HISTORIA MATHEMATICA **25** (1998), 89–93
ARTICLE NO. HM972191

LETTER TO THE EDITOR

On Hooke's 1685 Manuscript on Orbital Mechanics

In a paper published in the May 1997 issue of this journal [1], Herman Erlichson criticizes my analysis [2] of a manuscript by Robert Hooke, dated September 1685, which describes a geometrical construction for orbital dynamics. Erlichson claims that I "misunderstood what Hooke was doing" [1, 176], and he then argues for a very different interpretation of Hooke's work. However, I will show that Erlichson confuses the technique employed by Hooke to draw a figure in his manuscript with the dynamics, described in the accompanying text, which this figure serves to illustrate. I will also provide additional evidence not included in my previous analysis, which shows that Erlichson's interpretation does not conform with Hooke's explicit words.

The text of Hooke's manuscript gives a geometrical construction for a polygonal orbit, in the case where the force is central and impulsive. This construction is based on dynamical principles which Hooke had been pursuing at least since 1666 when he first presented his ideas to the Royal Society, and which he communicated directly to Newton in 1679 [2]. Hooke's construction can readily be seen to correspond to Newton's geometrical construction for his proof that Kepler's area law is a consequence of central forces, which appeared first as Theorem 1 in the *De Motu*, and later became Proposition 1 of Book 1 of the *Principia*. In his description, Hooke considered the magnitude of the force impulse to be proportional to the distance from the center,[1] and he then asserted that "... the motion of this body shall therefore be polygonal in an ellipse."[2] In his text, Hooke did not give any proof of this assertion. However, the manuscript includes a figure containing a polygonal orbit and several auxiliary lines which shows that Hooke related the vertices of this orbit and the displacements associated with the impulses to equally spaced

[1] Hooke explicitly described the linear dependence of the impulse on radial distance: "... the second pulse of gravity shall meet the body at β ... driving it towards the center o with the velocity $\beta\gamma$ which has the same proportion to the radius βo that $\alpha\delta$ [the change of velocity due to the first pulse of gravity at α] has to the [radius] αo." While the text does not give the magnitude of $\alpha\delta$, the figure indicates clearly that $\alpha\delta/\alpha o = \mu$, where μ is a constant determined by the corresponding ratio for motion on the surrounding circle. Then he repeated this linear proportion explicitly: "For as αo is to $\alpha\delta$, soe βo to $\beta\gamma$ ..." [2, 346].

[2] Notice that the word *therefore* (for this reason) in this sentence leaves no doubt that Hooke meant that elliptic motion occurs as a consequence of a central force which varies linearly with the distance from the center, and not the converse as Erlichson claims in his paper. Hooke also described the initial conditions for elliptical motion to occur in the following manner: "... but because the [initial] velocity $h\alpha$ is less in proportion to $\alpha\delta$ than it ought to make it move in a circle, therefore its motion shall be in an ellipse" [2, 346]. It is interesting to note that already in 1666 Hooke had experimented with the conical pendulum which gives nearly elliptical orbits, and demonstrated mathematically that in this case the radial force increases linearly with the radial distance to the center of the ellipse [2, 335].

0315-0860/98 $25.00

vertices and displacement lines on a surrounding circle. This circle represents the orbit for special initial conditions (position and velocity) described elsewhere in Hooke's manuscript. From this figure one can attempt to reconstruct the basis of Hooke's assertions. In particular, the figure shows how Hooke drew the displacement lines representing motion due to force impulses proportional to the distance from the center with the aid of horizontal lines from the vertices associated with the circle, which are shown in the figure, but not mentioned in the text [2, 340]. Moreover, this graphical construction fixes the constant of proportionality or strength of the impulses to the value required for uniform motion on the surrounding circle [2, 340]. The vertices of the resulting polygonal orbit then lie on an ellipse, because the construction is an affine or scale transformation of the corresponding construction on this circle, as was pointed out to me by D. T. Whiteside, and provides a rigorous proof of Hooke's assertions [2, 340]. This transformation can be visualized by supposing the impulse geometrical construction for circular motion to be drawn on a rubber sheet which has been stretched horizontally. Then a corresponding construction for elliptical motion emerges when the sheet relaxes to its original state with impulses which now depend linearly on the radial distance.

In his paper, Erlichson argues from the figure in Hooke's manuscript that Hooke "took the ellipse as a given" [1, 176], and that he superimposed on it Newton's force impulse construction. However, the precise manner in which Hooke drew the figure in his manuscript is irrelevant, because the resulting vertices and displacement lines are *unique*[3] as a consequence of the affine transformation. This uniqueness is not under dispute. What matters is the evidence—which the figure provides—that Hooke was aware of this transformation, and consequently that he had an original proof for his assertion that a central force which depends linearly on the distance leads to an elliptical orbit.[4] Erlichson claims that if Hooke had used the geometrical

[3] Erlichson ignores the fact that in my paper I had already stated that "it is possible that Hooke may have drawn the polygonal path on an ellipse in this diagram by effectively applying an affine transformation to the polygonal path on the circumscribed circle, rather than by following the equivalent geometrical construction described in the text" [2, 341]. In fact, I described the same graphical construction discussed by Erlichson, but presented it as a graphical test that the polygonal vertices lie on an ellipse, under the assumption that the vertices had been obtained by a graphical method related to the force impulse. I also evaluated the scale factor of the transformation in Hooke's figure [2, 347, footnote 33] which Erlichson does not mention in his presentation of the same result [1, 170].

[4] According to Erlichson's interpretation, "Since Hooke took the ellipse as given he was clearly working on the direct problem. ... In reading this [Hooke's] description one notices immediately that Hooke has assumed that he is going from one on-orbit ellipse point, a, to another, β. ... of his ellipse vertices construction. As we have already noted, Nauenberg misunderstood what Hooke was doing in describing it as a 'graphical evaluation of the indirect problem'" [1, 176]. But the "indirect or inverse problem" is by definition the problem of finding the orbit given the force law, and what Hooke's text gives is an explicit description for the graphical construction of an orbit in the case in which the force depends linearly on the radial distance (recall note 1). A separate problem is to relate the resulting orbit, which is a polygon, to a known mathematical curve. In general, a graphical construction can give only an approximation to such a curve, but in this case the resulting vertex points of the polygon lie on an ellipse. While Hooke's text does not state how he arrived at this conclusion, the accompanying figure indicates clearly that Hooke related his graphical construction to the special case for circular motion [2, 340]. This relation corresponds to an affine transformation of the circle, and therefore gives a rigorous proof that the vertices obtained by his graphical construction must lie on an ellipse. It is easy to see from similarity of triangles how Hooke would have reached this conclusion [2, 340].

construction based on such impulses to determine the vertices of the orbit, as he had explained in the text of his manuscript, ‘‘then there would be no requirement that the vertex points lie on-orbit’’ [1, 176]. However, this statement ignores the fact that the affine transformation applies to the displacement lines as well as to the vertices on the circle [2, 340]. Moreover, even an approximate graphical or numerical construction of the orbit based on such impulses gives vertices that lie very closely on an ellipse [2, 339]. Near the end of his paper, Erlichson gives a proof that the displacements due to the impulses in Hooke’s graphical construction of the figure are proportional to the radial distance [1, 181], ignoring the description of this proof in my paper [2, 340]. Finally, Erlichson concedes that ‘‘it does not fundamentally matter that Hooke constructed his vertices first from geometrical knowledge [an affine transformation] and then went about moving from vertex to vertex with an inertial leg and a centripetal leg [linear force law]’’ [1, 183]. Yet this conclusion is not reflected in the bulk of Erlichson’s paper, where he claims that Hooke’s construction ‘‘has no general power’’ [1, 170], and that the affine transformation ‘‘is crucial for proving that Hooke’s points are geometrical and not dynamical’’ [1, 172].

Here and elsewhere, Erlichson does not make a distinction between the method by which Hooke’s drawing was carried out, presumably with a compass and ruler, and the dynamics which it illustrates according to the accompanying text. Hooke’s description corresponds precisely to Newton’s dynamics for a force which consists of impulses at periodic intervals of time. Erlichson argues that Hooke ‘‘somehow knew (or guessed)’’ from his construction the correct linear dependence of the magnitude of the impulses, thus possibly solving a direct problem in dynamics (given an orbit to determine the centripetal force) [1, 176]. However, Hooke explicitly considered the linear dependence of the force as *given*, as may be seen also by his remark elsewhere in the manuscript [6, 510]:

> Suppose the single attraction to be in the same proportion with the Distances from the center, the body moved by any degree of velocity with any Inclination to the Ray shall Describe an ellipse about the attracting point as its center.

While Hooke’s drawing implies that he had also obtained the solution of the direct problem by the affine transformation, the text is clear that this is not a problem which he had formulated or believed that he had solved.

At the end of his paper, Erlichson says that it ‘‘does not fundamentally matter how Hooke constructed his vertices,’’ even though this has been the main basis of his critique of my analysis. He then shifts to a new argument stating that what ‘‘does matter’’ is that Hooke ‘‘had depended heavily on symmetry’’ [1, 183]. In support of this new argument, he considers the case in which the direction of the initial velocity is not perpendicular to the radius vector, and concludes that in this case ‘‘the centripetal legs are no longer proportional to the distances to the center of the ellipse’’ [1, 178]. However, the illustration in his Fig. 6 [1, 178] shows that Erlichson reaches this conclusion because he has chosen an incorrect ellipse for the graphical construction of the displacement due to the central impulses. In fact, the vertices of the actual elliptical orbit can also be obtained by applying the geometrical construction described in Hooke's text. The correct ellipse will then have an axis tilted relative to that shown by Erlichson, and it can be related to a corresponding circular orbit by a more general affine transformation.

For other than a linear dependence of the force, the geometrical construction for impulses gives a polygonal orbit which approximates—but does not lie on—the correct continuum limit orbit. In my paper, I showed that for the inverse square force law with an initial scaled velocity comparable to that used by Hooke, the resulting polygonal orbit diverges from the elliptic orbit near the center of force [2, 344]. Yet Erlichson states in the abstract of his paper, and elsewhere in the text, that I claimed that "Hooke, on his own, had developed a quantitative theory of centripetal force" [1, 167]. On the contrary, what I demonstrated is that Hooke could not have extended his ideas successfully to "handle an elliptic orbit with the [inverse square] force center at the focus," which Erlichson raises as a question at the end of his paper [1, 183], without reference to the fact that it had been answered already in my paper [2, 344].

Concerning the relation of Hooke's work to Newton's, I quoted in my paper two letters, one from Flamsteed and another from Halley to Newton, to the effect that Hooke was "acquainted" with Newton's *De Motu* [2, 334], which was registered in the Royal Society by the end of 1684. In contrast, Erlichson presents only some conjectures as to why Hooke would have known or heard about the contents of Newton's manuscript. I also stated that "if Hooke had indeed seen the 1684 version of *De Motu*, he would have recognized that Newton had implemented geometrically his dynamical principle [for orbital motion] of compounding a tangential velocity with an impressed radial velocity due to a center of attraction" [2, 334], which Hooke had communicated to Newton in 1679. It is recognized by most Newtonian scholars that Newton considered this approach to general orbital motion only after his correspondence with Hooke, which subsequently led him to Proposition 1, according to his own account to Halley in 1686. This correspondence and other manuscripts give evidence that before 1679 Newton had analyzed orbital motion by a different method based on his concept of curvature [3].

In conclusion, I would like to point out that in the 1690s Newton considered a radical revision of the first edition of the *Principia*. According to a memorandum from David Gregory, written in July 1694, "he [Newton] deduces the computation of the centripetal force of a body tending to the focus of a conic section from that of a centripetal force tending to the center, and this again from that of a constant centripetal force tending to the center of a circle . . ." [4, 384]. While Newton's first deduction appeared in the second edition of the *Principia* as an alternate proof of Proposition 11 ("the same otherwise"), his second deduction was never included in any of the editions of the *Principia*. However, one of his private manuscripts [5, 574–577] reveals that this deduction for the radial dependence of the centripetal force tending to the center of an elliptic orbit is based precisely on the affine transformation of a circular orbit inferred from Hooke's 1685 manuscript. Thus, it was not Hooke, as Erlichson claims, but Newton who first formulated this as a direct problem, and solved it by applying the affine transformation some years after Hooke.

ACKNOWLEDGMENTS

I thank I. Bernard Cohen, J. Bruce Brackenridge, and Curtis Wilson for helpful comments and suggestions.

REFERENCES

1. Herman Erlichson, Hooke's September 1685 Ellipse Vertices Construction and Newton's Instantaneous Impulse Construction, *Historia Mathematica* **24** (1997), 167–184.
2. Michael Nauenberg, Hooke, Orbital Motion and Newton's *Principia, American Journal of Physics* **62** (1994), 331–350.
3. Michael Nauenberg, Newton's Early Computational Method for Dynamics, *Archive for History of Exact Sciences* **46** (1994), 221–252.
4. Isaac Newton, *The Correspondence of Isaac Newton*, ed. H. W. Turnbull, vol. 3, 1688–1694, Cambridge: Cambridge Univ. Press, 1961.
5. Isaac Newton, *The Mathematical Papers of Isaac Newton*, ed. D. T. Whiteside, vol. 6, Cambridge: Cambridge Univ. Press, 1974.
6. Trinity College Library, Cambridge, Ms. O.11a.1/16,f.5r; or Patri Pugliese, *The Scientific Achievement of Robert Hooke*, Ph.D. dissertation, Harvard University, 1982.

Michael Nauenberg
Department of Physics
University of California at Santa Cruz
Santa Cruz, California 95064