

Lemma 2

$\sigma = \text{max}$ $\tau - \eta \in \text{dom}(\pi)$ (2) Σ

$$(\sigma, \eta) \models \varphi \quad \nexists \tau = (\sigma, \eta) \vdash \neg \varphi$$

Midterm 2
11/19/97
Physics 114B

1. Do an approximate calculation of

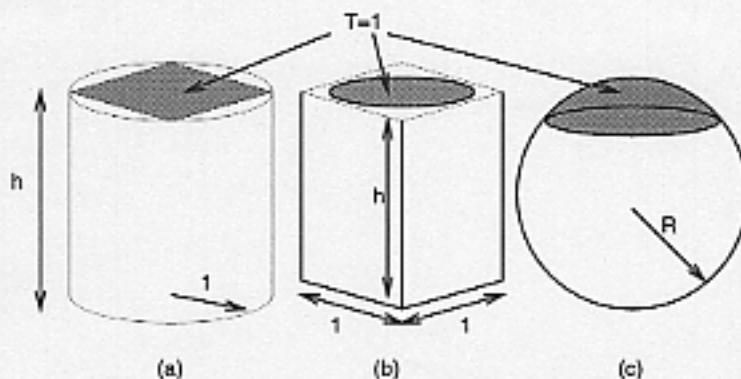
$$\int_{-\pi}^{\pi} \cos^N(\theta) d\theta$$

for large N using the same method used to derive Stirling's formula, the "method of steepest descent". Do so as follows.

- (a)(5 points) Plot the integrand for N both even and odd, and N large.
- (b)(5 points) Identify the point(s) that contribute mostly to the integral.
- (c)(10 points) Expand the integrand in the neighborhood of one of these points.
- (d)(10 points) From this, obtain the answer.

2. (40 points) Find the solution to Laplace's equation $\nabla^2 T = 0$ inside a sphere of radius 1 when the temperature on the surface is $T(r=1, \theta) = \cos(\theta) - 3 \sin^2(\theta)$.

- 3.(45 points) For the figures below write down the general form of solutions to Laplace's equation. The shaded boundary regions represents $T = 1$ and all other regions are held at $T = 0$. You need not evaluate the constants that appear in the separation of variables summation, but state which coordinate system should be used, rectangular, cylindrical, or spherical. You should also make use of any symmetry properties that allow you to throw away, or determine, some of the coefficients.



Useful Information:

Stirling's formula for large N : $N! \sim N^N e^{-N} \sqrt{2\pi N}$, $\int_{-\infty}^{\infty} e^{-x^2} dx = \sqrt{\pi}$.

$$P_0(x) = 1, \quad P_1(x) = x, \quad P_2(x) = \frac{3x^2 - 1}{2}$$

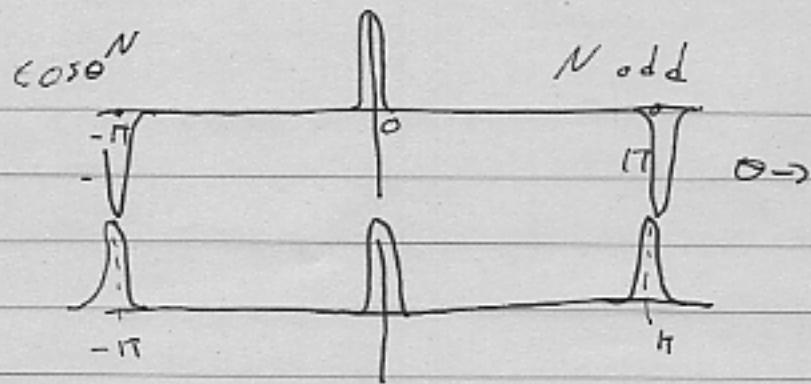
solutions to $\nabla^2 T = 0$:

$$\text{Rectangular : } T(\mathbf{r}) = e^{i\mathbf{q} \cdot \mathbf{r}}, \quad q_x^2 + q_y^2 + q_z^2 = 0$$

$$\text{Cylindrical : } T(r, \theta, z) = e^{\pm kz} \{ \cos(n\theta), \sin(n\theta) \} \{ J_n(kr), K_n(kr) \}$$

$$\text{Spherical : } T(r, \theta) = \{ r^l, r^{-l-1} \} P_l(\cos \theta)$$

1. (a)

(b) N large: $\theta = 0, \pm \pi$

$$(c) \text{ N odd } \int_{-\pi}^{\pi} \cos^N \theta \, d\theta = 0$$

Never $-\pi + \pi$ points add to give
same answer as integral @ 0

$$@ 0: (\cos \theta)^N \doteq (1 - \frac{\theta^2}{2})^N \doteq (e^{-\frac{\theta^2}{2}})^N = e^{-N\theta^2/2}$$

$$(d) \text{ Never: } \int_{-\pi}^{\pi} \cos^N \theta \, d\theta \doteq 2 \int_{-\infty}^{\infty} e^{-N\theta^2/2} \, d\theta$$

$$\text{let } x = \sqrt{\frac{N}{2}} \theta \Rightarrow d\theta = \sqrt{\frac{2}{N}} dx$$

$$\therefore \int_{-\pi}^{\pi} \cos^N \theta \, d\theta \doteq 2 \sqrt{\frac{2}{N}} \int_{-\infty}^{\infty} e^{-x^2} \, dx = 2 \sqrt{\frac{2\pi}{N}}$$

Never,

0 N odd.

$$1. \text{ Check: } \cos\theta = \frac{e^{i\theta} + e^{-i\theta}}{2}$$

$$\Rightarrow \cos^N \theta = \frac{1}{2^N} \sum_n \binom{N}{n} e^{im\theta} e^{i(N-m)\theta}$$

$$= \frac{1}{2^N} \sum_n \binom{N}{n} e^{i(N-2m)\theta}$$

$$\int_{-\pi}^{\pi} \cos^N \theta d\theta = \sum \text{ terms } d\theta \quad \text{but only } N=2m \text{ contributes}$$

↙

$$\Rightarrow N \text{ odd gives } 0.$$

$$= \frac{1}{2^N} \left(\frac{N}{2} \right) \int_{-\pi}^{\pi} 1 d\theta = \frac{2\pi}{2^N} \frac{N!}{\left(\frac{N}{2} \right)!^2}$$

$$\therefore \frac{3\pi}{2^N} \frac{N^N e^{-N} \sqrt{2\pi N}}{\left(\left(\frac{N}{2} \right)^{\frac{N}{2}} e^{-\frac{N}{2}} \right)^2 \left(2\pi \frac{N}{2} \right)} = \frac{2}{2^N} \sqrt{\frac{2\pi}{N}} \frac{N^N e^{-N}}{\left(\frac{N}{2} \right)^N e^{-N}}$$

$$= 2 \sqrt{\frac{2\pi}{N}} \quad N \text{ even}$$

0 N odd.

$$2. \text{ Inside sphere } T(r, \theta) = \sum_{l=0}^{\infty} a_l r^l P_l(\cos\theta)$$

$$@ r=1 : T(1, \theta) = \sum_{l=0}^{\infty} a_l P_l(\cos\theta)$$

$$= \cos\theta - 3\sin^2\theta = \cos\theta - 3 + 3\cos^2\theta$$

let $x = \cos\theta$. we want this as $\sum P_l(x)$'s

$$\text{i.e. } -3 + x + 3x^2 = a_0 + a_1 P_1(x) + a_2 P_2(x)$$

$$\Rightarrow a_1 = 1, a_0 + a_2 \frac{3x^2 - 1}{2} = -3 + 3x^2$$

$$\Rightarrow a_2 = 2 \Rightarrow a_0 + 3x^2 - 1 = -3 + 3x^2 \Rightarrow a_0 = -2$$

$$0 = a_3 = a_4 = a_5 = \dots$$

$$\therefore T(r, \theta) = -2 + 1 \cdot r \cos\theta + (3 \cos^2\theta - 1) r^2$$

$$\text{Check: } @ r=1 \quad T(1, \theta) = -2 + \cos\theta + 3\cos^2\theta - 1 = \cos\theta - 3\sin^2\theta \quad \checkmark$$

$$\text{In rectangular coords: } T(r, \theta) = -2 + z + 3z^2 - (x^2 + y^2)$$

$$= -2 + z + 2z^2 - x^2 - y^2 = T(x, y, z)$$

$$\begin{aligned} D^2 T(x, y, z) &= \left[\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2} \right] [-2 + z + 2z^2 - x^2 - y^2] = \\ &= \underbrace{\frac{\partial^2}{\partial x^2} [-2 + z + 2z^2 - x^2 - y^2]}_{-2} + \underbrace{\frac{\partial^2}{\partial y^2} [-2 + z + 2z^2 - x^2 - y^2]}_{-2} + \underbrace{\frac{\partial^2}{\partial z^2} [-2 + z + 2z^2 - x^2 - y^2]}_{+4} = 0 \quad \checkmark \end{aligned}$$

3. (a) Cylindrical: No singularities $\Rightarrow k_n$ term = 0

$$\left\{ \frac{e^{kz}}{e^{-kz}} \right\} = \sinh kz \quad \text{since } T=0 \text{ on bottom}$$



Sinh theta term 0

Cosh theta term only nonzero ~~nonzero~~
for $n = 0, 4, 8, 12, \dots$

because square has 4 fold symmetry

also $J_n(k_n z) = 0$ because of
B.C. on sides, so

$$T = \sum_n \sum_m a_{nm} J_n(k_{nm} z) \sinh(k_{nm} z) \cos(4\theta)$$

\uparrow
 $0, 4, 8, \dots$ \uparrow
 m^{th} zero of J_n .

rectangular:

(b) \vee Again, in z direction $Z(z) = \sinh(k_z z)$

since $T=0$ on sides, $X(x) = \sin\left(\frac{n\pi x}{l}\right)$

$Y(y) = \sin\left(\frac{m\pi y}{l}\right)$

$$k_z = \pi \sqrt{n^2 + m^2}$$

$$\Rightarrow T(x, y, z) = \sum_{n, m} a_{nm} \sinh\left(\pi \sqrt{n^2 + m^2} z\right) \sin(n\pi x) \sin(m\pi y)$$

By symmetry, $T(x, y, z) = T(y, x, z) \Rightarrow a_{nm} = a_{mn}$.

spherical:

$$3(c) \quad \text{No singularity} \Rightarrow r^{-\ell-1} \text{ term} = 0$$

$$\Rightarrow f(r, \theta) = \sum_{\ell} a_{\ell} r^{\ell} P_{\ell}(\cos \theta)$$