpha c}{n} \ , \quad r_n = n^2 \frac{4\pi\epsilon_0 \hbar^2}{e^2} = n^2 \alpha^{-1} \frac{\hbar}{m_e c} \quad .$$

The Bohr radius $a_0 = r_1 = \alpha^{-1}\hbar/(m_e c) = 0.529 \times 10^{-10}$ m, and the radii of the higher orbits are $r_n = n^2 a_0$.

[Note that we have neglected the motion of the proton since its mass m_p is nearly 2000 times that of the electron m_e. Including the proton motion just replaces m_e by the reduced mass $m_r = m_e/(1 + m_e/m_p)$.]

(b) Show that the ground state (i.e., $n = 1$) energy is $E_1 = \frac{1}{2}\alpha^2 m_e c^2$ and express E_1 in units of electron volts (eV).

Answer The energy is the sum of the kinetic and potential energies:

$$E_1 = \mathrm{KE}_1 + \mathrm{PE}_1 = \frac{1}{2} m_e v_1^2 - \frac{e^2}{4\pi\epsilon_0 r_1} = \frac{1}{2}\alpha^2 m_e c^2 - \alpha^2 m_e c^2 = -\frac{1}{2}\alpha^2 m_e c^2 \ .$$

Numerically, $m_e c^2 = 0.511$ MeV and $E_1 = -13.6$ eV. (That the energy is negative just means that the binding energy of the ground state of the hydrogen atom is 13.6 eV.)

(c) Find an expression for the energy levels E_n of the states of the hydrogen atom labeled by Bohr's quantum number n.

Answer Using r_n and v_n in $E_n = \mathrm{KE_n} + \mathrm{PE_n}$ above gives $E_n = E_1/n^2$.

Note that $\mathrm{KE_n} = -(1/2)\mathrm{PE_n}$. That $< \mathrm{KE} >= -(1/2) < \mathrm{PE} >$ (where $< ... >$ means time average) for an inverse-square-law (i.e., $1/r^2$) force is a consequence of a general theorem of mechanics known as the "Virial theorem". The **Virial theorem** states that $< \mathrm{KE} >= (n/2) < \mathrm{PE} >$ where n is the power of r in the potential: $V(r) = ar^n$. An r^{-2} force corresponds to an r^{-1} potential, so here $n = -1$.

2. (15 points) The light emitted by the hydrogen atoms transition from the $n = 3$ to the $n = 2$ energy level is called Hα.

(a) Find the energy E of the photons of this light in eV.

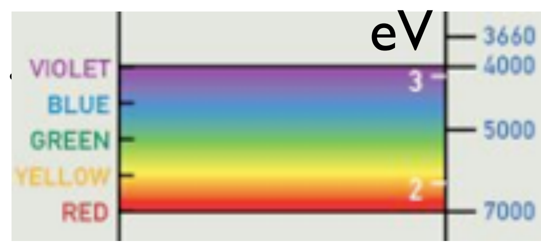
Answer $E_{\mathrm{H}_\alpha} = E_3 - E_2 = (1/9 - 1/4)E_1 = (-5/36)E_1 = 1.89$ eV .

(b) Find the frequency ν of this light by using Planck's formula $E = h\nu$.

Answer $\nu_{\mathrm{H}_\alpha} = E_{\mathrm{H}_\alpha}/h = (1.89\ \mathrm{eV})(1.602 \times 10^{-19}\mathrm{J})/(6.63 \times 10^{-34}\mathrm{J \cdot s}) = 4.56 \times 10^{14}$ Hz.

(c) Find the wavelength λ. What is the color of this light?

Answer $\lambda_{\mathrm{H}_\alpha} = c/\nu_{\mathrm{H}_\alpha} = 6.58 \times 10^{-7}$ m $=$ 658 nm, which is *red* light. Visible light ranges from 400 nm (blue) to 700 nm (red)



3. (10 points) (a) What is the energy required to reach the n = 3 energy level from the ground state (the n = 1 energy level)? What sort of light has this much energy?

Answer $E_3 - E_1 = (1/9 - 1)E_1 = -(8/9)E_1 = 12.1$ eV $= 1.94 \times 10^{-18}$ J. This is ultraviolet light.

(b) What is the minimum energy required to ionize a hydrogen atom [in its ground state] (i.e., free the electron)?

Answer The binding energy in the ground state, 13.6 eV $= 2.18 \times 10^{-18}$ J, is the minimum energy required to free the electron in its ground state. If the electron in the ground state absorbs more energy than 13.6 eV from a photon, it also has kinetic energy.

4. (5 points) Suppose that, on average, a hydrogen atom will exist in the $n = 2$ state for about 10^{-8} second. How many revolutions does the electron make in this time.

Answer The time per revolution in the n$^{\text{th}}$ Bohr energy level is $T_n = 2\pi r_n / v_m$. Recall from Problem 1 above that

$$v_n = \frac{e^2}{4\pi\epsilon_0 n\hbar} = \frac{e^2}{4\pi\epsilon_0 \hbar c}\frac{c}{n} = \frac{\alpha c}{n} \,, \quad r_n = n^2 \alpha^{-1} \frac{h}{m_e c} \,.$$

Thus

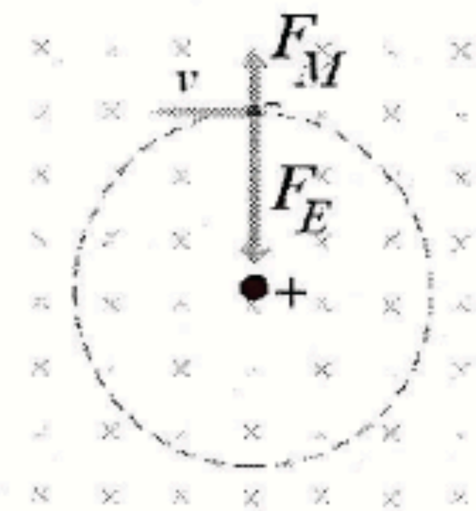
$$T_n = \frac{2\pi r_n}{v_n} = \frac{2\pi n^2 \hbar}{\alpha m_e c}\frac{n}{\alpha c} = n^3 \alpha^{-2} \frac{\hbar}{m_e c^2} = n^3 (4.55 \times 10^{-8}) \text{ s} \,.$$

Then $T_2 = 3.64 \times 10^{-7}$ s, and the number of revolutions of an electron in the $n = 2$ Bohr orbit in 10^{-8} s is $(10^{-8} \text{ s})/T_2 = 0.027$ revolutions.

[Note that the electron can't slow down, since as long as it is in the $n = 2$ state it's velocity is v_2. What actually happens as the electron transitions to the ground state remained a mystery in the Bohr theory. Then quantum mechanics showed that the electron wavefunction becomes a sum of the $n = 1$ and $n = 2$ wavefunctions during the transition; the interference of their respective frequencies is the frequency of the light emitted during the transition.]

5. (10 points) Giancoli, problem 27-72 on the **Zeeman effect**. In the Bohr model of the hydrogen atom, the electron is held in its circular orbit of radius r about its proton nucleus by electrostatic attraction. If the atoms are placed in a weak magnetic field $\mathbf{B}$, the rotation frequency of electrons rotating in a plane perpendicular to $\mathbf{B}$ is changed by an amount $\Delta f = \pm eB/4\pi m$ where e and m are the charge and mass of an electron.

(a) Derive this result, assuming the force due to $\mathbf{B}$ is much less than that due to electrostatic attraction of the nucleus.



As the electron orbits the nucleus in the absence of the magnetic field, its centripetal acceleration is caused solely by the electrical attraction between the electron and the nucleus. Writing the velocity of the electron as the circumference of its orbit times its frequency, enables us to obtain an equation for the frequency of the electron's orbit.

$$\frac{ke^2}{r^2} = m\frac{v^2}{r} = m\frac{(2\pi r f_0)^2}{r} \to f_0^2 = \frac{ke^2}{4\pi^2 m r^3}$$

When the magnetic field is added, the magnetic force adds or subtracts from the centripetal acceleration (depending on the direction of the field) resulting in the change in frequency.

$$\frac{ke^2}{r^2} \pm q(2\pi r f)B = m\frac{(2\pi r f)^2}{r} \to f^2 \mp \frac{qB}{2\pi m} f - f_0^2 = 0$$

We can solve for the frequency shift by setting $f = f_0 + \Delta f$, and only keeping the lowest order terms, since $\Delta f \ll f_0$.

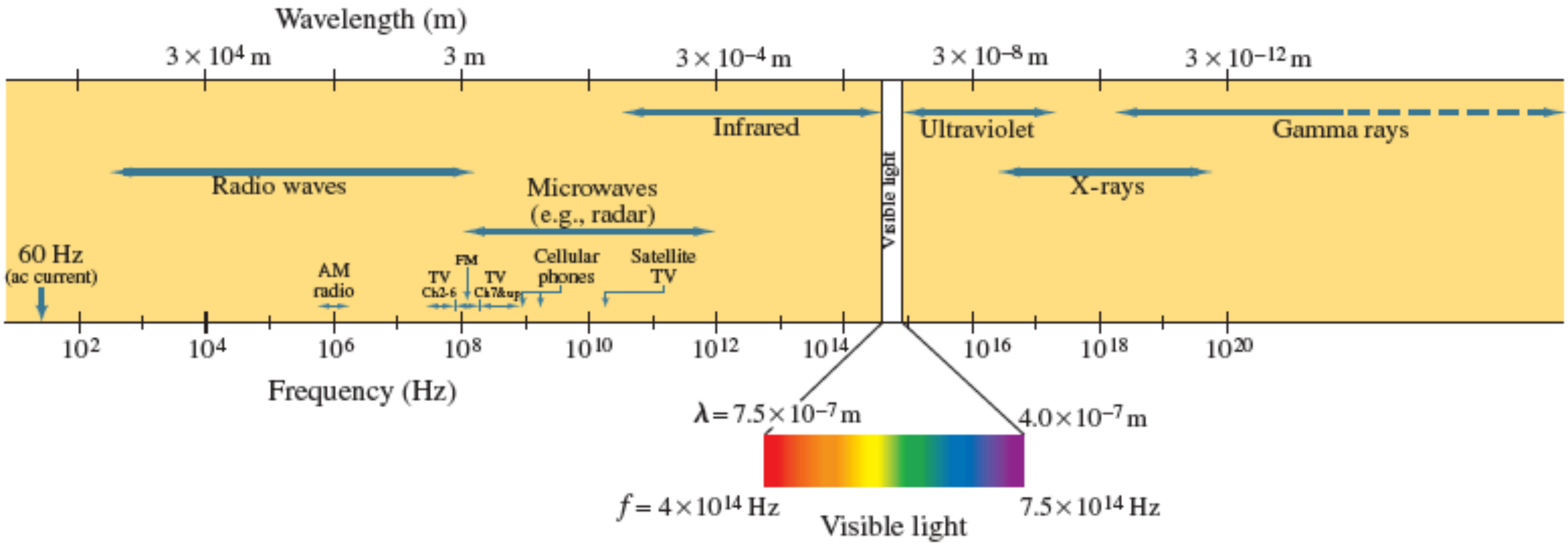
$$(f_0 + \Delta f)^2 \mp \frac{qB}{2\pi m}(f_0 + \Delta f) - f_0^2 = 0$$

$$\cancel{f_0^2} + 2 f_0 \Delta f + \cancel{\Delta f^2} \mp \frac{qB}{2\pi m} f_0 \mp \cancel{\frac{qB}{2\pi m}\Delta f} - \cancel{f_0^2} = 0 \to \boxed{\Delta f = \pm\frac{qB}{4\pi m}}$$

(b) What does the ± sign indicate?

The "±" indicates whether the magnetic force adds to or subtracts from the centripetal acceleration. If the magnetic force adds to the centripetal acceleration, the frequency increases. If the magnetic force is opposite in direction to the acceleration, the frequency decreases.

Giancoli **FIGURE 31–12**
Electromagnetic spectrum.



6. (15 points) It is possible for a muon to be captured by a proton to form a muonic atom. A muon is identical to an electron except for its mass, which is 105.7 MeV/c^2, and the fact that it is unstable, with a lifetime of about 2.2 microseconds.

(a) Calculate the radius of the first Bohr orbit of a muonic atom.

Answer To a first approximation, the radius is $R_1 = \alpha^{-1}\hbar/(m_\mu c) = a_0(m_e/m_\mu) = 0.529 \times 10^{-10}\text{m})(m_e/m_\mu) = 2.55 \times 10^{-13}$ m.

3 points Extra Credit: A more accurate treatment uses the reduced mass of the muon $m_{\mu,r} = m_\mu/(1+m_\mu/m_p) = m_\mu/(1+105.7/938) = 95.0\text{MeV}/c^2$. This is a bigger correction here than using the reduced electron mass was in Problem 1, since the muon mass is so much larger. The first Bohr radius of a muonic atom is thus more accurately 2.85×10^{-3} m

(b) Calculate the magnitude of the lowest energy state in eV.

Answer Replacing m_e by m_μ in the expression for E_1 in Problem 1 gives $E_1 = -(1/2)\alpha^2 m_\mu c^2$ so $E_1 = (m_\mu/m_e)(-13.6\text{eV}) = 2.81$ keV.

3 points Extra Credit: Using instead the reduced muon mass gives the more accurate answer 2.53 keV.

(c) What is the wavelength of the $n = 2$ to $n = 1$ transition in a muonic atom?

Answer The energy of the transition is $(1/4 - 1)E_1 = (3/4)E_1 = 2.11$ keV. The frequency $\nu = E/h$ and the wavelength is $\lambda = c/\nu = hc/E = (1.24 \times 10^{-6}\text{eVm})/E = 5.88 \times 10^{-10}$ m = 0.59 nm. This is an x-ray wavelength. (See Fig. 31-12 on p. 823 of Giancoli for a picture of the entire electromagnetic spectrum.)

3 points Extra Credit: Using instead the reduced muon mass gives the more accurate answer 0.655 nm.