Solutions to 1992 114B exam

1.

(a)Following hint, try $y = x^p$. Plugging into ODE:

$$p(p-1)x^{p-2} + \omega^2 x^{p-2} = 0$$

which implies $p(p-1) + \omega^2 = 0$. Solving quadratic equation yields

$$p_\pm = \frac{1 \pm \sqrt{1 - 4\omega^2}}{2}$$

which we write (since $\omega > 1/2$) as $p_\pm = \frac{1 \pm i\sqrt{4\omega^2 - 1}}{2}$.

(b) The general solution is then $y = C_+ x^{p_+} + C_- x^{p_-}$, and at $x = 1$, $y = 0$. Using this we obtain $C_+ + C_- = 0$. Therefore $y = C_+(x^{p_+} - -x^{p_-}) = C\sqrt{x}\sin(\sqrt{\omega^2 - \frac{1}{4}}\ln x)$.

2. Write $y(x,t) = X(x)T(t)$ and plugging into

$$\frac{1}{x^2}\frac{\partial^2 y}{\partial t^2} = \frac{\partial^2 y}{\partial x^2}$$

gives

$$\frac{T''}{T} = \frac{x^2 X}{X}$$

which then must equal a constant, call it $-\omega^2$. With the boundary condition from above, i.e. $y(1) = 0$, we have

$$T(t) = e^{\pm i\omega t}$$

and

$$X(x) = \sqrt{x}\sin(\sqrt{\omega^2 - \frac{1}{4}}\ln x).$$

(b) At $x = 2$, $y = 0$, so $\sin(\sqrt{\omega^2 - \frac{1}{4}}\ln 2) = 0$, which implies $\sqrt{\omega^2 - \frac{1}{4}}\ln 2 = n\pi$.

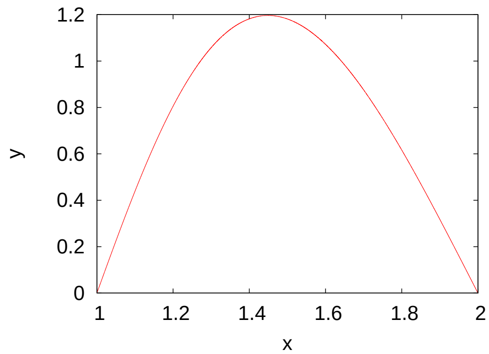
(c) Solving for ω, $\omega_n = \sqrt{(\frac{n\pi}{\ln 2})^2 + \frac{1}{4}}$ for $n = 1, 2, 3, \ldots$.

(d) Summing over all solutions:

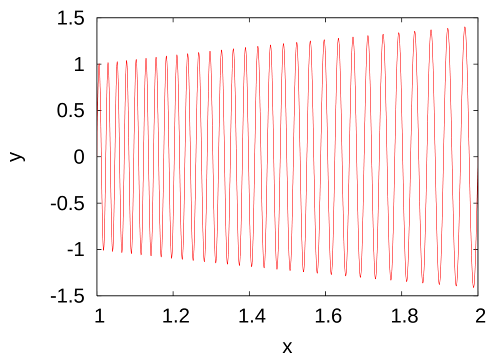
$$y = \sum_{n=1}^{\infty}(A_n\cos(\omega_n t) + B_n\sin(\omega_n t))\sqrt{x}\sin(\frac{n\pi}{\ln 2}\ln x).$$

(e)

n=1:



n=60:



3. $f(x,t) \equiv e^{-i\omega t} f(x)$ and

$$\frac{1}{x^2}\frac{\partial^2 y}{\partial t^2} = \frac{\partial^2 y}{\partial x^2} + F(x,t)$$

Try $y(x,t) = e^{-i\omega t} X(x)$. This implies

$$X''(x) + \frac{\omega^2}{x^2} X = f(x).$$

4. (a) From 1(b) $y = A(x')\sqrt{x}\sin(\sqrt{\omega^2 - \frac{1}{4}}\ln x)$ for $x < x'$.

(b) Rescale x: $y = C_+(\frac{x}{2})^{p_+} + C_-(\frac{x}{2})^{p_-}$, with the boundary condition $y(x/2) = 0$. Therefore $C_+ + C_- = 0$, and so $y = B(x')((\frac{x}{2})^{p_+} - -(\frac{x}{2})^{p_-}) = B(x')\sqrt{x}\sin(\sqrt{\omega^2 - \frac{1}{4}}\ln(x/2))$. for $x > x'$.

(c)

$$\int_{x'-\epsilon}^{x'+\epsilon} (G'' + \frac{\omega^2}{x^2} G) dx = \int_{x'-\epsilon}^{x'+\epsilon} \delta(x - x') dx$$

and taking $\lim_{\epsilon\to 0}$ gives

$$G'(x+\epsilon,x') - G'(x-\epsilon,x') = 1$$

(d)Define $D \equiv \sqrt{\omega^2 - \frac{1}{4}}$. From continuity $G'(x+\epsilon,x') = G'(x-\epsilon,x')$, so

$$A(x')\sin(D\ln x') = B(x')\sin(D\ln(x'/2)).$$

and from 4(c)

$$B(x')\cos(D\ln(x'/2)) - A(x')\cos(D\ln(x')) \;=\; \sqrt{x'}/D.$$

Solving for A and B gives

$$A(x') = \frac{\sqrt{x'}}{D}\frac{\sin(D\ln(x'/2))}{\sin(D\ln(2))}$$

and

$$B(x') = \frac{\sqrt{x'}}{D}\frac{\sin(D\ln(x'))}{\sin(D\ln(2))}$$

(e) $y = \int_1^2 G(x,x')f(x')dx'$.

5. (a) The combination of 52 taken 5 at a time:

$$\binom{52}{5} = \frac{52!}{5!47!} = 2598960$$

(b) There are 4 Royal Flushes, so the probability of this happening is

$$\frac{4}{\binom{52}{5}} = 4/2598960 = 1/649740 \approx 1.54\times 10^{-6}.$$