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DETERMINATION OF MASSES AND OTHER PROPERTIES OF EXTRASOLAR PLANETARY SYSTEMS WITH MORE THAN ONE PLANET

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ABSTRACT

Recent analysis of the Doppler-shift oscillations of the light from extrasolar planetary systems indicates that some of these systems have more than one large planet. In this case, it has been shown that the masses of these planets can be determined without the familiar ambiguity due to the unknown inclination angle of the plane of the orbit of the central star provided, however, that the star mass is known. A method is presented here that also determines a lower limit to the mass of this star from these observations. As an illustration, our method is applied to the Keck and Lick data for GJ 876.

Subject headings: methods: analytical — planetary systems

1. INTRODUCTION

During the past six years, a large number of extrasolar planetary systems have been discovered by observations of Doppler-shifted oscillations of the light emitted by the central star (Udry 2000). Recent analysis of the data has shown that some of these systems have more than one Jupiter-sized planet circulating the central star (Marcy et al. 2001a, 2001b). It is well known that when there is only a single planet, an ambiguity occurs in the determination of the mass of this planet when the inclination angle of the plane of the orbit is not known. For systems with more than one sizable planet, however, it turns out that this ambiguity can be removed when the gravitational interaction between these planets is important to the evolution of the planetary orbits, as has been pointed out by Laughlin & Chambers (2001). By the equivalence principle, this interaction is proportional to the mass of the planets, thus providing an additional dependence on these masses that is absent when there is only a single planet. In the case of multiple planets, only approximate analytic solutions of the gravitational equations of motion exist, and one must resort to numerical integrations to analyze the data. In this paper, we present a method based on such an integration to obtain the masses and other properties of the extrasolar planetary system. As an illustration, we apply our method to the Keck and Lick data for GJ 876 obtained by Marcy et al. (2001a). Some previous analyses of this data depended on the approximation that each of these planets is traveling on a Keplerian elliptical orbit either with constant orbital elements (Marcy et al. 2001a) or on variable orbital elements[1] (Laughlin & Chambers 2001), although these latter authors have also implemented an exact numerical integration of the equations of motion . While we have been able to verify their results with our method, we have found a second solution that differs significantly from theirs. The value of the reduced χ^2 of these two fits is of the order of 2.5–3, although the mean should be approximately unity, indicating that some fundamental physics in the analysis of the data has not been taken into account. Indeed, an additional source for velocity fluctuations in the light emitted by the central star may be due to the convective motion and turbulence of the chromosphere, which has been shown to be correlated to increase magnetic activity in some stars (Saar, Butler, & Marcy 1998; Saar & Fischer 2000). An estimate of the magnitude of these fluctuations can be obtained from the requirement that the reduced χ^2 be of the order of unity, as will be shown in § 3.

2. METHOD FOR ANALYSIS OF EXTRASOLAR PLANETARY DATA

Our starting point is to rescale the gravitational equations of motion by introducing a length scale l and timescale τ that satisfies the Kepler-Newton relation

$$\frac{l^3}{\tau^2} = Gm_s \ , \qquad (1)$$

where m_s is the mass of the central star. This mass is taken as one of the parameters in a least-squares fit to the data, while either l or τ can be chosen as another parameter. In these rescaled variables, the magnitude of the force per unit mass due to a planet with mass m_j^g at a distance r_j is

$$\frac{m_j^g}{m_s}\frac{1}{r_j^2} \ . \qquad (2)$$

For clarity, we have labeled this planetary mass with a superscript g to indicate that we are referring here to the *gravitational* masses. On the other hand, by momentum conservation, the velocity of the central star is related to the velocities v_j of n planets according to the relation

$$\boldsymbol{v}_s = -\sum_{j=1}^{j=n}\frac{m_j^i}{m_s}\boldsymbol{v}_j + \boldsymbol{v}_0 \ , \qquad (3)$$

which depends on the *inertial* masses m_j^i of the planets, which we label with the superscript i. Here, the velocity v_0 is proportional to the uniform velocity of the center of mass relative to the observer and becomes another parameter in our fit. By the equivalence principle, the gravitational and inertial masses are equal, $m_j^g = m_j^i = m_j$, which implies that

[1] When there is more than one planet, such an approximation is not unique. For example, one can locate the center of attraction and the corresponding focus of each of the ellipses either at the center of mass or at the position of the star, leading to somewhat different values for the orbital elements.

the same mass ratio m_j/m_s appears both in this kinematical relation and in the dynamical interaction between planets (eq. [2]). This is the key that opens a new way to obtain these mass ratios in extrasolar planetary systems with more than one planet.[2] According to our scaling (eq. [1]), the observed velocities are obtained by multiplying these velocities by a scale factor

$$V = \frac{l}{\tau} = \left(\frac{Gm_s}{\tau}\right)^{1/3} . \qquad (4)$$

The observed Doppler variations, however, also depend on the inclination angle i of the mean plane of the orbit of the star relative to the observer; therefore, the relevant scale for observations is the product $V \sin i$. Consequently, the independent parameters in a fit to the observations are the product $m_s \sin^3 i$ and the mass ratios m_j/m_s.

The additional parameters that are required to determine the evolution of the extrasolar system are those parameters that determine the initial conditions, e.g., the position and velocities of each of the planets at the start of the observations and the relative uniform velocity of the center of mass of the system. The initial velocity of the central star is then given by the conservation of momentum relation (eq. [3]). A useful approach currently in practice is to parameterize the orbital elements of the initial osculating Keplerian ellipse for each planet. It must be remembered, however, that when interplanetary perturbations are important, these orbital elements evolve in time. As will be shown below, some of these elements have periodic variations, while others change continuously. Hence, in general, these initial parameters do not describe the mean properties of the system when interplanetary perturbations are important.

The values of the parameters are obtained by a least-squares fit to the data obtained by numerically integrating the gravitational equations of motions. In particular, we consider the dependence of this fit as a function of our parameter $m_s \sin^3 i$. According to equations (3) and (4), when this parameter is increased above the value at which a minimum has been found, the planetary mass ratios m_j/m_s are expected to decrease inversely with the cube root of $m_s \sin^3 i$ in order to fit the observed velocity of the star. Hence, if the interplanetary interactions are important, then the χ^2 of the fit should increase; otherwise, it approaches a constant. Correspondingly, when this parameter is decreased, the χ^2 also increases because now the interplanetary interactions increase above their actual value. Since $\sin i \leq 1$, the value of the parameter $m_s \sin^3 i$ at the minimum χ^2 then gives a *lower limit* for the mass m_s. When this mass can also be obtained from the mass-luminosity relation, our method provides a novel check on the validity of this relation as well as on the self-consistency of the least-squares fit.

3. APPLICATION TO GJ 876

We illustrate our method by an application to the Keck and Lick data for the velocity modulations of GJ 876 obtained by Marcy et al. (2001a). These authors have fitted their data assuming that there are two large planets orbiting the center of mass on Keplerian ellipses that have *constant* orbital elements, and they found that the periods were in a nearly 2 : 1 ratio. The occurrence of this resonance, however, indicates that the interactions between these planets cannot be neglected, as was originally assumed. Recently, Laughlin & Chambers (2001) also fitted this data by integrating the equations of motion numerically, but we have found that their solution is not unique, and we present here another fit that gives planetary masses and other properties of the system that differ significantly from their results.

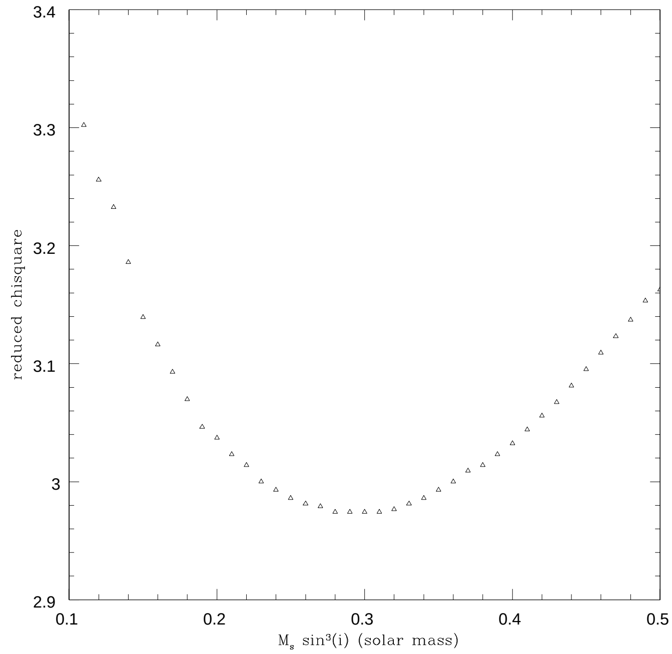

FIG. 1.—Eleven parameter least-squares fitS to the Keck data as a function of the parameter $m_s \sin^3$ i.

In Figure 1 we show the dependence of the reduced χ^2,[3] obtained by a least-squares fit to the Keck data, as a function of the parameter $m_s \sin^3 i$. The minimum in the parameter space was found by a simplex program. Under the assumption that the two planets, and consequently the central star, all move on the same plane, we have 11 parameters in our fit. These correspond to the two planetary masses, the eight parameters that determine the initial positions and velocities of the planets in a plane at the time of the first Keck observation, and the component of relative uniform velocity of the center of mass of the system v_0 along the line of sight. Alternatively, these parameters can be chosen to be the initial orbital elements of the osculating ellipses for the inner and outer planets. These elements are given in Table 1 with $v_0 = 77.7$ m s^{-1}.

We find that the minimum of the χ^2 lies at $m_s \sin^3 i \approx .3$ $M_\odot$, which is remarkably close to the value of the mass of the central star obtained by Marcy et al. (2001a) from the mass-luminosity relation, $m_s = 0.32(0.05)$ $M_\odot$. The value of our reduced χ^2 of about 3 at the minimum, however, would seem to indicate a very poor fit to the Keck data. If the observational errors have not been underestimated, then

[2] In the future, with increasing data, these two mass ratios could be taken as independent parameters in a fit to provide a new test of the equivalence principle for extrasolar planetary systems.

[3] We use here the conventional definition of reduced χ^2, which corresponds to the square of the quantity called "reduced χ^2" in papers on GJ 876 listed in the references.

TABLE 1

INITIAL OSCULATING ORBITAL ELEMENTS AND PLANETARY MASS RATIOS FOR $m_s \sin^3 i = 0.32\ M_\odot$ AND $v_0 = 77.7$ m s^{-1}

Parameter	Inner	Outer
Period (days)	29.24	59.90
Mass ratios (M_j/M_s)	0.00180	0.00557
Major axis (AU)	0.127	0.205
Eccentricity	0.228	0.00185
Pericenter (deg)	−60.88	29.39
Eccentricity anomaly	269.83	43.85

this must be due to physical sources for fluctuations of the Doppler-shifted light that have not been taken into account yet. It is known that velocity fluctuations can occur due to the convective motion and turbulence in the chromosphere of a star (Saar, Butler, & Marcy 1998; Saar & Fischer 2000). Therefore, the mean square of these fluctuation should be added to the square of the observational errors to obtain the actual reduced χ^2. Assuming that the reduced χ^2 for our fit should be of the order of unity, we obtain an estimate for the magnitude of these fluctuation of 4–6 m s^{-1}, comparable to the observational errors in the Keck data, which is 3–5 m s^{-1}. These fluctuations may also limit the accuracy with which the gravitational effects due to planets on the motion of the star can be observed.

An independent confirmation of our fit is obtained by applying the parameters from the Keck data to evaluate the reduced χ^2 for the Lick data points. This fit, represented in Figure 2, has only a single parameter corresponding to the relative zero-point velocity between the Keck and Lick telescope systems. It can be seen that the reduced χ^2 also increases rapidly for $m_s \sin^3 i$ below 0.3, but above this value it now approaches a constant. Since the statistical errors in the Lick data are about 3 times larger than those in the Keck data, evidently this fit is not very sensitive to interplanetary perturbations of the order of, or smaller than, its actual value. For $m_s = 0.32\ M_\odot$, these results imply that $\sin i \approx 1$. In contrast, Laughlin & Chambers (2001) found that $\sin i = 0.55$ from a fit to the Keck data, and $\sin i = 0.78$ for a corresponding fit to the combined Keck and Lick data.

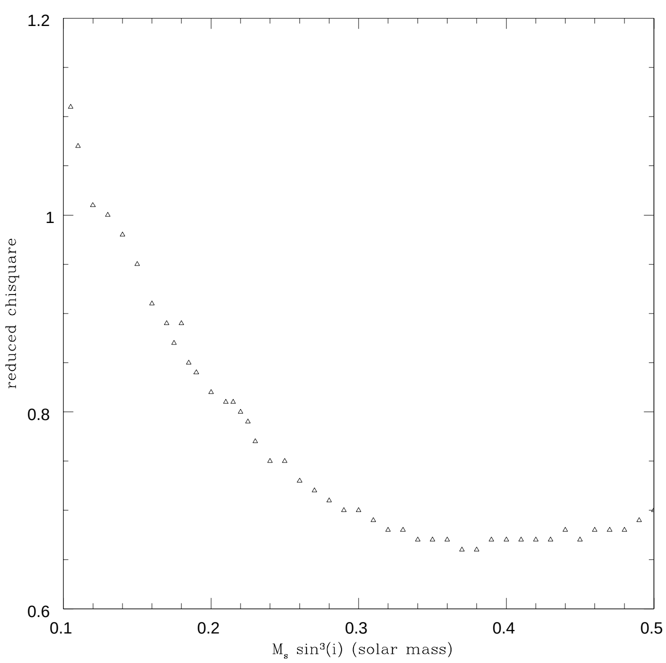

FIG. 2.—Single-parameter least-squares fit to the Lick data based on the best fit to the Keck data.

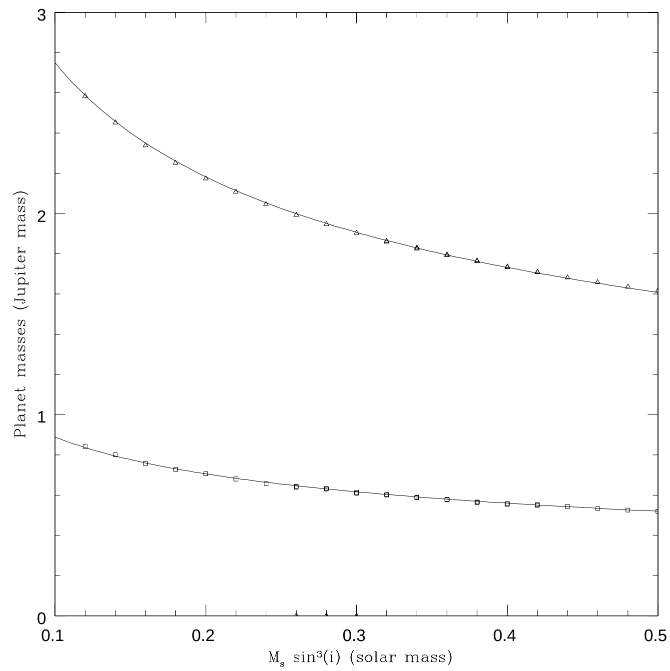

FIG. 3.—Masses of the inner and outer planet as a function of the parameter $m_s \sin^3 i$.

In Figure 3, we show the dependence of the resulting masses of the two planets as a function of $m_s \sin^3 i$, where we have chosen for the planetary mass scale the observed magnitude of the central star. This dependence is well fitted by the relation

$$m_j = m_{j0}\left(\frac{0.32}{m_s \sin^3 i}\right)^{1/3}, \qquad (5)$$

where m_{j0} is a constant that is normalized at $m_s(\sin i) = 0.32$. From the minimum of our χ^2, we obtain $m_1 = 0.6$ and $m_2 = 1.9$ in units of Jupiter mass, which is comparable to the results of Marcy et al. (2001a) but in disagreement with two different values for these planetary masses obtained by Laughlin & Chambers (2001), who found substantially larger values due to their smaller values of $\sin i$.

In Figure 4, we show our results for the velocity modulations of the central star with the Keck and Lick data points superposed as squares and pentagons, respectively. A blowup of this plot is represented in Figure 5 that exhibits the characteristic midperiod oscillations that are signatures of the inner planet. The zero point in the timescale has been chosen at the first Keck data point, and we have extended our calculation to 4000 days to show the occurrence of a long-term periodic modulation of about 3200 days of the rapid oscillations of about 60 days. These rapid oscillations are associated with the mean period of the outer planet, while the long-term modulation is associated with the nearly uniform rotation of the major axis of the orbit of the inner

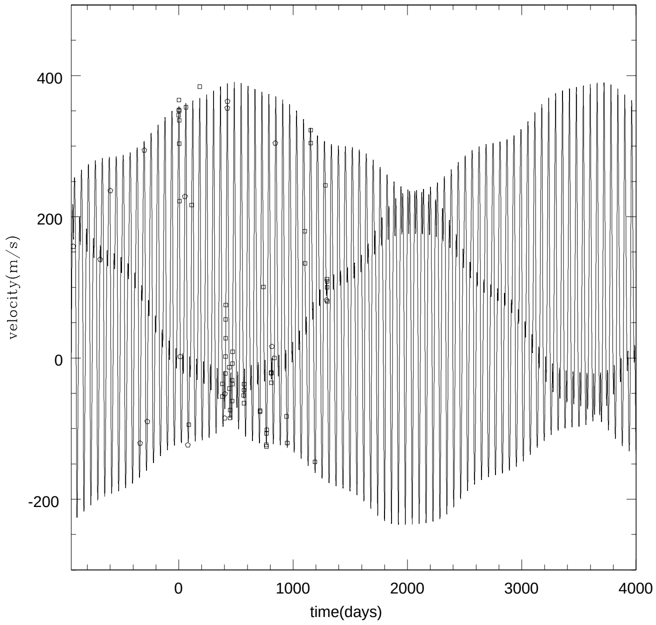

FIG. 4.—Calculated velocity oscillations of the central star in GJ 876. The Lick and Keck data are represented as pentagons and squares, respectively.

planet, as will be demonstrated in § 4. The apparent symmetry of these oscillations on reflection about an axis at $v_0 \approx 78$ m s^{-1} (eq. [3]), shifted by half the long modulation period, is due to the fact that this major axis rotates through 180° during this half-period. In addition, we see that the envelope also exhibits a medium-length modulation of about 660 days, which, as we shall see, corresponds to the mean period of oscillations of the major axis of the planets and the eccentricity of the inner planet and is associated with the oscillations from an exact 2 : 1 resonance that will be discussed later on.

4. PROPERTIES OF THE TWO PLANETARY ORBITS OF GJ 876

From the numerical solution of the equations of motion, one can directly determine the properties of the planetary orbits and the orbit of the central star. A typical example of the planetary orbits during a single period of the outer planet is represented in Figure 6. During this time interval, the inner planet turns approximately twice around the central star, traveling along two orbits that are slightly displaced relative to each other due to the interplanetary perturbations and the motion of the central star. At the time that the inner planet first reaches its pericenter (*triangle*), the outer planet (*triangle*) is nearly aligned with the central star, which is represented on this scale only as a dot. Approximately half a period later, the inner planet has completed a revolution and is again at pericenter (*square*), while the outer planet (*square*) comes again into conjunction with the central star but at the opposite side of the initial location on its orbit. Then, after the inner planet completes a second revolution, the outer planet also returns to conjunction with the star. This is the characteristic signature of a dynamical 2 : 1 resonance. The resulting motion of the central star with its position at 10 equal time intervals is represented in Figure 7. When the inner and outer planets are aligned on the same side of the star, we see that the motion of the star is accelerated, while when these planets are in conjunction on opposite sides of the star, the motion is slowed down, and a dimple appears in the orbit. These planets exchange angular momentum in an oscillatory fashion, as represented in Figure 8. The orbital periods of the planets are obtained by evaluating the time elapsed between successive passages at nearest distance, r_p, or largest distance, r_a, from the center of mass of the extrasolar planetary system. While the ratio of the mean periods of the outer and inner planets is approximately 2, as obtained previously by Marcy et al. (2001a),we

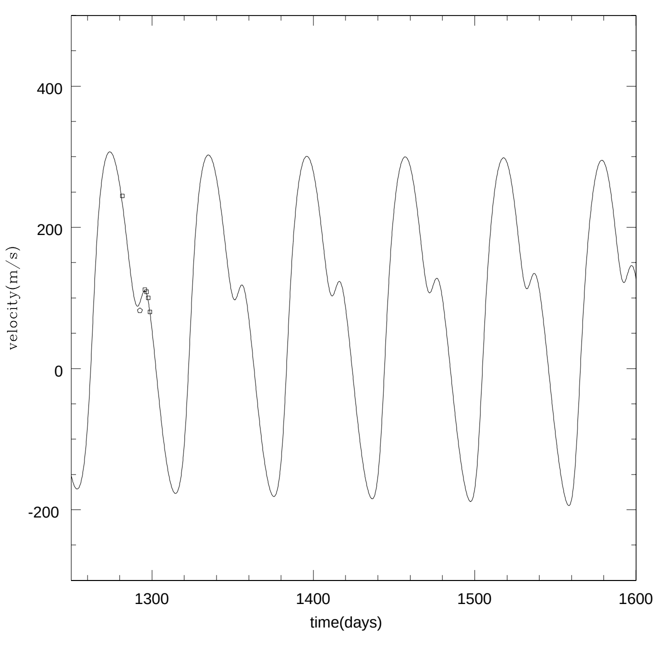

FIG. 5.—Blowup of Fig. 4 for the velocity oscillations of the central star in GJ 876, which exhibits the variations at half-period characteristic of the inner planet.

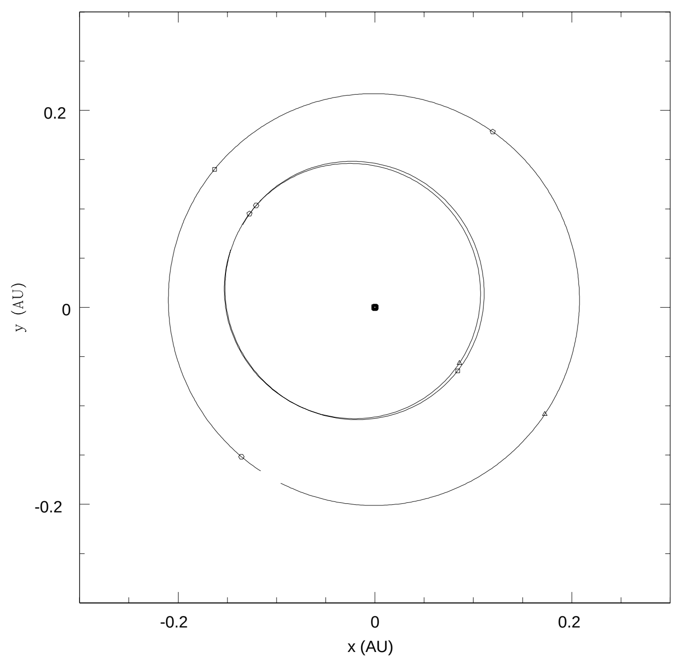

FIG. 6.—Orbits of the outer and inner planet showing the resonance alignment of these planets at resonance with the central star (*triangles and squares*) and their location when the outer planet is at quadrature (*pentagons*).

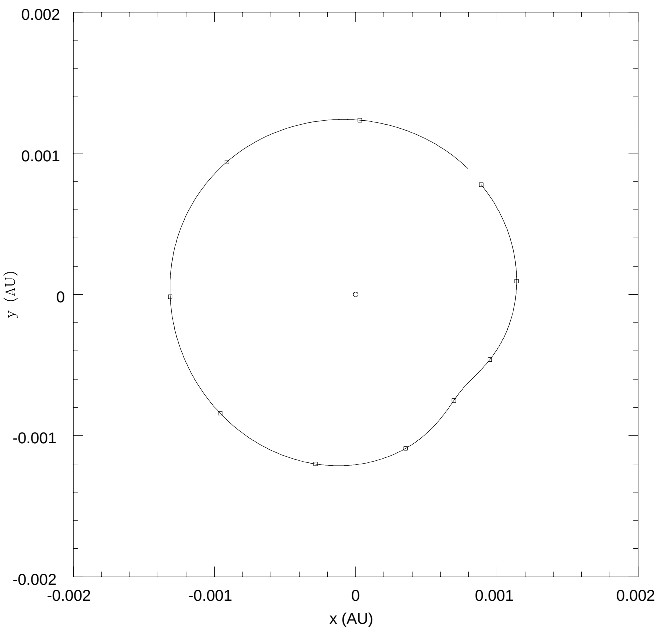

FIG. 7.—Orbits of the central star showing its position at 10 equal time intervals during a single period of the outer planet.

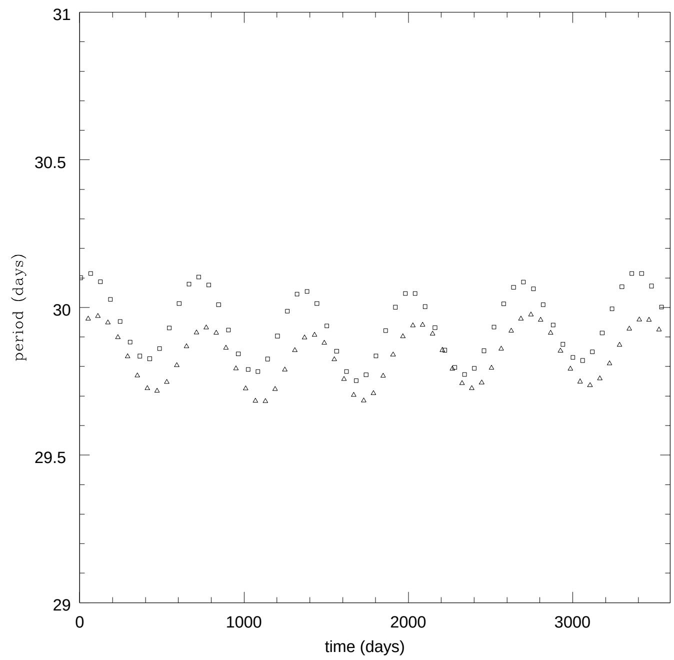

FIG. 9.—Period oscillations for two successive orbits of the inner planet.

find that these periods are not constants, as had been assumed previously. As we have seen during a single period of the outer planet, the inner planet travels around two slightly different orbits with periods that each have a periodic oscillation of 660 days, as represented in Figure 9. In contrast, the variations of the period of the outer planet, represented in Figure 10, exhibits a longer term periodicity of 3200 days, which is associated with the rotation period of the axis of the orbit of the inner planet. This period differs somewhat when it is defined relative to r_p or r_a.

Correspondingly, we can also define an effective major axis $a = (1/2)(r_a + r_p)$ and eccentricity $e = (r_a - r_p)/(r_a + r_p)$ for each planetary period. The results for the inner planet are represented in Figure 11 and Figure 12, which again exhibit periodicity around two orbits. While the major axis of the outer planet exhibits similar oscillations with a period of 660 days (Fig. 13), its eccentricity has a much longer periodicity of about 3200 days (Fig. 14), associated with the rotation period of the major axis of the inner planet, which give rise to the long-term mod-

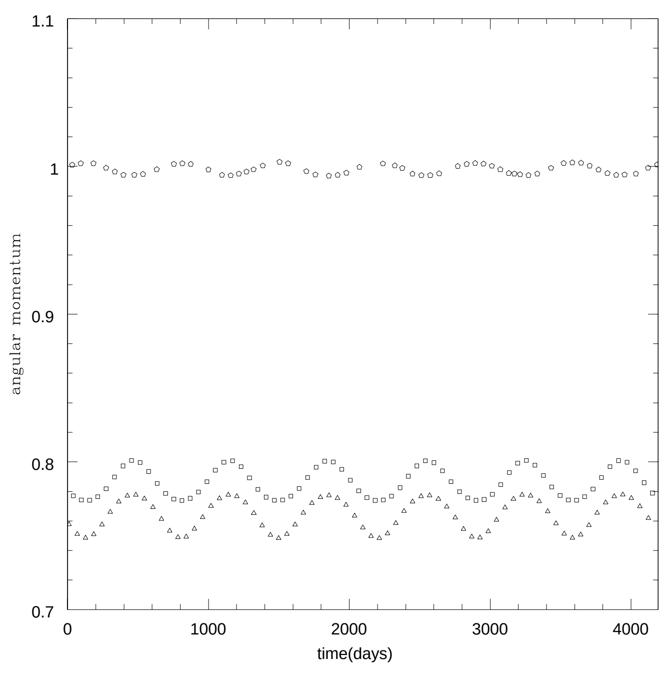

FIG. 8.—Angular momentum of the inner planet (*triangles and squares*) and outer planet (*pentagons*) at pericenter.

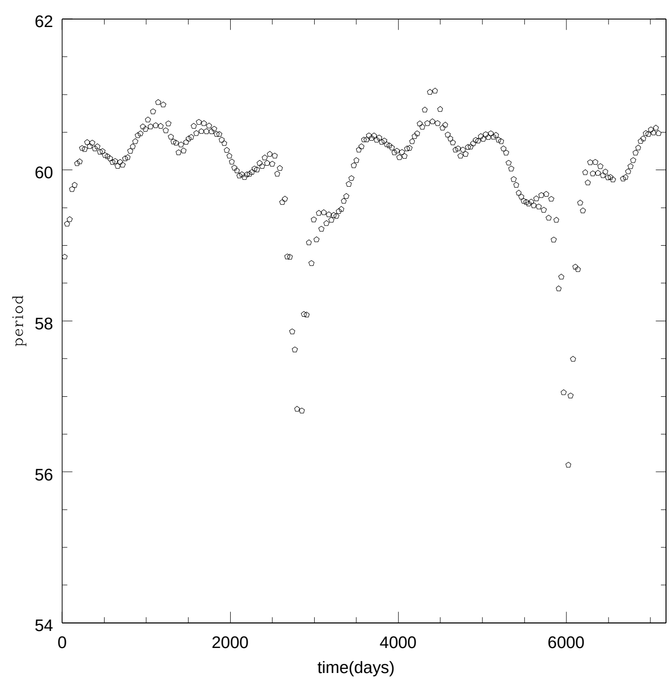

FIG. 10.—Oscillations for the period of the outer planet

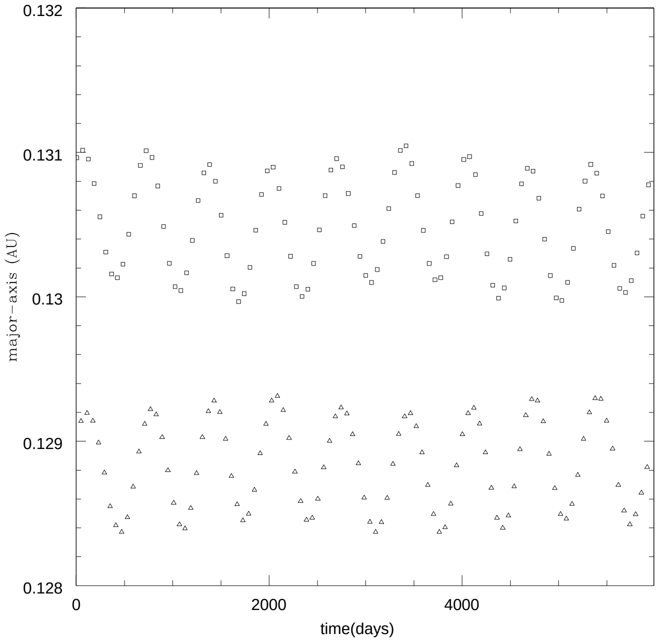

FIG. 11.—Major-axis oscillations of two successive orbits of the inner planet.

ulation period represented in Figure 4. This is represented in Figure 15, where we see that, apart from small 660 day oscillations, the longitude of the inner planet at pericenter rotates with nearly uniform angular velocity, and it is always approximately aligned at that time with the longitude of the central star and the outer planet in accordance with a dynamical 2 : 1 resonance. In Figure 16, we show the corresponding rotation of the pericenter of the outer planet, which exhibits rapid changes when the eccentricity decreases rapidly (see Fig. 14).

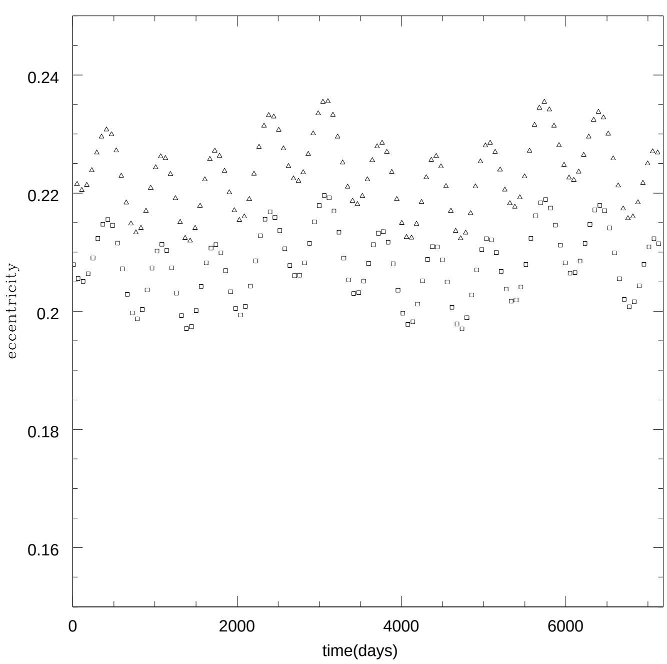

FIG. 12.—Eccentricity oscillations of two successive orbits of the inner planet.

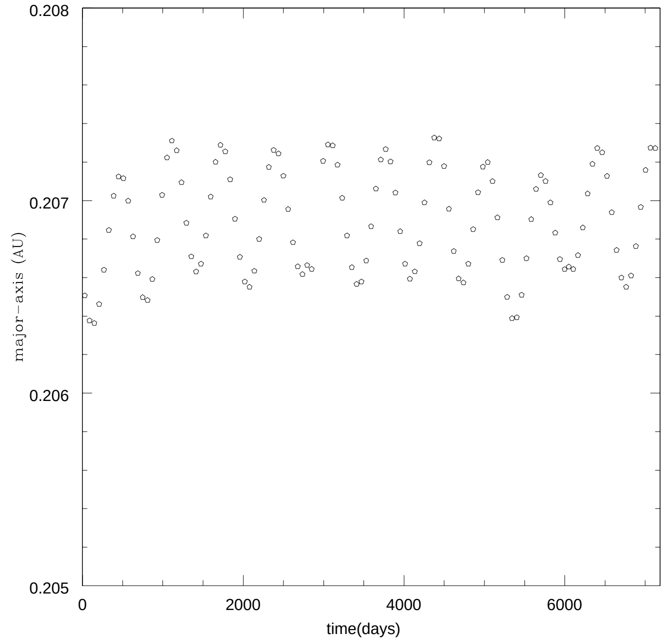

FIG. 13.—Major-axis oscillations of the outer planet

5. THE 2 : 1 RESONANCE IN GJ 876

The occurrence of a 2 : 1 resonance in GJ 876 keeps the inner and outer planets from getting too close to each other, which can cause large perturbations that would disrupt the system. In the present case, when the eccentricity of the outer planet is very small compared to that of the inner planet, we require that each time the inner planet completes two turns around the center of mass of the system and returns to pericenter, the outer planet turns around only once and then becomes aligned with the inner planet, the central star, and the center of mass of the system. That this condition is in fact approximately satisfied by GJ 876 can be

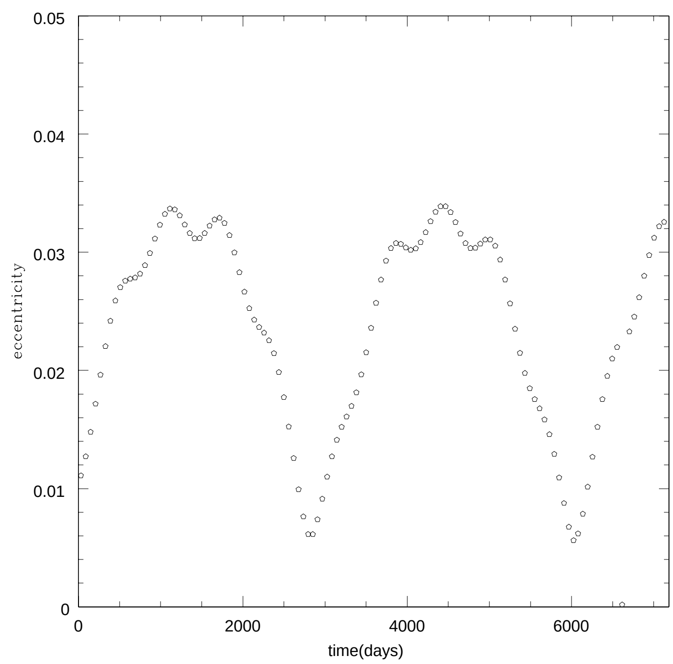

FIG. 14.—Eccentricity variations of the outer planet

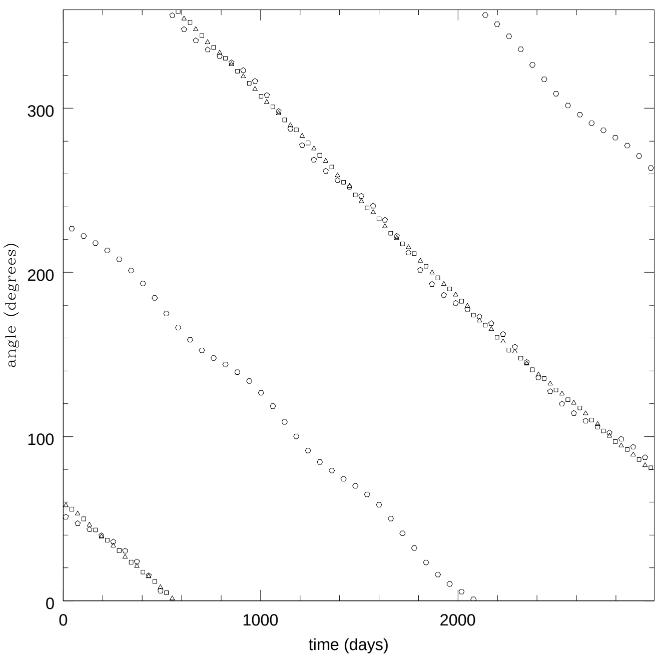

FIG. 15.—Longitude of the inner planet at pericenter for successive orbits (*triangles and squares*) and the corresponding longitudes of the outer planet (*pentagons and hexagons*).

seen in Figure 6, which shows typical orbits of the inner and outer planets for two revolutions of the inner planet and one revolution of the outer planet. Analytically, this condition for a 2 : 1 resonance can be written in the form

$$\int_0^{2P_i} \omega_o(t)dt = 2\pi - \int_0^{2P_i} [\Omega_i(t) - \Omega_o(t)]dt \; , \qquad (6)$$

where $\omega_o(t)$ is the angular velocity of the outer planet, $\Omega_i(t)$ and $\Omega_o(t)$ are the rotation rates of the major axis of the inner and outer planets described previously, and P_i is the mean period of two successive rotations of the inner planet. We assume here that the direction of rotation of the two planets is the same, but in the opposite direction to that of the major axis. For $2P_i \approx P_o$ where P_o is the mean period of the outer planet, we obtain

$$(P_o - 2P_i)\omega_o(P_o) = \int_0^{2P_i} [\Omega_i(t) - \Omega_o(t)]dt \; . \qquad (7)$$

From this expression, one can obtain the conventional relation for a 2 : 1 resonance (Murray & Dermott 1999) by replacing the rotation rates by their mean values and also by assuming that the rotation rate of the outer planet is uniform or that $w_o(P_o) = 2\pi/P_o$. This latter condition, however, is not valid in general. The deviation from exact resonance gives rise to librations that have a mean period of 660 days, while the rotation of the axis of the orbit of the planets has a mean period of 3200 days. These librations are the origin of the oscillations in the orbital elements of the planetary orbits represented in Figures 9–11 and Figures 12–13, and can also be seen directly from the data by observing the oscillations in the boundaries of the fit to the Keck data, as represented in Figure 4. Sinusoidal librations were introduced by Laughlin & Chambers (2001) in an analytic model to a fit to the Keck and the Lick data that included oscillations in the major axis of the orbit planets, but their model neglected the corresponding oscillations in the eccentricities and periods and the occurrence of a rapid oscillation with mean period P_i between two effective elliptic orbits that characterize the motion of the inner planet. The identification of two long-term periodicities indicates that the 2 : 1 resonance motion in GJ 876 is quasi-periodic.

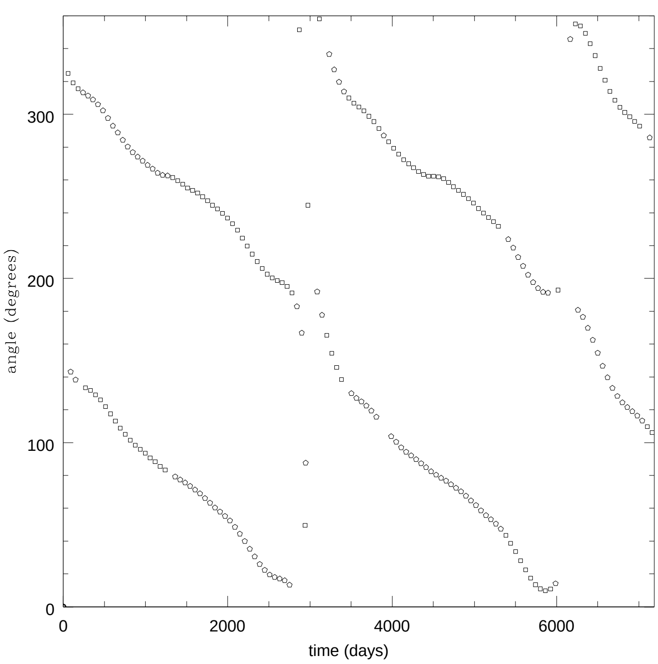

FIG. 16.—Longitude at maximum and minimum distance from the center of the outer planet

6. CONCLUDING REMARKS

The essential new feature in our analysis is the scale transformation (eqs. [1] and [4]), which shows that the mass m_s of the central star and the inclination angle i of the mean plane of its orbit are not independent parameters in a least-squares fit to the observational data. Instead, these two parameters appear as a single variable in the form of a product $m_s \sin^3 \, i$ that provides a lower limit to the mass of the star directly from observations of Doppler-shifted oscillations of the emitted light. We have emphasized that the scaled gravitational interactions depend only on the ratios m_j/m_s of the planetary masses to the mass of the central star, which is the reason why these ratios can be determined independent of a knowledge of the inclination angle i. When the effects of interplanetary interactions are important, the initial orbital elements of the osculating ellipses for the planetary orbits, which are commonly introduced in the analysis of extrasolar planetary systems, do not correspond to mean properties of these systems. We have shown that for a 2 : 1 resonance, the motion of the inner planet is actually characterized by a continuous switching back and forth between two elliptical orbits during a single period of the outer planet, as can be seen in Figures 8–11. For the case of GJ 876, the eccentricity for the outer planet oscillates between a minimum value of 0.006 and a maximum value of 0.034 (Fig. 14), while its initial value is 0.0018. The orientation of the major axis is not fixed but rotates nearly uniformly in the case of the inner planet with a period of 3200 days (see Fig. 15), while for the outer planet it exhibits a somewhat

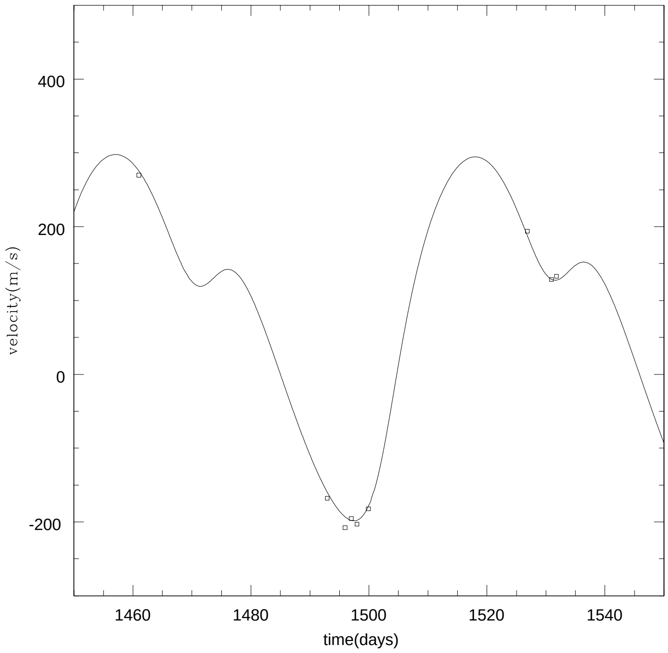

FIG. 17.—Least-squares fit to latest Keck data point (*squares*)

more complicated motion, represented in Figure 16. The rapid variations that appear here around $t = 3000$ days occur near the minimum of the eccentricity of the outer plane and are associated with the degeneracy for the major axis in the limit of a circular orbit. It is important to check the longtime stability of any numerical solution of the equations of motion, as has been pointed out by Laughlin & Chambers (2001), and this has been verified for the solution presented here (i.e., G. Laughlin 2001, private communication). In our analysis we neglected the difference between the inclination angles of the mean planes of the planetary orbits and possible effects due to tidal distortions of the planets. The differences between our fit to the data for GJ 876 and the corresponding fit of Laughlin & Chambers (2001) indicates that at present it is not yet possible to determine uniquely the properties of this system, but we expect that this ambiguity will be resolved by additional data from future observations.

Note added in manuscript.—After the completion of this work, G. W. Marcy et al. (2001, private communication) released nine new data points and revised the values for the velocities of the central star published previously. Our fit to this data gives a similar reduced χ^2 as before, and the result showing the last nine data points is given in Figure 17. In addition, I have been informed that two additional papers analyzing GJ 876 have appeared recently, one by Rivera & Lissauer (2001), and the other by Lee & Peale (2002).

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