Practice problems for final

1. Solve the three dimensional wave equation inside a cube $0 < x, y, z < L$

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} + \frac{\partial^2 u}{\partial z^2} = \frac{1}{v^2}\frac{\partial^2 u}{\partial t^2}$$

with the boundary conditions that $u = 0$ on the faces of the cube. At $t = 0$ $u(x, y, z) = \delta(x - L/2)\delta(y - L/2)\delta(z - L/2)$ and

$$\left.\frac{\partial u}{\partial t}\right|_{t=0} = 0$$

everywhere inside the cube.

2. Consider the diffusion equation for $T(x, t)$ in one dimension on the interval $0 < x < L$.

(a) Given that the boundaries are insulated, that is for all time,

$$\left.\frac{\partial T}{\partial x}\right|_{x=0} = \left.\frac{\partial T}{\partial x}\right|_{x=L} = 0$$

show that

$$\int_0^L T(x, t)\, dx$$

does not change with time.

(b) Suppose instead the boundaries are held fixed in temperature with $T(x = 0, t) = 0$ and $T(x = L, t) = 1$. Find the temperature of the bar as a function of x in the limit of long times.

3. Calculate the solution to $\nabla^2 T(x, y) = 0$ for an isosceles right triangle whose hypotenuse is of length $\sqrt{2}$ and is held at $T = 1$. The other two sides are held at $T = 0$. Hint: First subtract 1 from all temperatures in the problem. Now consider how the solution of this problem can be obtained from a square. Use symmetry to argue that the temperature along the diagonal of the square is 0 and hence corresponds to a solution to the triangle problem. Then solve the square problem using superposition and separation of variables.