

Homework 4
Due 5/20/10

The first problem of this homework calculates the radius of gyration of a ring polymer using some mathematical techniques that are common in theoretical physics. The use of continuous variables simplifies the analysis, and allow one to apply results from the theory of Fourier series. The equipartition theorem from statistical mechanics is also used. This provides a good introduction to many other kinds of problem that use a lot of the same techniques. The last problem is on a very interesting concept used in many areas of physics, that of “multifractals”

1. Consider a polymer chain described by coordinates $\mathbf{r}(s)$, where s is the arclength, and is a real variable with $0 < s < L$. You are used to the coordinates being described as \mathbf{r}_i where i is an integer, but it is easier sometimes to turn the index i into a continuous variable.

We want to calculate the radius of gyration of an ideal *ring* polymer, that is, one with $\mathbf{r}(0) = \mathbf{r}(L)$. We take the Hamiltonian (or energy) at inverse temperature β , and step length l , to be:

$$\beta H = \frac{3}{2l} \int_0^L \left| \frac{d\mathbf{r}(s)}{ds} \right|^2 ds$$

Calculate the radius of gyration of this chain as a function of L and l . Use the following steps.

(a) Write $\mathbf{r}(s)$ in terms of a Fourier series

$$\mathbf{r}(s) = \sum_{k=-\infty}^{\infty} \hat{\mathbf{r}}_k \exp(-2\pi i k s / L)$$

and then express H in terms of the new Fourier variables $\hat{\mathbf{r}}_k$. H should decouple when expressed in terms of these new variables.

(b) Write the radius of gyration as

$$R_g^2 = \left\langle \frac{1}{L} \int_0^L \left| \mathbf{r}(s) - \frac{1}{L} \int_0^L \mathbf{r}(s) ds \right|^2 ds \right\rangle$$

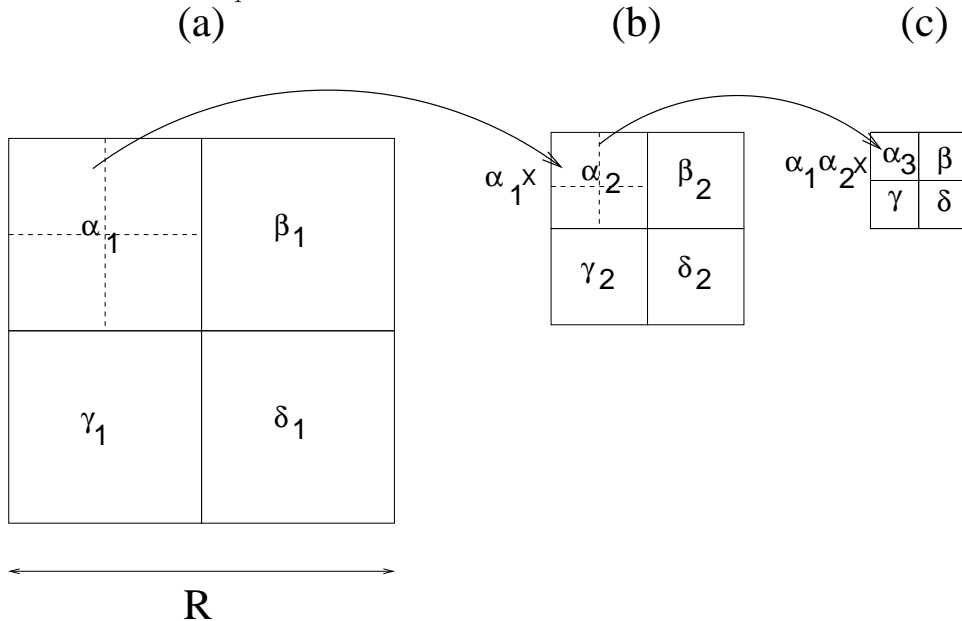
As in part (a), express this also in terms of the new Fourier variables $\hat{\mathbf{r}}_k$. *Hint: write down the inverse formula for $\hat{\mathbf{r}}_k$ in terms of $\mathbf{r}(s)$. You should notice that the second integral inside of the average for R_g^2 is simply related to one of the $\hat{\mathbf{r}}_k$'s.*

- (c) $\hat{\mathbf{r}}_k$ is complex and has a real and imaginary part. Because $r(s)$ is real, use this to relate $\hat{\mathbf{r}}_k$ to $\hat{\mathbf{r}}_{-k}$.
- (d) Use the equipartition theorem to calculate $\langle Re(\hat{\mathbf{r}}_k)^2 \rangle$ and $\langle Im(\hat{\mathbf{r}}_k)^2 \rangle$.
- (e) Now calculate R_g^2 using the results of parts (b) and (d).

2. *Multifractals* Multifractals appear in a large variety of systems: turbulence, chaotic dynamical systems, nonequilibrium aggregation models, and are thought to even appear in financial markets. In this problem we'll start exploring these fascinating systems. The following model was first introduced in understanding turbulence, see R. Benzi, G. Paladin, G. Parisi and A. Vulpiani J. Phys. A: Math. Gen. 17 (1984) 3521-3531.

Consider the field $\phi(\mathbf{r})$, in this case it could be the energy dissipation as a function of position for a turbulent fluid. Turbulence is believed to be self similar so we model the physics to be the same at all length scales down to a cutoff length a . We think of large scale eddies breaking up into smaller scale ones, and those in turn breaking up. Although the model here seems very crude in many ways, it captures essential features of the physics that are inaccessible by many other involved mathematical calculations.

The model is depicted below.



We start off in (a) by breaking up the field into four parts (eddies) so that the proportion of the total dissipated power per unit volume (power density) in the four sub-boxes are α_1 , β_1 , γ_1 , and δ_1 (which are all positive and add up to unity). We assume these numbers are randomly drawn from some probability distribution, so that the probability distribution for α_1 is $P(\alpha_1)$. Many choices can be made for this distribution and will effect the physics that follows. Now

we assume that system breaks itself up further in a manner similar to what it just did but now at a smaller scale. So in (b), the upper left hand box then divides its dissipated power in the same way among four boxes. α_2 , β_2 , γ_2 , and δ_2 are drawn randomly in the same way as before. However the power density for this sub-box is scaled by an overall factor of α_1 . Therefore the total power in the upper left hand box of (b) at this new level is $\alpha_1\alpha_2$. We can continue to subdivide this way. The final box, (c), shows the eddies breaking up at a further level. Now the total power density in the upper left hand box in (c) is $\alpha_1\alpha_2\alpha_3$.

This process continues all the way down to a cutoff length a . If there are N generations of subdivision, The ratio of the initial box length R in (a) to a is 2^N . That is

$$R/a = 2^N. \quad (1)$$

Call the complete field for all space constructed using the above procedure $\phi(\mathbf{r})$. Its values depend on the realizations of the random variables (e.g. α_i) used to construct it. We would like to know various averages of ϕ averaged over all realizations of these random variables.

- a** Calculate $\langle \phi^m \rangle$. Take m to be any positive exponent. Express your answer as a function of the number of generations N , and in terms of moments of the distribution $P(\alpha)$ mentioned above, that is $\langle \alpha^m \rangle$.
- b** Using eq. 1, write $\langle \phi^m \rangle$ as a function of R , the size of the system. Your answer should still depend on the moments of the same distribution as in (a).
- c** Calculate the moments $\langle \phi^m \rangle$ explicitly in the case where the α 's can only take on two distinct values both with equal probabilities. Note that $\langle \alpha \rangle = 1$ because the total power injected at a larger scale is accounted for by the power present at smaller scales. So this model has one free parameter characterizing the width of $P(\alpha)$.