

THE RECEPTION OF NEWTON'S *PRINCIPIA*

ABSTRACT. Newton's *Principia*, when it appeared in 1687, was received with the greatest admiration, not only by the foremost mathematicians and astronomers in Europe, but also by philosophers like Voltaire and Locke and by members of the educated public. In this account I describe some of the controversies that it provoked, and the impact it had during the next century on the development of celestial mechanics, and the theory of gravitation.

INTRODUCTION

The first edition of Newton's Mathematical Principles of Natural Philosophy, commonly known as the Principia, appeared in 1687, transforming our understanding of celestial mechanics and gravitational theory. In his magisterial book, Newton gave a physical description and mathematical proof for the origin of Kepler's three laws for planetary motion. These laws were empirical laws based on the careful observations of the Danish astronomer Tycho Brahe, which summarized most of the astronomical knowledge about planetary motion that was known at the time. To accomplish this feat, Newton assumed the validity of the principle of inertia which he formulated in the Principia as the first law of motion¹, and he introduced two fundamental concepts: that the main attractive gravitational force acting on a planet is a central force directed towards the sun, and that the magnitude of this force is proportional to the planet's acceleration². From Kepler's third law³, and the approximation that planets move with uniform motion in a circular orbit, Newton then deduced that this gravitational force varied inversely with the square of the radial distance of a planet from the sun. In addition, Newton proposed that the magnitude of the gravitational force between any two bodies was proportional to the product of their *inertial* masses, and by applying Kepler's third law to the motion of the moon and the satellites of Jupiter and Saturn, he determined the masses of these planets and the mass of the earth

¹The principle of inertia was formulated in the form:

Every body perseveres in its state of being at rest or of moving uniformly straight forward except insofar as it is compelled to change its state by forces impressed (Cohen 1997, 416).

²These concepts had been formulated also by Robert Hooke who discussed his ideas with Newton in a lengthy correspondence in 1679 (Turnbull 1960. 297-314) (Nauenberg 1994) (Nauenberg 2005).

³The square of the period of the planet is proportional to the cube of its distance from the sun.

relative to the mass of the Sun ⁴. In his essay *La cause de la pesanteur*, the great Dutch scientist Christiaan Huygens remarked that he was very pleased to read how Newton, by

supposing the distance from the earth to the sun to be known, had been able to compute the gravity that the inhabitants of Jupiter and Saturn would feel compared with what we feel here on earth, and what its measure would be on the Sun (Cohen 1997, 219).

The importance of these developments was appreciated not only by astronomers and mathematicians who read the *Principia*, but also by philosophers and by the educated public. The French philosopher, François Marie Voltaire encapsulated this recognition with a succinct comment,

Avant Kepler tous les hommes toient aveugles, Kepler fut borgne, et Newton a eu deux yeux ⁵(Besterman 1968, 83)

and shortly after Newton's death the English poet Alexander Pope wrote

Nature, and Nature's Laws lay hid by night
God said, let Newton be! and all was light.

As the reputation of the *Principia* grew, even people who had little or no mathematical ability attempted to understand its content. The English philosopher John Locke, who was in exile in Holland, went to see Huygens who assured him of the validity of Newton's propositions. He was able to follow its conclusions, and later befriended Newton, referring to him "as the incomparable Mr. Newton" in the preface of his essay *Concerning Human Understanding* (Locke 2001,13). While in exile in England, Voltaire became acquainted with Newton's work, and after his return to France he wrote the *Eléments de la Philosophie de Neuton* which popularized Newton's ideas in France. In this enterprise he was fortunate to have the collaboration of a gifted companion, Gabrielle Le Tonnelier de Breteuil, better known as the Marquise du Châtelet, who translated the

⁴This result was found among some early papers by Newton (Herivel 1965, 196). Newton and, independently, Christiaan Huygens, had deduced that a body moving with uniform velocity v in a circular orbit of radius r , has a radial acceleration a towards the center of the orbit given by $a = v^2/r$. Since $v = 2\pi r/T$, where T is the period, he obtained $a = 4\pi^2 r/T^2$. Substituting for T Kepler's third law for planetary motion, $T^2 = cr^3$, where c is a constant, gives $a = (4\pi^2/c)(1/r^2)$. In this way Newton found that the acceleration a depends inversely on the square of the radial distance r between a planet and the sun.

In the *Principia*, Newton proposed that the same relations apply also to the motion of the satellites around a planet. According to his principle of universal gravitation, a is proportional to the mass M of the center of force, and therefore $a = GM/r^2$, where G is a universal constant, now called Newton's constant. Hence $M = 4\pi^2/Gc$, where c is Kepler's constant, and by determining the value of c for the satellites of Jupiter, Saturn and the earth, Newton obtained the ratio of the mass of each of these planets relative to the mass of the sun, given in Prop. 8, Book 3 of the *Principia*.

⁵Before Kepler all men were blind; Kepler was one-eyed, and Newton had two eyes (Feingold 2004,99)

Principia into French. Francesco Algarotti, who was in communication with Voltaire, published his *Newtonianismo per le dame* which became fashionable in Italy (Feingold 2004).

Initially, there was considerable reluctance to accept Newton's general principles, particularly because an action at a distance was generally regarded as due to occult forces, in contrast to contact forces. According to Descartes, gravitational forces were due to vortices of unobserved celestial dust, and this explanation had been accepted by most Continental astronomers. At the end of Book 2 of the *Principia*, Newton gave a proof that Cartesian vortices were incompatible with Kepler's second and third laws for planetary motion, but his proof was based on somewhat arbitrary assumptions about the frictional properties of these vortices, and in an essay, 'Nouvelles pensée sur le système de M. Descartes', the Swiss mathematician Johann Bernoulli gave several objections to this proof (Aiton 1995, 17). In his *Discourse sur les différentes figures des astres*, Pierre-Louis Moreau de Maupertuis openly defended Newton's views, pointing out its predictive power, and remarked that Cartesian impulsion was no more intelligible than Newtonian attraction (Aiton 1995, 19), but that universal gravitation was "metaphysically viable and mathematically superior" to vortices as an explanation of celestial mechanics (Feingold 2004, 98). In a remarkable tour de force, Newton had applied his gravitational theory to determine the shape of the earth, and he found that the centrifugal force due to the daily rotation about its axis deformed the earth into an oblate spheroid flattened at the poles⁶. This prediction was contrary to the observations of the Italian astronomer Gian Domenico Cassini, who had joined the French Academy of Sciences, and his son Jacques Cassini. They had obtained faulty geodetic

⁶Assuming that the earth can be regarded as a rotating fluid held together by its own attractive gravitational forces, in Prop. 19, Book 3, Newton gave a proof that the shape of the earth is an oblate spheroid corresponding to an ellipse of revolution about its short axis. He calculated the ellipticity $\epsilon = a/b - 1$, where a and b are the major and minor axis of the ellipse, by the requirement that two channels of water to the center of the earth, one from the pole and another from the equator would be in pressure equilibrium at the center. Remarkably, in his calculation Newton also took into account the variation of the gravitational force inside the earth due to the shape distortion which he discussed in Prop. 91, Cor. 3, Book 1 (For a modern discussion see (Whiteside 1974, 225-226) and (Chandrasekhar 1995, 313-317) . Newton obtained for the ellipticity, $\epsilon = (5.04/4)\delta$, where δ , the ratio of centrifugal acceleration to the acceleration of gravity g , is $\delta = (4\pi^2 r_e)/(gT^2)$, r_e is the mean radius of the earth and T is the period of rotation (one sidereal day). This gives $\delta = 1/289$, and Newton found that $\epsilon = 1/229$ and announced that the distance to the center of the earth at the equator exceeds the value at the poles by $\epsilon r_e = 17$ miles. The present observed value is 13 miles, because the actual density of the earth is not homogeneous. A similar calculation was carried out by Huygens who included, however, only the effect of the centrifugal forces, because he did not accept Newton's principle of universal gravitation. Hence, Huygens obtained $\epsilon = (1/2)\delta = 1/578$. Newton's result was first derived by Clairaut, who showed that the relation $\epsilon = (5/4)\delta$ is correct to first order in ϵ (Todhunter 1962, 204).

measurements, indicating that the earth is a prolate spheroid ⁷. To resolve this conflict, Maupertuis together with the French mathematician Alexis-Claude Clairaut led a scientific expedition commissioned by the French Academy of Sciences that left for Lapland on April 20, 1736 to measure the length of a degree of the meridian at that latitude, in order to compare it with the corresponding length at the latitude of Paris. Maupertuis became famous for confirming Newton's prediction, and Voltaire called him the 'aplatisseur du monde et de Cassini', remarking sarcastically that

Vous avez confirmé dans des lieux pleins d'ennui
Ce que Newton connut sans sortir de chez lui ⁸

Another expedition, headed by La Condamine, Bouguer and Godin, also members of the French Academy of Sciences, went about a year earlier to Peru to measure a corresponding arc of the meridian near the equator. But they ran into considerable difficulties and delays due to personal animosities between the leaders of the expedition, and only ten years later, were they able to report their results which were consistent with the conclusions of Maupertuis' expedition (Todhunter 1962). Subsequently, the problem of evaluating theoretically the shape of the earth became the subject of intense efforts by Continental mathematicians who studied Newton's *Principia*, and its difficulty spurred major advances in mathematical physics.

In his *Lettres philosophiques*, Voltaire reported these controversies with his characteristic wit,

For your Cartesians everything is moved by an impulsion you don't really understand, while for Mr. Newton it is by gravitation, the cause of which is hardly better known. In Paris you see the earth shaped like a melon, in London it is flattened on two sides (Voltaire 1734)

The contrast between the methods of Descartes and Newton were neatly contrasted by Bernard le Bovier Fontenelle, the secretary of the French Academy of Sciences, who wrote in his *Eloge de Newton*,

Descartes proceeds from what he clearly understands to find the cause of what he sees, whereas Newton proceeds from what he sees to find the cause, whether it be clear or obscure (Fontenelle 1728)

Newton, however, left open the question of the origin of the gravitational force. During a correspondence with the Reverend Richard Bentley, he made his reservations clear,

It is unconcievable that inanimate brute matter should (without the mediation of something else which is not material) operate and affect other

⁷For example, Jacques Cassini found that the length of a degree of longitude in the parallel of St. Malo, France, is 36,670 *toises*, but on the supposition of a spherical earth it should be 37,707 *toises* (Todhunter 1962, 111) (The length of a *toise* can be obtained from Newton's remark in Prop. 19, book 3, that 367,196 *London* feet, the mean radius of the earth obtained by a Mr. Norwood, is equal 57,300 *Parisian* *toises*).

⁸You have confirmed in these difficult locations what Mr. Newton knew without leaving his home.(Florian 1934, 664)

matter without mutual contact; as it must if gravitation in the sense of Epicurus be essential and inherent in it ... That gravity ... may act at a distance through a vacuum ... is to me so great an absurdity that I believe no man who has in philosophical matters any competent faculty of thinking can ever fall into it. Gravity must be caused by an agent acting constantly according to certain laws, but whether this agent is material or immaterial is a question I have left to the consideration of my readers (Westfall 1995, 505).

This view led to accusations by some readers of the Principia that Newton had left the physics out of his mathematics.

Kepler had shown that the planets travel along elliptical orbits with the sun located at one of the foci ⁹, moving with a non-uniform velocity satisfying his second law or area law¹⁰. To account for such an orbit, Newton had to extend the rigorous geometrical methods developed by Greek mathematicians to encompass the limit of ratios and sums of vanishing quantities (Nauenberg 1998). In the Principia such quantities were represented by lines and arcs of curves of arbitrarily small length, a procedure that had been introduced by Apollonius, and applied by Ptolemy for calculations in his geocentric model of celestial motion, and by Archimedes for calculations of lengths and areas encompassed by curves. In the 17-century, this procedure was developed further by several mathematicians including in particular Renè Descartes, whose work Newton had studied carefully ¹¹ (Whiteside 1967). Since motion occurs with the passage of time, it was necessary for Newton to express time as a geometrical variable, but this was a major stumbling block (Nauenberg 1994a). It was only after a lengthy correspondence with Robert Hooke (Turnbull 1960, 297-314) (Nauenberg 1994b), that Newton was able to give a proof of the validity of Kepler's area law for any central force (Nauenberg 2003). Newton recalled that

In the year 1679 in answer to a letter from Dr. Hook ... I found now that whatsoever was the law of the forces which kept the Planets in their Orbs, the area described by a radius from them to the Sun would be proportional to the times in which they were described. And by the help of these two

⁹This subtlety was not always appreciated by the general public. For example, the Bank of England issued a two pound note, now retracted, showing incorrectly a figure from Newton's Principia, with the sun at the center of the ellipse.

¹⁰In the Principia, Book I, Prop. 1, Newton gave the following formulation of the area law:

The areas which bodies made to move in orbits describe by radii drawn from an unmoving center of forces lie in unmoving planes and are proportional to the times (Cohen 1999).

¹¹Newton studied Frans van Schooten's second edition (1659) of his translation of Descartes *Geometrie* from French into Latin, with appended tracts by Hudde, Heurat and de Witt. This translation was crucial to Newton's education because he could not read French.

propositions I found that their Orbs would be such ellipses as Kepler had described... (Lohne 1960)

Thus, Newton was able to geometrize the passage of time by the change of an area - a concept without which writing the Principia would not have been possible. He emphasized its importance by starting the Principia with a mathematical proof of the generalization of Kepler's second law described in Prop. 1, Book 1. (Brackenridge 1995) (Nauenberg 2003).

The style of the Principia followed the mathematical format of the Horologium Oscillatorium (Huygens 1673) (Nauenberg 1998) by Christiaan Huygens, who was the most prominent scientist in the Continent during the later part of the 17-th century. In 1673, when Newton received a copy of Huygens' book from Henry Oldenburg, he promptly responded that

I have viewed it with great satisfaction finding it full of very subtile and useful speculations very worthy of the Author. I am glad, we are to expect another discourse of the Vis Centrifuga [centrifugal force] which speculation may prove of good use in natural Philosophy and Astronomy, as well a Mechanicks. (Huygens 1897, 325)

In the preface of his biography, A view of Sir Isaac Newton's Philosophy, Henry Pemberton, who was the editor of the the third edition of the Principia, wrote that

Sir Isaac Newton has several times particularly recommend to me Huygens' style and manner. He thought him the most elegant of any mathematical writer of modern times, and the most just imitator of the ancients. Of their taste, and form of demonstration Sir Isaac always professed himself a great admirer. I have heard him even censure himself for not following them yet more closely than he did; and speak with regret of his mistake at the beginning of his mathematical studies, in applying himself to the works of Des Cartes and other algebraic writers, before he had considered the elements of Euclide with that attention, which so excellent a writer deserves (Pemberton 1728).

In turn Huygens greatly admired Newton's work, and in the summer of 1689 he came to England to meet Newton and discuss with him the current theories of gravitation. Like Leibniz, Huygens did not accept Newton's concept of an action at a distance which was regarded as an occult force by followers of Descartes vortex theory of gravitation, but he accepted the inverse square dependence on distance of the gravitational force.

In 1689 the British mathematician David Gregory visited Newton in Cambridge, and reported that

I saw a manuscript [written] before the year 1669 (the year when its author Mr. Newton was made Lucasian Professor of Mathematics) where all the foundations of his philosophy are laid: namely the gravity of the Moon to the Earth, and of the planets to the Sun. And in fact all these even then are subject to calculation (Herivel 1965, 192).

This manuscript, which Newton never published, revealed that already sixteen years before writing the *Principia*, Newton had carried out the “moon test”, later described in Book 3, Prop. 4, where he compared the gravitational force of the earth on the moon to the force of gravity on an object on the surface of the earth (Herivel 1965, 192-198)(see Appendix). In order to make this comparison, however, Newton had to assume that the gravitational force varied inversely proportional to the square of the distance from the center of the earth to any distance above its surface. Previously, he had deduced the inverse square dependence of this force from planetary motion (see footnote 2), where the distance between the planets and the sun is very large compared to their sizes, and it was reasonable to treat these celestial bodies as point masses. But to assume that this radial dependence was still valid for much shorter distances, and in particular down to the surface of a planet had to be justified. Apparently it was only after Newton already had started writing the *Principia*, that he was able to provide such a justification, by assuming that the gravitational attractive force due a finite size body can be compounded by adding the contribution of each of its elements. In Prop. 71, Book 1 of the *Principia* he gave a remarkable proof that the gravitational force of a spherical distribution of mass acts at any distance from its surface as if the total mass is concentrated at its centre (Chandrasekhar 1995, 269-272). Furthermore, in Prop. 91, Book 1, he considered also the force acting along the axis of any solid of revolution, and in Cor. 2 he applied the result to evaluate the special case of an oblate ellipsoid which he needed to determine the eccentricity of the earth due to its daily rotation (see footnote 6).

In Book 3 of the *Principia*, Newton applied his mathematical theory of orbital dynamics to planetary and lunar motion and to the motion of comets in order to provide evidence for the universal law of gravitation - that the attractive gravitational force between two bodies is proportional to the product of their masses and inversely proportional to the square of the distance between them ¹². He persuaded the Royal Astronomer, John Flamsteed,

¹²In ‘Rules of Reasoning in Philosophy’, *Principia*, Book 3, in Rule 3 Newton concluded :

Lastly, if it universally appears, by experiments and astronomical observations, that all bodies about the earth gravitate towards the earth, and that in proportion to the quantity of matter [mass] which they severally contain; that the moon likewise, according to the quantity of its matter, gravitates towards the earth; that on the other hand, our sea gravitates towards the moon; and all the planets one towards another; and the comets in like manner towards the sun; we must in consequence of this rule, universally allow that all bodies whatsoever are endowed with a principle of mutual gravitation.

Compare Newton’s formulation of universal gravitation with the earlier one of Robert Hooke, who wrote,

That all Celestial Bodies whatsoever, have an attraction or gravitating power towards their own Centers, whereby they attract not only their own parts, and keep them from flying from them, as we may observe the Earth

to provide him with the best available observational data at the time for the periods and major axis of the planets, and for the Jovian and Saturnian satellites. Then he showed that these observations were in good agreement with Keplers third law, Book 3, Phenomenon 2 - 4, which for circular motion he had considered some 20 years earlier as formulated in Cor.6 of Prop.4, Book 1, (see Appendix)

If the periodic times are as the three half powers of the radii, the centripetal force will be inversely as the squares of the radii.

In Prop. 15, Book 3, he extended this proof to elliptical motion, applying Cor.1 of Prop.45, Book I, to show that the near immobility of the aphelia of the planets, Book 3, Prop. 14, implied that the gravitational force between the planets and the sun satisfied the inverse square law. This was Newton's best proof for the inverse square law, because he had shown that the smallest deviation from this law would give rise to a precession of the planetary aphelia which over the centuries would have accumulated to give an observable effect.

Newton was aware, however, that astronomical observations had shown that there were deviations from Kepler's laws in the motion of the planets and the moon. In the preface to the Principia, he wrote:

But after I began to work on the inequalities of the motion of the moon, and ...the motions of several bodies with respect to one another ...I thought that publication should be put to another time so that I might investigate these other things...

Remarkably, a large part of these investigations apparently took place during the time that Newton was composing his book, when he developed methods to calculate the perturbation of the solar gravitational force on the lunar motion around the earth, and the effects due to the interplanetary gravitational forces on the motion of the planets around the sun. In Prop. 45 he presented his simplest perturbation approximation for the lunar orbit by assuming that it was a Keplerian elliptic orbit, but with its major axis rotating uniformly. In characteristic fashion, first he solved the problem of obtaining the exact law of force which would give rise to such an orbit, and found that this force was a linear combination of inverse square and inverse cube forces . Then he determined the rotation rate of the lunar apse by considering the effect of the component of the solar gravitational force along the earth-moon radial distance averaged over a period. But this approximation gave a

to do, but that they do also attract all the other Celestial Bodies that are within the sphere of their activity; and consequently that not only the Sun and Moon have an influence upon the body and motion of the Earth, and the Earth upon them, but that Mercury, also Venus, Mars Saturn and Jupiter by their attractive powers, have a considerable influence upon its motion as in the same manner the corresponding attractive power of the Earth hath a considerable influence upon every one of their motions also (Hooke 1674) (Nauenberg 1994a).

precession of the major axis of the lunar ellipse of only half the observed rate. In the first two editions of the Principia Newton was somewhat ambivalent about this large discrepancy, and only in the third edition, which appeared in 1726, did he add a remark, in Corollary 2 of Prop. 45, that “ the apse of the Moon is about twice as swift ” as the value that he had calculated. This discrepancy became one the first major challenges for the mathematicians and astronomers who studied the Principia, and it took another 20 years before a solution to this problem was first found by Clairaut and the French mathematician Jean le Rond d’Alembert.

THE RECEPTION OF THE PRINCIPIA BY THE MATHEMATICIANS IN THE CONTINENT

When Continental mathematicians and astronomers, primarily from Holland, Germany, Switzerland, and France, first read the Principia, they had some difficulties understanding Newton’s novel mathematical concepts, with its combination of geometrical quantities in the tradition of Greek mathematics and his concept of limits of ratios and sums of infinitesimals - quantities which become vanishingly small (Nauenberg 2010). After introducing three “Laws of Motion”, Newton presented ten mathematical “Lemmas” on his geometrical differential method of “ first and last ratios”. These lemmas constitute the basis for his calculus, and he referred to them in the proof of his propositions. Except for Lemma 2 in Book 2 of the Principia, Newton did not explain his analytic differential calculus in much detail, and European mathematicians, who already had been introduced to an equivalent calculus ¹³ by the German philosopher and mathematician, Gottfried Wilhelm Leibniz, first had to translate Newton’s mathematical language into Leibniz’s language before they could make further progress. Indeed, Leibniz was the first to express Newton’s formalism for orbital motion in the form of a differential equation based on his calculus (Nauenberg 2010). Leibniz claimed to have achieved his results having only read a review of Newton’s Principia, but in 1969 E.A. Fellman (1973) obtained a copy of Newton’s work which contained abundant Marginalia by Leibniz, indicating that he had carefully studied the text before undertaking his own work. Moreover, recently discovered manuscripts show Leibniz preparatory work for his 1689 essay Tentamen de motuum coelestium causis based on a reading of the Principia (Aiton 1995), 10) (Bertoloni Meli, 1991). But Leibniz obtained a differential equation of motion for celestial objects that was remarkably original ¹⁴

¹³In 1696 the Marquis de l’ Hospital published Analyse des infiniment petits, based on lectures about the calculus of Leibniz, given to him by Johann Bernoulli who he had hired as his private tutor in mathematics.

¹⁴Applying Prop. 1 in Book 1 of the Principia, Leibniz derived an expression for the second order differential ddr for the radial distance. This led him to a genuine discovery which is not found in the Principia: that this differential is proportional to an effective centrifugal force minus the central attractive force $f(r)$. In modern notation Leibniz’s result corresponds to the equation $d^2r/dt^2 = h^2/r^3 - f(r)$, where $h = r^2d\theta/dt$ is a constant corresponding to the angular momentum. For the case that the orbit is an ellipse he found that $f(r) = \mu/r^2$, where μ is the strength of the gravitational interaction (Aiton 1960) (Aiton1995) (Bertoloni Meli 1991) (Guicciardini 1999). Leibniz, however, assumed

Another mathematician who applied Leibniz's version of the differential calculus was Jacob Hermann, a member of a group around Jacob and Johann Bernoulli, two of Europe's leading mathematicians who had formed a school in Basel (Guicciardini 1999). Expressing Prop. 1 and Prop. 2 in Book 1 of Newton's *Principia* in the language of this calculus, he obtained a differential equation for the motion of a body under the action of central force. Then he gave a proof that conic sections were the only solutions for the case that the central force varied inversely with the square of the distance from the center of force (Herman 1710), (Nauenberg 2010) This was an important result, because in the first edition of the *Principia*, in Cor. 1 to Prop. 13, Newton had asserted, without proof, that conic sections curves were the *unique* solutions to orbital motion under inverse square forces. Johann Bernoulli criticized Hermann's solution for being incomplete(Hermann had left out a constant of the motion), and then derived the elliptic orbit by solving, via a suitable transformation of variables, the general integral for orbital motion in a central field force given in Prop. 41 Book 1 of the *Principia*, for the special case of an inverse square force (Bernoulli 1710), (Nauenberg 2010) Remarkably, Newton did not include this fundamental solution in the *Principia*, giving rise to a gap that has caused considerable confusion in the literature that remains up to the present time. Instead, Newton gave as an example the orbit for an inverse cube force^{15 16}.

Bernoulli also communicated to Newton an error he had found in Prop. 10 of Book 2. In both cases Newton made corrections in the next edition of the *Principia* (1713) without, however, acknowledging Bernoulli's important contributions (Guicciardini 1999). Some British mathematicians like David Gregory were able to contact Newton, and get help from him to overcome obstacles in understanding the *Principia*, but this appears not to have been possible for Continental mathematicians.

After Leibniz, the first Continental mathematician who undertook the reformulation of Newton's mathematical concepts into the language of Leibniz's calculus, was Pierre Varignon (Aiton 1960, 1955) (Bertoloni-Meli 1991) (Guicciardini 1999) . Varignon introduced an alternative expression for a central force in terms of the curvature of the orbit¹⁸, without being aware that Newton's earliest understanding of non-circular orbital motion was also based on curvature (Nauenberg 1994). In a cryptic note written in his 1664 Waste book, Newton remarked that

without justification that $h = \mu =$ latus rectum of the ellipse, and he incorrectly attributed Kepler's area law to a property of celestial vortices which leads to a physically inconsistent interpretation of his equation.

¹⁵Prop. 41, Cor. 3

¹⁶In Prop. 11-13 Newton gave a proof that if the orbit for a central force is a conic section, then the force varies inversely as the square of the radial distance. Johann Bernoulli criticized the incompleteness of Cor. 1 of Prop. 13, Book 1, where Newton claimed to give a proof to the solution of the *inverse* problem: given the gravitational force to show that the resulting orbit is a conic section¹⁷

¹⁸Varignon (1701) called the radius of curvature 'le rayon de Développé', and obtained his expression for the central force by recognizing that a small arc of the orbit corresponds to that of a circle with this radius, called the *osculating* circle by Leibniz but originating in the work of Huygens (Kline 1972) (Nauenberg 1996),

If the body b moved in an ellipse, then its force in each point (if its motion in that point be given) [can] be found by a tangent circle of equal crookedness with that point of the ellipse (Herivel 1965, 130)

Here the word “crookedness” refers to curvature which is measured locally by the inverse radius ρ of the tangent or osculating circle (as it was named later by Leibniz) at any point on an ellipse. Curvature was also a mathematical concept that had been introduced earlier by Huygens in his *Horologium Oscillatorum* (Huygens 1673) , (Nauenberg 1996) Evidently, Newton was aware that the central force or acceleration a for non-uniform orbital motion can be obtained from a generalization of the relation for uniform circular motion, $a_c = v^2/\rho$, where v is the velocity, which he, and independently Huygens, had obtained earlier (see Appendix). Then $a = a_c/\cos(\alpha)$ where α is the angle between the direction of the central force and that of the radius of curvature. The problem, however, is that the motion or velocity v , which is a variable along the orbit for non-circular motion because then there is a tangential component of the central force, had to be known (Brackenridge 1995) , Brackenridge 2002). But 15 years later, Newton found a proof for the area law, which implies that for any central force, the area swept by the radial line per unit time, $(1/2)vr\cos(\alpha)$ is a constant (proportional to the conserved angular momentum) and r is the radial distance. By substituting this expression for v , Newton had an explicitly expression for the central acceleration $a \propto 1/\rho r^2 \cos^3(\alpha)$. Indeed, for conical sections, the quantity $\rho \cos^3(\alpha)$ is a constant (the semi-latus rectum of an ellipse) which provided Newton with a succinct proof that for such orbits the force depends inversely on the square of the radial distance (Nauenberg 1994a). This relation was also found by Abraham DeMoivre¹⁹ (Guicciardini 1999, 226), and applied by John Keill and Roger Cotes who were members of the school of British mathematicians²⁰. In the first edition of the *Principia*, however, the curvature expression for the force does not appear explicitly, although Newton applied it in a few instance without any explanation²¹, while in the second edition curvature is discussed in a new Lemma, Lemma 11, and the curvature measure for force is derived as corollaries to Prop. 6. Subsequently, Newton applied it to obtain “another solution” to the fundamental problem formulated in Prop. 11, Book 1,

Let a body revolve in an ellipse, it is required to find the law of the centripetal [central] force tending towards a focus of the ellipse.

According to Newton’s recollections, as told to DeMoivre in 1727,

19

After having found this theorem, I showed it to M. Newton and I was proud to believe that it would have appeared new to him, but M. Newton had arrived at it before me; he showed this theorem to me among his papers that he is preparing for a second edition of his *Principia Mathematica* ...

²⁰For a brief history of this important development see (Whiteside 1974, 548-549)

²¹Prop. 15, Book 2, and Prop. 26-29 Book 3.

In 1684 Dr. Halley came to visit him at Cambridge and after they had been some time together, the Dr. asked him what the thought the Curve would be that would be described by the Planets supposing the force of attraction towards the Sun to be reciprocal to the square of their distance from it. Sir Isaac replied immediately that it would be an *Ellipsis*, the Dr. struck with joy and amazement asked him how he knew it, why said he, I have calculated it, whereupon Dr. Halley asked him for his calculation without delay. Sir Isaac looked among his papers but could not find it, but he promised him to renew it, and then to send it to him. (Westfall 1995).

However, the solution which Newton eventually sent to Edmund Halley in a manuscript entitled De Motu (Whiteside 1974), and that three years later he presented in the same form in Prop. 11 of the Principia, treated instead the inverse to the problem posed by Halley, namely given that the orbit is an ellipse, to prove that the central force obeys the inverse square law, or as Newton formulated in 1687,

Let a body revolve in an ellipse; it is required to find the law of the centripetal force tending towards a focus of the ellipse (Cohen 1999, 462)

Relations with Bernoulli and his school were further aggravated when the notorious priority dispute on the invention of the calculus erupted between Newton and Leibniz in 1711. By the 1740's, serious reservations arouse regarding the general validity of the inverse square law for gravitational force because of the failure of Newtons approximation of the solar perturbation to account for the rate of precession of the lunar apside. One of the first to question on this ground the validity of this law was the great mathematician Leonhard Euler. He remarked that,

having first supposed that the force acting on the Moon from both the Earth and the Sun are perfectly proportional reciprocal to the squares of the distances, I have always found the motion of the apogee to be almost two times slower than the observations make it: and although several small terms that I have been obliged to neglect in the calculation may be able to accelerate the motion of the apogee, I have ascertained after several investigations that they would be far from sufficient to make up for this lack, and that it is absolutely necessary that the forces by which the Moon is at present solicited are a little different from the ones I supposed. (Waff 1995, 37).

He concluded that

all these reason joined together appear therefore to prove invincibly that the centripetal force in the Heavens do not follow exactly the law established by Newton (Waff 1995, 37).

Clairaut had reached similar conclusions and was delighted to find that he was in agreement with Euler. He had also found

that the period of the apogee [i.e. the time it takes for the lunar apogee to return to the same point in the heavens] that follows from the attraction

reciprocally proportional to the squares of the distances, would be about 19 years, instead of a little less than 9 years which it is in fact (Waff 1995, 39)

a result that Newton had mentioned earlier in Prop. 45, Book 1. To account for this discrepancy, Clairaut proposed that an additional force was also in effect which varied with distance inversely as the fourth power, possible due to Cartesian vortices. Actually, suggestions for possible correction to the inverse square gravitational law had been considered by Newton in Query 31 of his *Opticks*, but he did not want to publicize them. Another mathematician, Jean le Rond d'Alembert, arrived at the same discrepancy for the motion of the lunar apogee, but in contrast to Euler and Clairaut, he did not question the mathematical form of Newton's gravitational law because of its successes in describing other inequalities of the lunar motion. Ultimately, the French Academy of Sciences propose a prize for the solution of this problem, and in 1749 Clairaut finally obtained a solution without altering the inverse square force, by considering higher order contributions to the solar perturbation, followed by d'Alembert with a more careful analysis which gave the same result.²² (Waff 1995). Previously, similar solution had been obtained by Newton, but it contained some errors (Nauenberg 2001a), and in a Scholium to Prop. 35, Book 3, inserted only in the first edition of the *Principia*, he declared that

...These computations, however, excessively complicated and clogged with approximations as they are, and insufficiently accurate we have not seen fit to set out .

The details of Newton's computations remained unknown until 1872 when they were found among his papers in the Portsmouth Collection (Whiteside 1974, 508-538) (Nauenberg 2000) (Nauenberg 2001a)

The importance of Clairaut's result can hardly be overestimated. In admiration Euler declared in a letter to Clairaut that

. . . the more I consider this happily discovery, the more important it seems to me. For it is very certain that it is only since this discovery that one can regard the law of attraction reciprocally proportional to the squares of the distance as solidly established, and on this depends the entire theory of astronomy (Waff 1995, 46)

In 1748 the French academy of sciences chose for its prize contest a theory that would explain the inequalities in the motion of Jupiter and Saturn due to their mutual gravitational interaction, which Newton had considered only semi-quantitatively in Prop. 13, Book 3²³

²²The title that Clairaut chose for his winning essay was 'Theory of the Moon Deduced from the Single Principle of Attraction Reciprocally Proportional to the Squares of the Distances'

²³From the action of Jupiter upon Saturn "...arises a perturbation of the orbit of Saturn at every conjunction of this planet with Jupiter, so sensible, that astronomers have been at a loss concerning it" (Cohen 1999, 818).

This problem was much more difficult than the lunar case, and Euler was the first to deal with it (Euler 1769), and now he declared that

... because Clairaut has made the important discovery that the movement of the apogee of the Moon is perfectly in accord with the Newtonian hypotheses ..., there no longer remains the least doubt about its proportions... One can now maintain boldly that the two planets Jupiter and Saturn attract each other mutually in the inverse ratio of the squares of their distance, and that all the irregularities that can be discovered in their movement are infallibly caused by their mutual action... and if the calculations that one claims to have drawn from the theory are not found to be in good agreement with the observations, one will always be justified to doubting the correctness of the calculations, rather than the truth of the theory (Waff 1995, 46)

After missing an expected lunar eclipse, Tycho Brahe had discovered a bi-monthly variation in the lunar speed, and Newton was able to account for this variation as an effect of the solar gravitational force. In Prop. 28, Book 3, Newton introduced a novel frame of reference where the earth is fixed at the center of a rotating frame with the period of one year. In this frame the sun stands still when the eccentricity of the earth-sun orbit is neglected. Then taking into account the solar gravitational force, Newton found an approximate periodic orbit of the moon which accounted for the periodic of the variation discovered by Brahe. In Prop. 29, Book 3, appropriately entitled 'To find the variation of the moon', he calculated the amplitude of this variation, and found it in very good agreement with Brahe's observation. In his review of Newton's work on lunar theory the great French mathematician and astronomer, Pierre-Simon Laplace, singled out this result, and remarked admiringly at Newton's insightful approximations,

Such hypothesis in calculations ... are permitted to inventors during such difficult researches ²⁴

RECEPTION OF NEWTON'S GRAVITATIONAL THEORY FOR PLANETARY AND LUNAR MOTION

Inspired by Newton's work, Leonhard Euler introduced his rotating frame to calculate the solar perturbation to the lunar motion (Euler 1772). Likewise, in 1836 this frame was considered also by Gustaf Carl Jacobi, who gave a proof for the existence of a constant of the motion in what became known as the restricted three body problem. Later, the American astronomer George Hill also obtained periodic solutions in this rotating frame (Hill 1783), and his work was extended by Henri Poincaré, which led him eventually to his profound discovery of chaotic orbital motion in Newtonian dynamics (Poincare 1892), (Barrow-Green 1991), (Nauenberg 2003b).

²⁴Ces hypothèses de calcul... sont permises aus inventeurs dans des reserches aussi difficiles (Laplace 1825, 391)

In twenty two corollaries to Proposition 66, Book 1, Newton described entirely in prose his perturbations methods, but his detailed calculations remained unpublished (Nauenberg 2000) (Nauenberg 2001a). Here Newton considered gravitational perturbations to the elliptical motion of a planet around the sun or the moon around the earth as a sequence of impulses, equally spaced in time, which instantaneously alters the velocity of the celestial body in its orbit without, however, changing its position when these impulses occur. In Prop. 17, Book 1, Newton had shown how the orbital parameters - the eccentricity, and the magnitude and direction of the principal axis of the ellipse - can be determined given the velocity and position at a given time (initial conditions). Hence, these impulses lead to periodic changes in the orbital parameters which are determined by the discontinuous change in velocity after the impulse has taken place. In corollaries 3 and 4 of Proposition 17, Newton gave a succinct description of his method of variation of orbital parameters . These corollaries were added to later drafts of the *Principia* ²⁵. indicating that Newton had developed this method during the period when he was writing his book. In the limit that the time interval between impulses is made vanishingly small, Newton's perturbation methods corresponds to the method of variational parameters developed much later by Euler ²⁶, Joseph Louis Lagrange and Pierre-Simon Laplace ²⁷. Now this method is usually credit to them.

Unpublished manuscript in the Portsmouth collection of Newton's papers, first examined in 1872 by a syndicate appointed by the University of Cambridge (Brackanbridge 1999) reveal that Newton had intended to include a more detailed description of his perturbation

²⁵In the initial revisions of the early manuscript for the *Principia*, Prop. 17 contained only Corollaries 1 and 2 (Whiteside 1974, 160–161)

²⁶Starting with the equations of motion as second order differential equations in polar coordinates, Euler assumes that the solution for the orbit is described by an ellipse with time varying orbital parameters p, e and ω , where p is the semilatus rectum of the ellipse, e is the eccentricity, and ω is the angle of the major axis. Then he obtained first order differential equations for e and ω by imposing two constraints: that $p = h^2/\mu$, where h is the angular momentuma, and that $E = \mu(e^2 - 1)/2p$ where E is the time varying Kepler energy of the orbit. In modern notation $\mu = GM$ where M is the sum of the mass of the earth, and the moon and G is Newton's gravitational constant (Euler 1769). It can be readily shown that Euler's constraints lead to the same definition of the ellipse described geometrically by Newton in the Portsmouth manuscript (Nauenberg 2000) (Nauenberg 2001a).

²⁷Laplace obtained the differential equations for the time dependence of the orbital parameters by evaluating the time derivate of the vector $\vec{f} = \vec{v} \times \vec{h} - \mu\vec{r}/r$, where \vec{f} is a vector along the major axis of the ellipse with magnitude $f = \mu e$. The construction of this vector was first given in geometrical form by Newton in Book 1, Prop. 17, and in analytic form by Jacob Hermann and Johann Bernoulli (Bernoulli,1710) (Nauenberg 2010). Laplace's derivation(Laplace 1822, 357-390) of the variation of orbital parameter is in effect the analytic equivalent of Newton's geometrical approach in the Portsmouth manuscript (Nauenberg 2000) (Nauenberg 2001a)

methods in the *Principia*, but neither the propositions and lemmas in these manuscripts nor the resulting equations, which in effect are non-linear coupled differential equations for the orbital parameters, appeared in any of its three editions (Nauenberg 2000) (Nauenberg 2001a). But some of his results for the inequalities of the lunar motion appeared in a lengthy Scholium after Prop. 35, Book 3, which includes numerical results obtained by approximate solutions to his equations. In this Scholium, for example, Newton stated that

By the same theory of gravity, the moon's apogee goes forwards at the greatest rate when it is either in conjunction with or in opposition to the sun, but in its quadratures with the sun it goes backwards; and the eccentricity comes, in the former case to its greatest quantity; in the latter to its least by Cor. 7,8 and 9, Prop. 66, Book 1. And those inequalities by the Corollaries we have named, are very great, and generate the principle which I call the semiannual equation of the apogee; and this semiannual equation in its greatest quantity comes to about $12^{\circ}18'$, as nearly as I could determine from the phenomena ²⁸ ²⁹.

²⁸In Cor. 7 and 8, Prop. 66, Newton gave a qualitative explanation for this motion of the moon's apogee due to the perturbation of the sun, stating it was based on results given in Book 1, Prop. 45, Cor. 1. However, these results were obtained for the case of radial forces only, and are therefore strictly not applicable to the solar perturbation which is not a purely radial force with respect to the earth as a center, and which depends also on the angle ψ . According to the differential equation for the motion of the lunar apogee which appears in the Portsmouth manuscript, its rate depends on the relative angle between the moon's apogee ω and the longitude θ of the sun, where $\omega - \theta = \psi - \phi$. It reaches a maximum value when $\omega - \theta = n\pi$ where n is an integer. and a minimum when n is an odd integer divided by 2, in accordance with Cor. 8. In fact, substituting Newton's numerical values $\beta = 11/2$, one finds that the maximum rate of advance is $21.57'$, and of retardation $14.83'$. This is in reasonable agreement with the values $23'$ and $16\ 1/3'$ given in the original (1687) Scholium to Prop. 35 corresponding to $\beta \approx 6$. In Cor. 9 Newton gave a qualitative argument for the variability of the eccentricity, but there is no evidence that he obtained this quantitative result from his "theory of gravity". According to his theory the maximum variability of the apogee is $15m/8 = 8^{\circ}2'$ instead of $12^{\circ}18'$ as quoted in the Scholium to Prop. 35. Although the lunar model of Horrocks was probably the inspiration for his Portsmouth method, in the end Newton was able to account partially for this model from his dynamical principles.

²⁹These anomalies in the orbit of the moon around the earth had been a major challenge to astronomers since Antiquity. Already by the second century B.C., Hipparchus had found that the moon's motion at quadrature deviated in longitude by over two and a half degree from the predictions of the Greek model of epicyclic motion, although this model accounted for the moon's position at conjunction and opposition from the sun. Subsequently, Ptolemy proposed the first mechanism to account for this anomaly, known as the evection, but his mechanism also predicted a near doubling of the apparent size of the moon during its orbit which is not observed. Nevertheless, Ptolemy's lunar model was not challenged until the

Newton's lunar work was received with immense admiration by those who were able to understand the profound mathematical innovations in his theory. An early reviewer of the second edition of the *Principia* stated that

the computations made of the lunar motions from their own causes, by using the theory of gravity, the phenomena being in accord, proves the divine force of intellect and the outstanding sagacity of the discoverer

Laplace asserted that the sections of the *Principia* dealing with the motion of the moon are one of the most profound parts of this admirable work³⁰, and the British Astronomer Royal, George Airy, declared "that it was the most valuable chapter that has ever been written on physical science" (Cohen 1972). The French mathematician and astronomer François Félix Tisserand in his *Traité de Mécanique Céleste* (Tisserand 1894) carefully reviewed Newton's lunar theory as it appeared in the *Principia*, and also compared some of Newton's results in the Portsmouth manuscript with the results of the variation of parameters perturbation theory of Euler, Laplace and Lagrange. For an arbitrary perturbing force, Tisserand found that Newton's equation for the rotation of the major axis of the ellipse was correct to lowest order in the eccentricity of the orbit, while his application to the lunar case differed only in the numerical value of one parameter, which Newton gave as $11/2$, instead of the correct value of 5 (Nauenberg 2001a). In particular, Tisserand concluded that

Newton derives entirely correctly that the average annual movement of the apogee is $38^{\circ}51'51''$, while the one given in the astronomical tables is $40^{\circ}41'5''$ ³¹

D'Alembert, however, doubted whether some of Newton's derivations were really sound, and complained that

there are some that M. Newton said to have calculated with the theory of gravitation, but without letting us know the road that he took to obtain

15-th century when the Arab astronomer Ibn-al Shatir developed an alternative mechanism for the lunar motion which was later adopted by Copernicus. Their model accounted for the evection without introducing the large unobserved variations of the lunar size in Ptolemy's model. In the 17-th century alternative models were developed by Tycho Brahe and Kepler who incorporated his law of areas for planetary motion into his lunar model. In 1640, Jermy Horrocks refined Kepler's model further predicting correctly the inequalities in the distance of the moon from the earth. These are some of the additional inequalities that Newton was also able to demonstrate to be caused by the gravitational force of the sun acting on the moon (Nauenberg 2001b)

³⁰Parmi les inégalités du mouvement de la Lune en longitude, Newton n'a développé que la *variation*. La méthode qu'il suivie me paraît être une des choses le plus remarquables de l'Ouvrage des Principes (Laplace 825, 409)

³¹Newton déduit, tout à fait correctement ...que le mouvement moyen annuel de l'apogée est de $38^{\circ}51'51''$, tandis que celui qui est donné dans les Tables astronomiques est de $40^{\circ}41'5''$ (Tisserand 1894, 45)

them. Those are the the ones like 11'49" that depend on the equation of the sun's center ³².

Here, d'Alembert was referring to Newton's calculation, in the Scholium mentioned previously, of the annual equation of the mean motion of the moon which depends on the earth's eccentricity ϵ in its orbit around the sun. Newton had taken ϵ equal to $16 \frac{7}{8}$ divided by 1000, and D'Alembert may have been aware that the amplitude of this perturbation is $3\epsilon m = 13'$ where m is the ratio of the lunar sidereal period to a period of one year.⁴⁷⁸ Hence, although in this Scholium Newton had stated that his results had been obtained by "his theory of gravity", it appears that he adjusted some of the perturbation amplitudes to fit the observational data .

For the next two centuries after the publication of the *Principia*, Newton's approach to what became known as the three body problem³³ in dynamical astronomy stimulated the work of mathematicians and astronomers, and this problem remains a challenge up to the present time ³⁴ . By the late 1700's Lagrange and Laplace had written major treatises on analytic mechanics (Lagrange 1811), and celestial mechanics (Laplace 1878) containing the mathematical progress that had been made. There is an often repeated tale ³⁵ that Napoleon once asked Laplace why God did not appear in his work, and that Laplace famously responded "I didn't need that hypotheses", but in print he declared that

These phenomena and some others similarly explained, lead us to believe that everything depends on these laws [the primordial laws of nature] by relations more or less hidden, but of which it is wiser to admit ignorance, rather than to substitute imaginary causes solely in order to quiet our uneasiness about the origin of the things that interest us (Morando 1995, 144)³⁶.

³²en est quelques-unes que M. Newton did avoir calculées par la Theorie de la gravitation, mais nous apprendre le chemin qu'il a pris pour y parvenir. Telle son celles de 11' 49" qui dépend de l' équation du centre du soleil.

³³Given the initial conditions (position and velocities) for three celestial bodies moving under the action of their mutual gravitational attraction, to determine their motion at all times in the future.

³⁴For example, in 1772 Lagrange discovered an exact solution of the three body problem where each of the celestial bodies move in elliptic orbits with the same period, and with a common focus located at their center of mass. The stability of these orbits, however, was not examined fully until much later, first by Routh (1875) for the special case of circular orbits, and later for elliptic orbits by Danby (1964). These studies were restricted to linear instabilities, and a non-linear instability analysis has been undertaken only recently (Nauenberg 2002).

³⁵Se non è vero, è ben trovato.

³⁶Ces phénomènes et quelques autres semblablement expliqués autorisent à penser que tous dépendent de ces lois, par des rapport plus ou moins cachés, mais dont il es plus sage d'avouer l'ignorance que d'y substituer des cause imaginées par le seul besoin de calmer notre inquiétude sur l'origine des choses qui nous intéressent (Laplace 1835, 478).

Newton claimed God needed to interfere from time to time in order to maintain the stability of the solar system, but Laplace asserted that he had been able to give a mathematical proof of this stability. Later, however, this proof was shown to be flawed ³⁷ by the work of Henri Poincaré (1892).

The overall impact of Newton's *Principia* in astronomy was best summarized by Laplace's conclusion,

This admirable work contains the germs of all the great discoveries that have been made since about the system of the world: the history of its development by the followers of that great geometer will be at the same time the most useful comment on his work, as well as the best guide to arrive at new discoveries ³⁸

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Appendix, Newton's moon test. Newton's assumption that the inverse square law for gravitational forces applies on the surface on the earth, requires the relation $a_m/g = (r_e/r_m)^2$, where a_m is the radial acceleration of the moon towards the earth, g is the gravitational acceleration at the surface of the earth, r_m is the radius of the moon's orbit, and r_e is the radius of the earth. Since Newton had found that $a_m = 4\pi^2 r_m / T_m^2$, he tested the inverse square law by calculating the ratio $(g/2)/d_m$, where $g/2$ is the distance a body falls in one second on the surface of the earth, and $d_m = a_m/2 = 2\pi^2 r_m / T_m^2$ is the corresponding distance that the moon "descends towards the earth in one second". Pendulum experiments had established that $g = 32$ feet/ sec^2 , but to obtain d_m , Newton first had to calculate $d_e = 2\pi^2 r_e / T_e^2$, which is the corresponding distance of fall for a body on the surface of the earth co-rotating with the earth's period T_e of one day.

Taking for the earth's radius $r_e = 3500$ miles, and assuming that a mile is 5000 feet, he obtained $d_e = 5/9$ inches, and $(g/2)/d_e = 345.6$ which he rounded to 350. Huygens had carried out a similar calculation, but taking a different value of the earth's radius, $r_e = 3711$ miles, and $g = 27.33$ feet/ sec^2 . he obtained for this ratio the value 265 (Huygens 1929), while the correct value is 290. This result answered the long standing question why, if the earth was spinning about its axis once a day, objects on its surface do not fly off:

The force of gravity is many times greater that what would prevent the rotation of the earth from causing bodies to recede from it and raise into the air (Herivel 1965, 196)

³⁷Laplace's proof that secular variations of the mean solar distances of the planets do not occur were based on perturbation expansions up to third order in the eccentricities, but these expansion were shown not to be convergent.

³⁸Cet admirable Ouvrage contient les germes de toutes les gandes dècouvertes qui ont ètès faits depuis sur le système de monde: l'histoire de leur développement par les successeurs de ce grand géomètrè serait à la fois le plus utile commentaire de son Ouvrage, ce le meilleur guide pur arriver à de nouvelles dècouvertes.

Since $d_e/d_m = (r_e/r_m)(T_m/T_e)^2$, where $T_m/T_e = 27.3216$, and $r_m/r_e \approx 60$ which was already measured by Greek astronomers, one obtains $d_e/d_m=12.44$ (Newton rounded it to 12.5). Hence, $(g/2)/d_m = 16(9/5)12.5 = 4320$ which differs appreciable from the expected value $(r_m/r_e)^2 \approx 3600$. Newton's only comment about this discrepancy was that the force of gravity at the surface of the Earth

is 4000 and more times greater than he endeavor of the Moon to recede from the Earth,

but he must have been gravely disappointed with this result. The reason for the failure of Newton's early *moon test* is that in his calculations he had used an incorrect value for the radius of the earth based on a value of about 61 English miles per degree of latitude, and also that he had assumed that a mile corresponds to 5000 feet instead of the correct value 5280 (in this manuscript Newton stated that " 1/30 of a mile is 500/3 feet"). Apparently he did not become aware of his errors until 1683, when he substituted in his relation a much better value for the earth's radius r_e obtained in 1669 by Picard from his measurement for a degree of latitude of 69.2 English miles (see Prop. 19, Book 3). This measurement gives $r_e = 3965$ miles, close to the modern value. In this case $(g/2)/d_m = 4320(61/69.2)(5000/5280) = 3606$, in excellent agreement with the result predicted by Newton's theory.

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