Physics 5K INTRODUCTORY PHYSICS – HONORS SECTION Spring 2012

Homework 4 – Solutions

Due Friday June 1 in Class

The following problems concern relativity and particles moving in magnetic fields. We discussed relativity in class May 25, including using the Einsteins Rocket website. We showed that when an object is moving a velocity v, its length is contracted in the direction of motion by the factor $\gamma = (1 - v^2/c^2)^{-1/2}$ (i.e., the length is divided by γ), its clocks are slowed by the same factor, and its momentum and energy are increased by the same factor. Thus the momentum and energy of a particle of rest mass m are given by $p = mv\gamma$ and $E = mc^2\gamma$.

1. (15 points) An electron is in the ground ($n = 1$) state of Bohr's model of a hydrogen atom.

(a) Find the velocity v of the electron in the $n = 1$ state of the Bohr atom in terms of its mass m_e, the speed of light c, and the fine structure constant $\alpha = e^2/(4\pi\epsilon_0\hbar c) = 1/137.036$.

Answer In problem 1 of Homework 3, we found that the speed of the electron in the n^{th} Bohr orbit is $v_n = \alpha c/n$, so $v_1 = \alpha c$. It's interesting that the electron mass doesn't enter in this expression.

(b) Find the numerical value of the electron's velocity v in m/s.

Answer $v_1 = \alpha c = 2.19 \times 10^6$ m/s.

(c) Determine the fractional error that you make if you use the classical expression for the kinetic energy $K_c = m_e v^2/2$ rather than the relativistic expression $K_r = m_e c^2(\gamma - 1)$.

Answer Here $\gamma = (1-v^2/c^2)^{-1/2} = (1-\alpha^2)^{-1/2}$. Expanding for $x << 1$, $(1+x)^a = 1+ax+(1/2)a(a-1)x^2 + \mathcal{O}x^4$, so here $\gamma = 1 + \alpha/2 + 3\alpha^4/8 + \mathcal{O}\alpha^6$. It follows that the relativistic expression $K_r = m_e c^2(\gamma-1) = m_e c^2(v^2/2c^2 + 3v^4/8c^4)$. The first term is just the classical expression $K_c = m_e v^2/2$, so the difference $K_r - K_c = m_e c^2(3v^4/8c^4)$, and the fractional error if you use the classical expression is $(K_r - K_c)/K_c = (3/8)m_e\alpha^4/(1/2)m_e\alpha^2 = (3/4)\alpha^2 = 4.0 \times 10^{-5}$.

Or, by direct calculation, $(\gamma - 1 - \alpha^2/2)/(\alpha^2/2) = 1.068 \times 10^{-9}/2.66 \times 10^{-5} = 4.0 \times 10^{-5}$.

2. (10 points) (a) Show that the speed v of a particle of mass m and energy E is given by $v/c = [1-(mc^2/E)^2]^{1/2}$ and that, if E is much greater than mc^2, it is a good approximation to take $v/c = 1 - (1/2)(mc^2/E)^2$.

Answer Since $E = mc^2\gamma$, $[1(mc^2/E)^2]^{1/2} = [1 - \gamma^{-2}]^{1/2} = [1 - 1 + v^2/c^2]^{1/2} = v/c$. If $mc^2/E << 1$ then Taylor expand this expression to find $v/c = 1 - (1/2)(mc^2/E)^2$.

(b) Find the speed of an electron of kinetic energy $K = 0.511$ MeV and that of an electron of kinetic energy $K = 10MeV$.

Answer Since an electron's rest energy $m_e c^2 = 0.511$ Mev, if it has the same kinetic energy $K = m_e c^2(\gamma - 1) = m_e c^2$ then its total energy $E = m_e c^2 + K = 2m_e c^2$. Using the formula in part (a), $v/c = [1 - (1/2)^2]^{1/2} = (3/4)^{1/2} = 0.866$ so $v = 0.866c = 2.60 \times 10^8$ m/s.

Using the same formula, if $K = 10$ MeV, then $E = 10.511$ MeV, so again using the formula in part (a), $v/c = [1 - (1/10.511)^2]^{1/2} = 0.995$ so $v = 0.995c = 2.986 \times 10^8$ m/s. Alternatively, using the second (approximate) formula, $v/c = 1 - (1/2)(1/10.511)^2 = 0.995$ as before.

3. (15 points) A muon is identical to an electron except for its mass, which is $m_\mu = 105.7\ \mathrm{MeV}/c^2$, and the fact that it is unstable, with a lifetime of about 2.2 microseconds. Suppose that muons of energy 10 GeV are produced when cosmic rays strike nitrogen nuclei at an altitude of 35 km, and that the muons are traveling straight downward.

(a) If we neglect relativistic time dilation (the relativistic slowing of moving clocks), how far would the muons go in 2.2 microseconds?

Answer Since $m_\mu c^2/E << 1$ we can use the approximate formula of the previous problem to see that $v \approx c$. Then $c\tau = (3 \times 10^8 \mathrm{m/s})(2.2 \times 10^{-6} \mathrm{s}) = 6.6 \times 10^2$ m.

(b) Now taking into account time dilation, calculate how far the muon will go in 2.2 microseconds.

Answer Here $\gamma = E/m_\mu c^2 = 10\ \mathrm{GeV}/105.7\ \mathrm{MeV} = 94.6$, so because of time dilation the effective lifetime of the 10 GeV muon is 94.6τ and the distance travelled in that time is $94.6(660\ \mathrm{m}) = 62$ km. Thus time dilation explains how these energetic muons reach sea level.

(c) Calculate the fraction of the muons produced at 35 km altitude that reach sea level.

Answer The fraction of muons that survive is $f = \exp(-t/\gamma\tau)$, where $t = 35\ \mathrm{km}/c = 1.167 \times 10^{-4}$ s. Then $f = \exp[-1.167 \times 10^{-4}/(94.6)(2.2 \times 10^{-6})] = \exp(-0.56) = 0.57$, so 57% survive.

4. (15 points) If it is found that charged particles of energy 100 TeV are coming from a particular direction in space, how far away can their source be so that they are not deflected by more than about 10 degrees? Assume that the galactic magnetic field in which they move is 3 microgauss, and express your estimate of the distance in both meters and parsecs (pc).

Answer As I showed in the Physics 5K May 21 lecture (see especially pages 9-11), the particles of charge e and energy E move on a circle with Larmor radius

$$R = \frac{E}{ecB} = \frac{(10^{14}\ \mathrm{eV})(1.6 \times 10^{-19}\ \mathrm{J/eV})}{(1.6 \times 10^{-19}\ \mathrm{Coul})(3 \times 10^8\ \mathrm{m/s})(3 \times 10^{-10}\ \mathrm{T})} \approx 10^{15}\ \mathrm{m}\ .$$

If the particles move a distance fR in the magnetic field (which we are assuming is constant), they get deflected by f radians. Since $10^o = 0.17$ radian, the particles must move less than $0.17R = 1.7 \times 10^{14}$ m. Since 1 pc $= 3.1 \times 10^{16}$ m, this is 5.5×10^{-3} pc.

5. (15 points) An alternative explanation for the charged particles of 100 TeV energy that come from the same direction is that they actually come from the decay of neutrons with 200 TeV energy, since neutrons are neutral and therefore not deflected by magnetic fields. The lifetime of a free neutron is $\tau = 885$ seconds. How far would such neutrons travel before only 1/8 of the original neutrons are left?

Answer Since the hypothetical neutron energy is $E = 200$ TeV and the neutron rest energy is approximately 1 GeV = 0.001 TeV, $\gamma \approx 200/0.001 = 2 \times 10^5$. Since the neutron lifetime at rest $\tau = 885$ s, the neutron half-life $T_{1/2} = \tau \ln 2 = 613$ s. Then at rest 1/8 of the neutrons will survive to $3T_{1/2}$ since $2^{-3} = 1/8$. The effective lifetime of the 200 TeV neutrons will be increased by the γ factor, so the distance they will go is $d = 3T_{1/2}\gamma c = 3(613 \text{ s})(2 \times 10^5)((3 \times 10^8 \text{ m/s}) = 5.5 \times 10^{17} \text{ m} = 18$ pc.

6. (10 points) Draw a diagram of charged particles spiraling around lines of magnetic field and drifting toward a region where the strength of the magnetic field is increasing (so that the field lines are getting closer together). Explain using the diagram why the charged particle drift direction reverses if the field becomes sufficiently strong.

Answer