Physics 5K Lecture 2 - Friday April 13, 2012

# Action Principles in Mechanics, and the Transition to Quantum Mechanics

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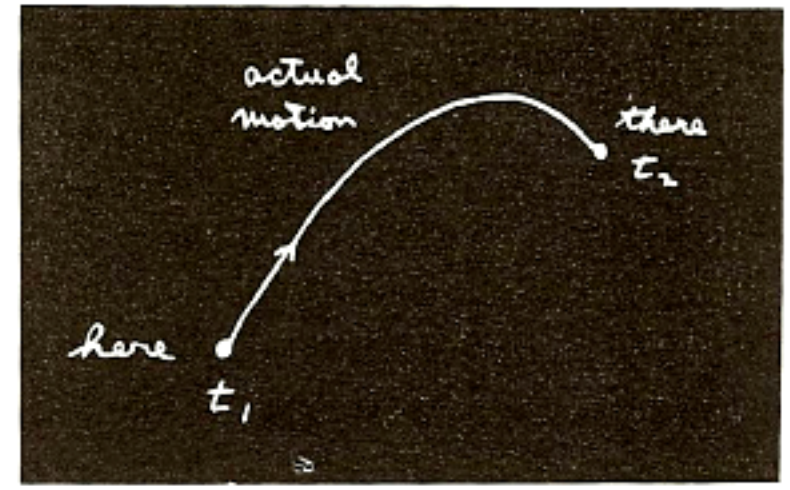
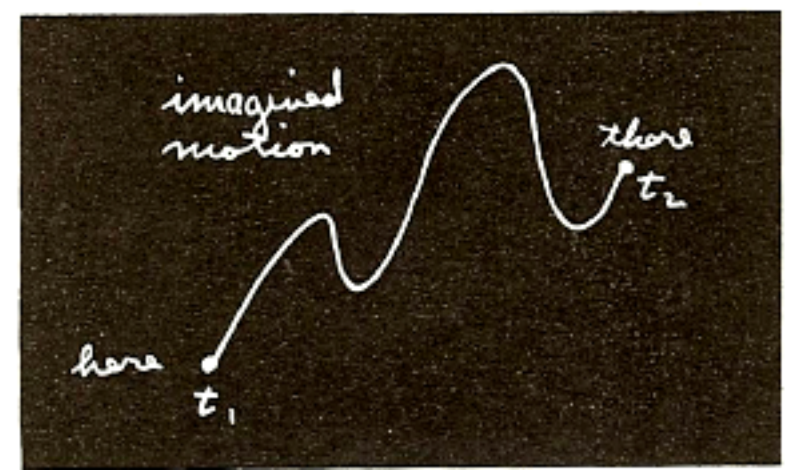
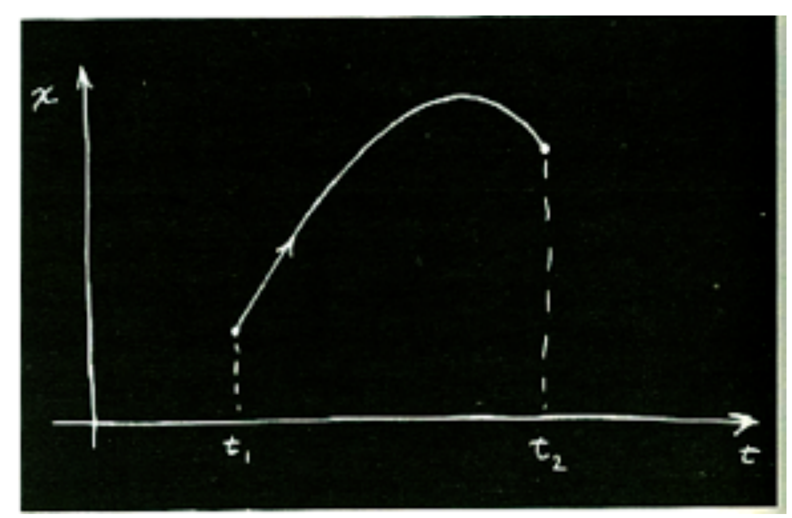
This lecture is about how the laws of classical mechanics follow from extremizing the Action, and how this generalizes to quantum mechanics: the path integral formulation of quantum theory. I follow the *Feynman Lectures on Physics*, Vol. 2, Chapter 19: “The Principle of Least Action.” Some of the material in this lecture will be covered in the UCSC course Phys 105, Mechanics. The material at the end follows B. R. Holstein and A. R. Swift, Am. J. Phys. 50, 829 (1982) and 50, 833 (1982).




In studying classical mechanics, you have thus far used Newton’s 2nd Law, F = ma, as the basic principle. However, there is a completely different mathematical approach which leads to the same equations and the same solutions, but which looks completely different. It turns out that to find the path that a particle travels from some initial starting point $(t_1, x_1)$ to some final point $(t_2, x_2)$, we can minimize a quantity called the Action. (Actually, what we want is the extremum of the Action -- either the minimum or the maximum.)

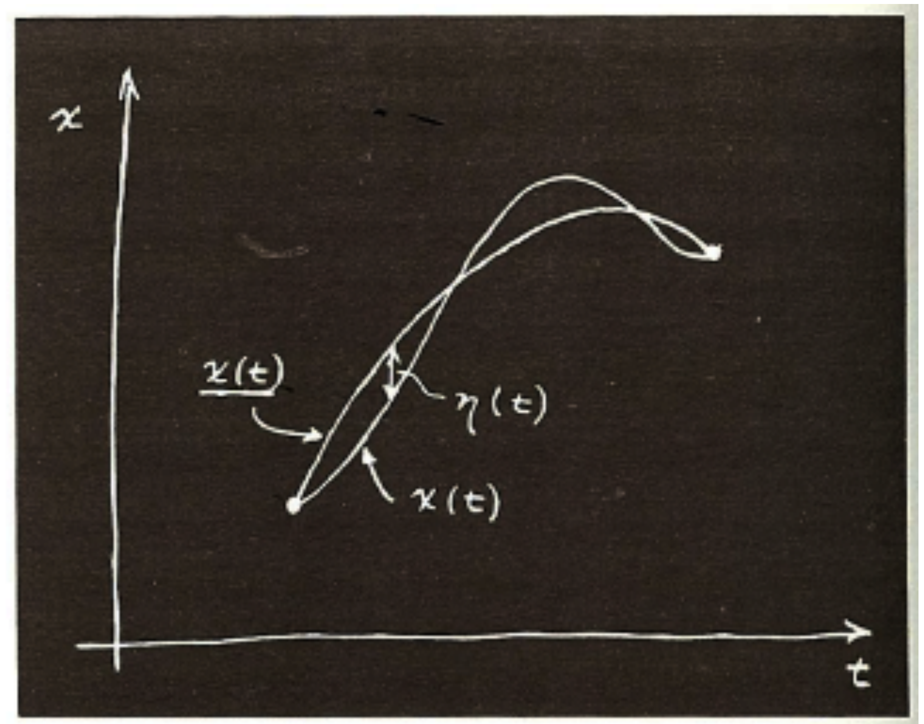



The Action is the integral of the difference between the kinetic and potential energy, Action = ∫(KE - PE) dt. To be specific, for a particle moving along the x axis from point $(t_1, \mathbf{x}_1)$ to point $(t_2, \mathbf{x}_2)$ under gravity, the integral is




$$\text{Action} = S = \int_{t_1}^{t_2} (\text{KE} - \text{PE})\, dt = \int_{t_1}^{t_2} \left[\frac{1}{2}\, m \left(\frac{dx}{dt}\right)^2 - mgx\right] dt.$$

This is a problem in the branch of mathmatics called "calculus of variations." There are fancy ways of dealing with this, but we will try to do it in a simple way.




Let's imagine that there is a true path x(t), and consider some other path x(t) + η(t) that differs by a small amount η(t), where $\eta(t_1) = \eta(t_2) = 0$ since the path must start and end at the fixed points $(t_1, x_1)$, $(t_2, x_2)$. A really hard way to find the true path would be to calculate the integral for all possible paths. Fortunately, there is an easier way.

If the Action is minimized, the first derivative vanishes. This means that any small variation will be not first order, but second order. That is, if the path x(t) is the true path, the Action is minimized, so for any small variation around this path, to first order there will be no difference in the Action -- any difference will arise at second order in η. (If there were a difference at first order, then in one direction the sign would be positive and in the other direction negative -- and we could therefore get the action to decrease, contrary to the assumption that it is minimum. A similar argument holds if instead the Action is maximized.)

If we consider a path x(t) + η(t), then the Action integral becomes

$$S = \int_{t_1}^{t_2} \left[ \frac{m}{2} \left( \frac{d\underline{x}}{dt} + \frac{d\eta}{dt} \right)^2 - V(\underline{x} + \eta) \right] dt.$$

Let's write out the squared term:

$$\left(\frac{d\underline{x}}{dt}\right)^2 + 2\,\frac{d\underline{x}}{dt}\,\frac{d\eta}{dt} + \left(\frac{d\eta}{dt}\right)^2 \quad \text{so} \quad \frac{m}{2}\left(\frac{d\underline{x}}{dt}\right)^2 + m\,\frac{d\underline{x}}{dt}\,\frac{d\eta}{dt} + \text{(second and higher order).}$$

For the potential energy, we expand in a Taylor series:

$$V(\underline{x} + \eta) = V(\underline{x}) + \eta V'(\underline{x}) + \frac{\eta^2}{2} V''(\underline{x}) + \cdots$$

Thus the Action becomes

$$S = \int_{t_1}^{t_2} \left[ \frac{m}{2} \left( \frac{d\underline{x}}{dt} \right)^2 - V(\underline{x}) + m \frac{d\underline{x}}{dt} \frac{d\eta}{dt} - \eta V'(\underline{x}) + \text{(second and higher order)} \right] dt .$$

Leaving out the "second and higher order" terms, we find that the change in the Action is

$$\delta S = \int_{t_1}^{t_2} \left[ m \frac{d\underline{x}}{dt} \frac{d\eta}{dt} - \eta V'(\underline{x}) \right] dt.$$

We now use "integration by parts" to simplify this further. Recall that, for any functions η and f,

$$\frac{d}{dt} (\eta f) = \eta \frac{df}{dt} + f \frac{d\eta}{dt} .$$

Integrating, we get the standard "integration by parts" result

$$\int f \frac{d\eta}{dt}\, dt = \eta f - \int \eta \frac{df}{dt}\, dt.$$

Applying this to our case,

$$\delta S = m \frac{d\underline{x}}{dt} \eta(t) \Big|_{t_1}^{t_2} - \int_{t_1}^{t_2} \frac{d}{dt} \left( m \frac{d\underline{x}}{dt} \right) \eta(t)\, dt - \int_{t_1}^{t_2} V'(\underline{x})\, \eta(t)\, dt.$$

The first term vanishes because $\eta(t_1) = \eta(t_2) = 0$, since the path must start and end at the fixed points $(t_1, x_1)$, $(t_2, x_2)$.

The change in the Action then becomes

$$\delta S = \int_{t_1}^{t_2} \left[ -m \frac{d^2 \underline{x}}{dt^2} - V'(\underline{x}) \right] \eta(t)\, dt.$$

But we know that if the Action was really extremized for path $\underline{x}(t)$, then the change in the Action must vanish. Since it must vanish for any possible value of η(t), it must be true that

$$\left[ -m \frac{d^2 \underline{x}}{dt^2} - V'(\underline{x}) \right] = 0.$$

Since V'(x) = - F(x), this is just our old friend F = ma.

So the assumption that the Action is extremized is just another way to do Newtonian mechanics! We can readily generalize this calculation to three dimensions, where η(t) will now have x, y, and z components, and the kinetic energy will be

$$\mathrm{KE} = \frac{m}{2} \left[ \left( \frac{dx}{dt} \right)^2 + \left( \frac{dy}{dt} \right)^2 + \left( \frac{dz}{dt} \right)^2 \right].$$

We can also generalize the argument to any number of particles: we just add the Actions for each particle, and we get Newton's 2nd law in three dimensions for any number of particles. There is also a straightforward generalization to special relativity.

The laws of optics can be derived from the "principle of least time". It turns out that **all the fundamental laws of physics that have yet been discovered can be written in the form of an Action-extremization principle**, so this approach is very powerful. It is also very useful, since it is often easier to solve problems this way -- and the method gives a very interesting way to go from classical to quantum mechanics.

There is quite a difference between a law that says that every particle just follows its nose, that is it just inches along determining the change in velocity from the force at each point divided by the mass, versus a law that says the entire path is determined by extremizing a certain integral. And yet these two different-sounding formulations are mathematically equivalent. So the lesson is: don't become too wedded to any particular formulation of a physical law!

But how does the particle know which path extremizes the Action? Does it feel out other paths in order to choose the right one? Amazingly, in quantum mechanics we learn that this is exactly what happens!

The probability that a particle starting at point $x_1$ and time $t_1$ will arrive at point $x_2$ at time $t_2$ is the square $|\Psi|^2$ of a probability amplitude $\Psi$. The total amplitude is the sum of the amplitudes for each possible path. The amplitude $\Psi$ for each path is proportional to $\exp(iS/\hbar)$ where "h-bar" $\hbar = h/2\pi = 1.054 \times 10^{-34}$ $m^2$ kg/s, and Planck's constant $h = 6.626 \times 10^{-34}$ $m^2$ kg/s. The Action S has dimensions of energy x time, and h has the same dimensions.

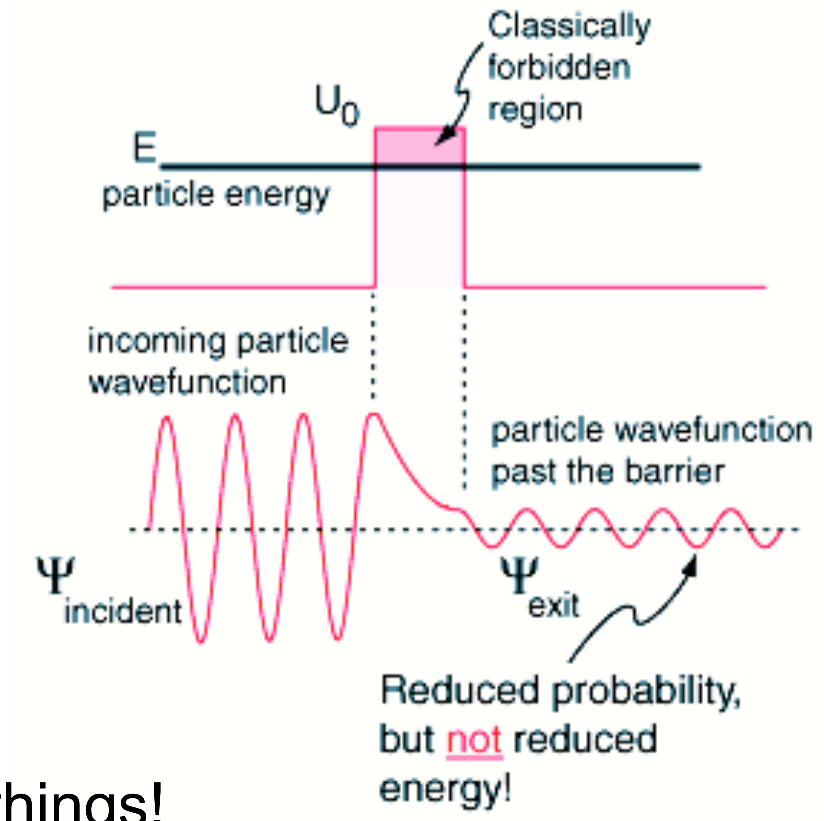
The size of the Action compared to $\hbar$ determines how important quantum effects are. If for all paths, S is large compared to $\hbar$, then the complex phase will be wildly different for all paths except for those that are extremely close to the true path, the one for which the Action is extremized, since any small variation about that path will not change the Action to first order. Then all the other paths will cancel out because of the wild change in phase, and the path that will be taken will be the classical path, the one for which the Action is extremized. But if the Action is comparable in size to $\hbar$, other paths besides the classical one can be important.

The fact that quantum mechanics can be formulated this way was discovered by Richard Feynman in 1942, when he was still a graduate student, based on earlier work by Dirac.

A particularly striking use of the path-integral formulation of quantum mechanics is its implications for "barrier penetration," particles moving through places that they are forbidden to be, according to classical mechanics. Here's a sketch of this approach.

According to classical mechanics, $ma = F = -V'(x)$ and $E = \frac{1}{2} mv^2 + V(x)$.
Note that $L \equiv KE - PE = \frac{1}{2} mv^2 - V(x) = mv^2 - E$, so

$$\text{Action} = S = \int_{t_1}^{t_2} (KE - PE)\, dt = \int_{t_1}^{t_2} (mv^2 - E)\, dt = \int_{t_1}^{t_2} mv \frac{dx}{dt}\, dt - \int_{t_1}^{t_2} E\, dt = \int_{x_1}^{x_2} p\, dx - E(t_2 - t_1)\ ,$$




where $p = mv = [2m(E-V)]^{1/2}$. Let's apply this to the quantum mechanical formalism where $\Psi = A \exp(iS/\hbar)$.

Consider a situation where V is greater than E, so that $p = [2m(E-V)]^{1/2}$ is imaginary. This describes a situation where the particle is forbidden to be, according to classical physics. But the wavefunction $\Psi$ doesn't vanish, rather it includes a term

$$\Psi = A \exp(iS/\hbar) = A \exp\left( - \int [2m(V-E)]^{1/2} dx/\hbar \right) .$$

Next week we will show how to use this to calculate important things!