

EDMUND C. STONER AND THE DISCOVERY OF THE MAXIMUM MASS OF WHITE DWARFS

MICHAEL NAUENBERG, University of California, Santa Cruz

The existence of a mass limit for white dwarfs is usually attributed solely to the late astrophysicist Subrahmanyan Chandrasekhar (1910–95), and this limit is named after him.¹ But as is often the case, the history of this discovery is more nuanced. In this paper I will show that the existence of a maximum mass was first established by Edmund C. Stoner (1899–1968), a physicist who began experimental research under the supervision of Rutherford at the Cavendish in Cambridge, but later switched to theoretical work. Rutherford recommended Stoner to a position at the Physics department of the University of Leeds where he spent his entire career.² According to G. Cantor, he was “probably the leading Cavendish-trained *theoretical* physicist of the 1920’s”,³ although he learned theory mostly on his own, and became known for his work on magnetism.⁴ Unfortunately, Stoner suffered from diabetes and poor health which restricted his travels, and this may account for the fact that he did not receive wider recognition for his achievements.

In 1924 Stoner wrote a paper on the distribution of electrons among atomic levels.⁵ In the preface of the fourth edition of his classic book, *Atomic structure and spectral lines*, Arnold Sommerfeld gave special mention to “einen grossen Fortschritt [a great advancement]” brought about by Stoner’s analysis, which then came to the attention of Wolfgang Pauli, and played an important role in his formulation of the exclusion principle in quantum physics.⁶ Therefore, it is not surprising that Stoner’s interest in white dwarfs was aroused by Ralph H. Fowler’s suggestion⁷ that the exclusion principle could be applied to solve a major puzzle, the origin of the extreme high density of white dwarfs,⁸ which could not be explained by classical physics. Arthur Eddington expressed this puzzle as follows: “I do not see how a star which has once got into this compressed state is ever going to go out of it.... *The star will need energy in order to cool....* It would seem that the star will be in an awkward predicament when its supply of subatomic energy fails. Imagine a body continually losing heat but with insufficient energy to grow cold!”⁹

At the time, the conventional wisdom was that the source of internal pressure which maintained all stars in equilibrium against gravitational collapse was the internal pressure of the matter composing the star which had been heated into a gas presumably, according to Eddington, by “subatomic energy”. But when this supply of energy is exhausted and the star cools, Fowler proposed that a new equilibrium would ensue, even at zero temperature, due to the “degeneracy” pressure of the electrons caused by the exclusion principle in quantum mechanics. Fowler, however, did not attempt to determine the equilibrium properties of such a star which he regarded as “strictly

analogous to one giant molecule in the ground state". Apparently he was unaware at the time that Llewellyn H. Thomas had developed a mathematical method to solve this problem in atomic physics.¹⁰ Subsequently, Stoner applied the minimum energy principle to obtain the equilibrium properties of such dense stars¹¹ in a constant density approximation, by substituting for the internal energy density Fowler's non-relativistic equation of state for a degenerate electron gas.¹² In particular, he found that the density increases with the square of the mass of the star.¹³ In such a gas the mean momentum of an electron is proportional to the cube root of the density, and Wilhem Anderson, a *Privatdozent* at Tartu University, Estonia, who had read Stoner's paper, noticed that for the mass of a white dwarf comparable to or higher than the mass of the Sun, the density calculated from Stoner's non-relativistic mass–density relation implied that the electrons become relativistic.¹⁴ Hence, Anderson concluded that in this regime, this relation gave “gröblich falschen Resultaten [gross false results]” for the properties of a white dwarf. He attempted to extend the equation of state of a degenerate electron gas to the relativistic domain, but he gave an incorrect formulation which, fortuitously, indicated that Stoner's minimum energy principle implied a maximum value for the white dwarf mass. Alerted by Anderson's paper, Stoner then derived the correct relativistic equation of state,¹⁵ and re-calculated, in a constant density approximation, the properties of white dwarfs for arbitrary densities.¹⁶ Thus, he obtained, now on solid theoretical grounds, the surprising result that when the density approaches infinity, the mass of the star reaches a *maximum* value.

Two years after the appearance of the first paper by Stoner on the “limiting density of white dwarfs”,¹⁷ Chandrasekhar published a paper¹⁸ with a similar title “arriving at the order of magnitude of the density of white stars from different considerations”, which was communicated by Fowler to the *Philosophical magazine*. Since the non-relativistic pressure–density relation for a degenerate electron gas is a power law with exponent $5/3$, Chandrasekhar realized — having read Eddington's book *The internal constitution of the stars*,¹⁹ which he had obtained as an essay prize — that the solution of the hydrostatic equation for gravitational equilibrium appropriate to a low mass white dwarf was the Lande-Emde polytropic solution with index $n = 3/2$. This solution leads to the same mass–density relation previously found by Stoner in the uniform density approximation, but with a proportionality coefficient smaller by a factor about two. Meanwhile, Stoner, in collaboration with Frank Tyler, had calculated the minimum energy of a white dwarf assuming a density distribution corresponding to the $n = 3/2$ polytrope,²⁰ obtaining the same result as Chandrasekhar, and somewhat earlier Edward A. Milne also had carried out this calculation.²¹ In his paper, Chandrasekhar ignored “relativistic-mass corrections”, because he did not yet know how to incorporate them, while Stoner already had shown, as an example, that for the white dwarf companion of Sirius these corrections gave a density almost an order of magnitude larger than the non-relativistic calculation. In “Some historical notes”,²² Chandrasekhar recalls that he had found that the degenerate electrons become relativistic²³ for white dwarfs with masses that are comparable or larger than the mass of the Sun. His calculation in the extreme relativistic limit appeared

separately in a very short paper (two pages long) on “the maximum mass of ideal white dwarfs”.²⁴ Again Chandrasekhar was able to obtain his result with great ease, because the relevant solution of the differential equation for gravitational equilibrium for the extreme relativistic equation of state of a degenerate electron, which has an exponent $4/3$, corresponds to the $n = 3$ Lane-Emde polytropic solution, which also appears in Eddington’s book. It turns out that in this case the mass is *independent* of the central or mean density of the star. Chandrasekhar acknowledged that his result was in surprising “agreement” with Stoner’s result, but he also claimed, without giving any proof, that it was a *maximum* mass for a white dwarf. Later, in an interview with Spencer Weart, Chandrasekhar acknowledged that “... at first I didn’t understand what this limit meant and I didn’t know how it would end,²⁵ and how it related to the $3/2$ low mass polytropes. But all that I did when I was in England and wrote my second paper on it”.²⁶

But a proof that the critical mass is a maximum mass already had been given in the uniform density approximation by Stoner, who also had shown analytically that the mass of a white dwarf is a *monotonically* increasing function of the density which is finite at infinite density, while it took Chandrasekhar several additional months before he found a rough argument to show that at the critical mass the density becomes infinite.²⁷ His awareness of Stoner’s analysis, however, was left unmentioned, although it is clear that it must have given him some confidence in the validity of his result.

At about this time, the physicist Lev D. Landau, who had recently finished his studies in Leningrad, was visiting the ETH in Zurich where Rudolf Peierls was Pauli’s *Assistant*.²⁸ During this visit, Landau worked with Peierls on relativistic quantum field theory, and he developed the quantum theory of diamagnetism associated with a degenerate electron gas in a metal. Thus, it is not surprising that he should also consider the role of quantum degeneracy of an electron gas, including the implications of special relativity, for the properties of stars. He was motivated by the work on stellar structure of Milne, whom he criticized for “making physical assumptions only for the sake of mathematical convenience”.²⁹ Like Stoner, Landau recognized that the equilibrium state of dense stars is a minimum of the energy. By applying this principle to the extreme relativistic equation of state, he found that the total energy $E = a\rho^{1/3}$, where ρ is the mean density and a is a constant that depends on the mass of the star. Hence, depending on the sign of a , the star would either “expand or collapse to a point” to attain the minimum value of E . The criterion separating these two regimes corresponds to $a = 0$, which in the uniform density approximation leads to Stoner’s solution for the critical mass. Instead, Landau solved this problem by considering the equation for the *chemical potential*, which in this case corresponds to the maximum energy or Fermi energy of an electron in a degenerate electron gas. The resulting differential equation is analogous to the Thomas-Fermi equation,³⁰ which in the relativistic regime is equal to the Lane-Emde $n = 3$ polytropic equation. Thus, starting from the same principles enunciated earlier by Stoner, but solving the resulting equations without making Stoner’s uniform density approximation, Landau obtained the exact value for the critical mass,³¹ but with a different numerical value

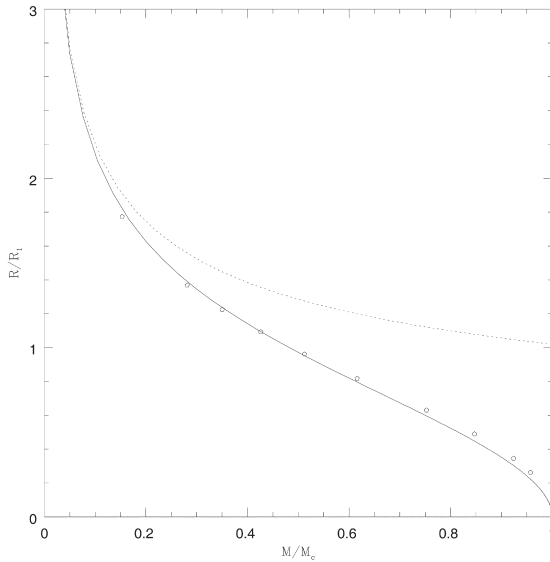


FIG. 1. The dark line is a plot of the scaled radius, R/R_1 v. scaled mass, M/M_c of Stoner's 1930 *analytic* solution in the uniform density approximation. The circles are the solutions published in 1935 by Chandrasekhar, who *numerically* integrated the equations of gravitational equilibrium using Stoner's pressure–density relativistic equation of state. The mass is given in units of the critical mass M_c and the radius in units of a length R_1 for which $(M/M_c)(R/R_1)^3 \approx 1$ in the non-relativistic limit, $(M/M_c) \ll 1$. The dashed line is the non-relativistic solution $R/R_1 = (M_c/M)^{1/3}$.

than that given by Chandrasekhar.³² Since there is some confusion in the literature concerning the dates³³ associated with Landau's work, it should be pointed out that Landau submitted his paper on dense stars for publication *five* months before the appearance of Chandrasekhar's first paper³⁴ on this subject, and therefore it is very unlikely that he would have been aware of Chandrasekhar's work.³⁵

Stoner's fully relativistic analytic solution, in the uniform density approximation,³⁶ for the mass–radius dependence of the dense stars is shown graphically in Figure 1. His result is compared with ten *numerical* calculations, shown by circles, which Chandrasekhar obtained *five* years later by integrating numerically the differential equations of gravitational equilibrium with Stoner's relativistic pressure–density equation of state.³⁷

This remarkable agreement is surprising, because Stoner's result was based on the uniform density approximation, while Chandrasekhar's was obtained by integrating the equations of gravitational equilibrium. The main difference is in the scales of mass and of length, e.g. Chandrasekhar's critical mass M_c is 20% smaller than Stoner's. Before 1935, following ideas of Milne,³⁸ Chandrasekhar had developed only a crude composite model for a white dwarf³⁹ in which the non-relativistic approximation was assumed to be valid for increasing mass until the central pressure became equal

to the pressure given by the extreme relativistic equation at the same density. For a larger mass, he applied this relativistic equation to a central region of the star, and the non-relativistic equation for an external region of the star bounded by a surface defined when these two equations gave the same pressure at equal densities.

Stoner was encouraged by Eddington, regarded as “the most distinguished astrophysicist of his time”,⁴⁰ to pursue the implication of his relativistic equation of state on the maximum density and temperature of white dwarfs, and he communicated Stoner’s two papers on this subject to the *Monthly notices of the Royal Astronomical Society*.⁴¹ Eddington’s 1932 correspondence with Stoner (see Appendix and Figure 2) deepens further the mystery why several years later, in a well known public attack⁴² on Chandrasekhar’s similar work on white dwarfs,⁴³ Eddington unexpectedly rejected the relativistic equation of state, and the profound implications of the existence of a white dwarf mass limit for the fate of stars with masses exceeding this limit.⁴⁴ Apparently Eddington had found that relativistic degeneracy was incompatible with his fundamental theory, and later confessed to Chandrasekhar that he would have to abandon this theory if relativistic degeneracy were valid.⁴⁵ Eddington’s criticisms⁴⁶ were entirely unfounded⁴⁷ but his enormous prestige led to the acceptance of his views by many in the astronomical community, and to an early rejection of Chandrasekhar’s work. After Eddington questioned the validity of the relativistic equation of state for a degenerate electron gas, Chandrasekhar went for support to several of the great pioneers of the modern quantum theory, including Dirac who was in Cambridge, and to Bohr and Rosenfeld whom he had met during a visit at Bohr’s Institute in Copenhagen. They assured him of the validity of the relativistic equation of state,⁴⁸ and advised him to ignore Eddington’s objections,⁴⁹ but Chandrasekhar continued relentlessly to pursue this matter, writing a paper with Christian Møller on relativistic degeneracy,⁵⁰ and persuading Rudolf Peierls to give another proof⁵¹ of its validity. During this controversy, however, Chandrasekhar apparently did not mention Stoner and his earlier derivation of this equation, which is referenced neither in his paper with Møller nor in the paper by Peierls. In an appendix to the first paper⁵² in which he applied Stoner’s equation, Chandrasekhar claimed to offer a “simpler derivation” of it, but it turned out to be essentially the same as the one given by Stoner. Here Chandrasekhar gave an acknowledgement to Stoner with the remark that “this equation has been derived by Stoner (among others)”, but the “others” remain unidentified, and in reality they don’t exist. He also mentioned “that Stoner had previously made some calculations concerning the (p, ρ) relations for a degenerate gas”, neglecting to give reference to a paper by Stoner⁵³ where a derivation of this pressure–density relation and his numerical tables appeared. For several more years Stoner continued to work on the equation of state for finite temperatures, publishing extensive tables of Fermi-Dirac functions⁵⁴ which later turned out to be also very useful for improved calculations of the properties of white dwarfs.⁵⁵ During his controversy with Eddington, Chandrasekhar also did not mention Landau’s independent derivation in 1931 of the critical mass of dense stars, although by then he had met Landau during his 1934 visit to Russia where he had presented his work. Nevertheless, in

his “Historical notes”,⁵⁶ Chandrasekhar complained of “the tendency in some current literature” to give Landau priority in this discovery, and he never gave reference to Landau’s work.

Later, in his 1939 book⁵⁷ on stellar structure where he reproduced his work on white dwarfs, Chandrasekhar mentioned that the “equation for the internal energy of an electron gas”⁵⁸ was derived by E. C. Stoner, but again he neglected to refer to Stoner’s explicit derivation of the pressure–density relation, and his numerical tables for such a gas, although in 1934 he had to reproduce these tables with higher accuracy, because these tables were essential for his numerical integrations of the differential equations for gravitational equilibrium.⁵⁹ He stated that “the existence of this limiting mass was first isolated by Chandrasekhar, though its existence had been made apparent from earlier considerations by Anderson and Stoner ...”.⁶⁰ One is left wondering, however, what he meant by this assertion,⁶¹ because it was Stoner and not Chandrasekhar who first “isolated” the limiting mass by giving explicitly the dependence of this mass on natural constants.⁶² In some “Biographical Notes”⁶³ in his book, Chandrasekhar gives a reference to two of Stoner’s five papers on the properties of white dwarfs,⁶⁴ but merely comments that in these papers “Stoner makes some further applications of Fowler’s ideas”,⁶⁵ not giving the reader any idea of the important concepts and results regarding the properties of white dwarfs contained in these seminal papers. By such obfuscation, Chandrasekhar gave rise to the current neglect of Stoner’s work.

In Kamesh Wali’s excellent biography of Chandrasekhar,⁶⁶ Stoner is not mentioned even once, nor does his name appear in Spencer Weart’s transcript⁶⁷ of his lengthy interview with Chandrasekhar in 1977. More recently, in his book *The empire of the stars*, Arthur Miller remarks that “it was indeed extraordinary that a nineteen-year-old Indian youth [Chandrasekhar] had managed to make a discovery that had eluded the great minds of European astrophysics”.⁶⁸ Although Miller briefly refers to Anderson and to Stoner, he claimed that they “had never examined the ramifications”⁶⁹ of the relativistic equation of state. But as we have shown here, with respect to Stoner, Miller’s claim is incorrect. In this connection, Miller also did not mention that Landau discovered the limiting mass when he was only twenty-three years old.

According to Chandrasekhar’s account of his discovery, which he repeated on numerous occasions,⁷⁰ both Fowler and Milne were at first not interested in this result, and five years later Eddington publicly ridiculed him for engaging in “stellar buffoonery”.⁷¹ This episode has become one of the best known legends in astronomy, told to generations of students in this field. They have been given, however, only a partial historical account, because Stoner’s important role has always been passed over in silence. Actually, the early reception of the discovery of the limiting mass also appears to have been more nuanced. When Chandrasekhar arrived in Cambridge and mentioned his discovery to Fowler, in effect Fowler responded that he had been scooped by Stoner.⁷² Likewise, from references in a paper by Milne,⁷³ it is clear that Milne also was aware of Stoner’s work, because he applied it to his own theory of stellar interiors, without, however, examining the implications of relativity. Therefore

Fowler and Milne's supposed lack of interest in Chandrasekhar's account of the limiting mass may partly have been due to the fact that they did not consider it to be a novel discovery. Moreover, early on both Milne and Eddington encouraged Chandrasekhar to do further research on the white dwarf problem, while at the same time, Eddington also encouraged Stoner to work on this problem. Surprisingly, Eddington even offered to collaborate with Stoner (see Appendix), who was in Leeds, rather than with Chandrasekhar, who was at his own institute in Cambridge. Evidently, Eddington recognized that Stoner could apply the fully relativistic equation of state for a degenerate electron gas at *arbitrary densities*, while at the time Chandrasekhar could consider only the non-relativistic (low density) and extreme relativistic (infinity density) limits. This prevented Chandrasekhar from carrying out a complete analysis of the properties of white dwarfs⁷⁴ until five years after Stoner had done a comparable analysis in the uniform density approximation.

There is no evidence in his writings that Chandrasekhar understood the relationship between his mathematical approach which was based on the hydrostatic equation for gravitational equilibrium,⁷⁵ and Stoner's minimum energy principle,⁷⁶ although already in 1931 this relationship had been elucidated by Landau's independent work.⁷⁷ In 1983 Chandrasekhar was awarded the Nobel prize, but in his acceptance speech, which mainly consisted of a historical review of his work on white dwarfs, he did not include a single reference to either Stoner or Landau. This general neglect of Stoner's seminal work on white dwarfs helps explain why, with a few notable exceptions,⁷⁸ Stoner's contributions and his priority in the discovery of the maximum mass of white dwarfs have now been forgotten.

APPENDIX: EDDINGTON'S LETTER TO STONER DATED 28 FEBRUARY 1932

In light of Eddington's famous controversy with Chandrasekhar at a 1935 meeting of the Royal Astronomical Society in which Eddington quipped, without giving any reference to Stoner, that the relativistic equation of state for a degenerate electron gas "is based on a combination of relativity mechanics and non-relativity quantum theory, and I do not regard the offspring of such a union as born in lawful wedlock ...",⁷⁹ it is remarkable that three years earlier Eddington had been in communication with Stoner about this equation of state, encouraging Stoner in his work, and even suggesting that they collaborate on an investigation of the effect of this equation on stellar structure. In a letter to Stoner written on 28 February 1932 (see Figure 2), Eddington said:

I have been thinking that a combination of your work and mine would make quite definite the state of the question as to upper limits to the temperature and density of a star of given mass. This is very important, e.g. in regard to theories of subatomic energy and does not seem to be as well understood by astronomers as it might be

He then added that

I suggest that it would be very useful to tabulate $f(\rho)$ [Stoner's relativistic equation for the pressure f as a function of the density ρ] or $f(\rho)/\rho^{5/3}$, others who have written on the subject seem to consider only the two extremes of ordinary [$f(\rho) \propto \rho^{5/3}$] and relativistic degeneracy [$f(\rho) \propto \rho^{4/3}$], whereas we are actually most concerned with intermediary conditions.

By "others" Eddington evidently was referring here to the work of Milne⁸⁰ and of Chandrasekhar⁸¹ who, at the time, had been taking into account such "intermediary conditions" by a crude interpolation scheme between two density regimes where either the non-relativistic or the extreme relativistic pressure–density relations were assumed to be applicable⁸². Eddington continued:

While the critical mass may have some interest of its own, it does not affect the more fundamental questions. It is useless to suggest a theory of subatomic energy involving temperatures of 10^{11} degrees which might be possible for Sirius but could not possibly apply to Krueger 60.

We have been fairly generous in upper limits, so that (especially if there is abundance of hydrogen) the critical mass is probably much greater than the sun's.

Evidently, at the time Eddington's primary interest was the applications of Stoner's relativistic equation of state to find limits on the temperatures required for the production of *subatomic energy* in stars. The passage of his letter quoted here reveals that in 1932 Eddington had no objections to Stoner's relativistic equation of state for a degenerate electron gas, which together with Stoner's minimum energy principle implied the existence of a critical mass. Moreover, he understood that the magnitude of this critical mass depended on the inverse square of the molecular weight μ , which had generally been assumed to be equal to 2.5. Hence, one can understand his remarks that for a hydrogen star, the critical mass would "probably be much greater than the sun's", because in this case $\mu = 1$, and the critical mass would be about six times larger than the mass of the Sun.

Stoner followed Eddington's suggestions by publishing additional numerical tables of his relativistic equation of state,⁸³ and by calculating the maximum density and temperature of dense stars in the uniform density approximation for arbitrary densities and for the polytropic density distribution in the non-relativistic and extreme relativistic limits.⁸⁴ In the last of his five papers on white dwarfs, Stoner took into account the effect of radiation pressure on the equilibrium state of white dwarfs. In the introduction he reviewed his previous work:

The question of limiting densities in connection with white dwarf stars has already been discussed in a series of papers. In the first of these — the relativity effect being considered in the second — the case of a sphere of uniform density was considered. The results may be considered as giving rough upper limits for the *mean* density. In the third paper the effect of non-uniform (polytropic) density distribution was discussed, some of the conclusions being similar to those reached

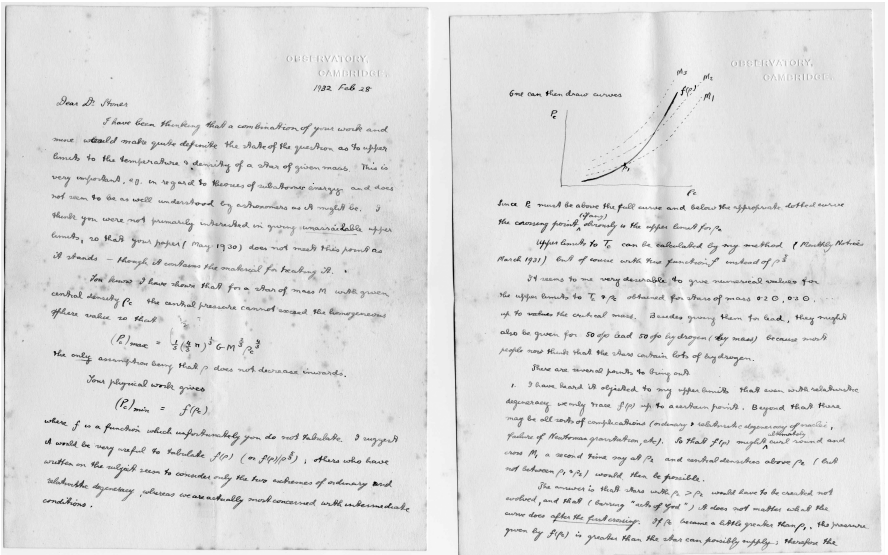


FIG. 2. Eddington’s letter of 28 February 1932 to Stoner encouraging Stoner to apply his relativistic equation of state to obtain upper limits to the density and temperature of dense stars of a given mass. In Eddington’s figure the dashed curves are plots of pressure $v.$ density to the power $4/3$ curves for different star masses M_i , $i = 1, 2, 3$, which he obtained under the assumption that the ratio of radiation and gas pressure inside a star is constant (*op. cit.* (ref. 9)). The solid curve is a sketch of Stoner’s relativistic pressure–density relation for a degenerate gas. Courtesy of the Master and Fellows of Trinity College, Cambridge, England, owners of the copyright to this letter, and the University of Leeds library, where this letter is located in the Stoner Archives, CSAC 6.73.

by Chandrasekhar at about the same time.

Here Stoner had applied an inequality, which had been published earlier by Eddington,⁸⁵ to obtain the maximum possible value of the density and the temperature of a star under the assumption that the central pressure was the sum of the pressure due to a degenerate electron gas and the pressure of radiation,⁸⁶ finding that “... the maximum values [of density and temperature] can be fixed by these considerations provided that the star has a mass *below a critical value*”, namely, the mass limit which Stoner had obtained previously in the absence of radiation.

Acknowledgements

I wish to thank Werner Israel, A. J. Miller, and K. C. Wali for useful comments and Malcolm MacGregor for helpful editorial comments.

REFERENCES

1. For example, see Freeman Dyson, “The death of a star”, *Nature*, cccxxxviii (2005), 1086.
2. L. F. Bates, “Edmund Clifton Stoner, 1899–1968”, *Biographical memoirs of Fellows of the Royal*

- Society*, xv (1969), 201–37.
3. G. Cantor, “The making of a British theoretical physicist: E. C. Stoner’s early career”, *The British journal for the history of science*, xxvii (1994), 277–90.
 4. E. C. Stoner, *Magnetism* (London, 1930). This was the first monograph containing the new quantum theory of magnetism .
 5. E. C. Stoner, “The distribution of electrons among atomic levels”, *Philosophical magazine*, xlvii (1924), 719–36.
 6. G. N. Fleming, “ The evolution of Pauli’s exclusion principle”, *Studies in history and philosophy of modern physics*, xxxviii (2007), 202–8. Fleming remarks that “in the paper [ref. 5] Stoner came within a hair’s breadth of enunciating the electron Pauli exclusion principle himself”. For an historical description of the origin of the exclusion principle and Stoner’s role in its formulation see J. L. Heilbron, “The origins of the exclusion principle”, *Historical studies in the physical sciences*, xiii (1982), 261–310. Heilbron’s perceptive comment, “of psychological interest is Pauli’s continual misstatement of the key observations he took from Stoner”, strikes a chord here, because Chandrasekhar’s continual neglect to acknowledge Stoner’s priority in the discovery of the unusual properties of white dwarfs is also the main reasons why Stoner’s contributions in this field have been forgotten.
 7. R. H. Fowler, “On dense matter”, *Monthly notices of the Royal Astronomical Society*, lxxxvii (1926), 114–22. Fowler died at the age of fifty-two, and E. A. Milne wrote that “this was the most original paper of his lifetime”, *Obituary Notices, ibid.*, cv (1945), 85. Fowler, and later Chandrasekhar, referred to the degeneracy pressure of electrons as due to Fermi-Dirac statistics, which is based on Pauli’s exclusion principle. But white dwarf calculations were done at zero temperature, and in this case quantum statistics does not play any role, and only the exclusion principle is required. Stoner’s work, *op. cit.* (ref. 5) , which led to Pauli’s formulation of the exclusion principle, indicates that R. H. Fowler’s first encounter with this principle occurred through his contact with Stoner.
 8. Actually, the density of the companion of Sirius, one of the only three white dwarfs that were known at the time, was underestimated by an order of magnitude. For the history of the early observation of a white dwarf, see J. B. Holberg and F. Wesemael, “The discovery of the companion of Sirius and its aftermath”, *Journal for the history of astronomy*, xxxviii (2007), 162–74.
 9. A. S. Eddington, *The internal constitution of the stars* (1st edn 1926; reprinted, New York, 1959), 82–83.
 10. L. H. Thomas, “The calculation of atomic fields”, *Proceedings of the Cambridge Philosophical Society*, xxiii (1927), 542–8. Thomas had been a student at Trinity College, Cambridge, where Fowler had been appointed a College Lecturer in Mathematics in 1920, but at the time Thomas wrote his paper he was visiting Bohr in Copenhagen. In the case of an atom, the forces are electrostatic which are repulsive between electrons and attractive between an electron and the nucleus. Treating the electrons as a degenerate gas, Thomas arrived at an equation equivalent to the Lane-Emde equation for a polytrope of index $3/2$, but with an opposite sign. A year later Enrico Fermi also derived the same equation which is now known as the Thomas-Fermi equation, see E. Fermi, “Über die Anwendung der statistischen Methode auf die Probleme des Atombaus”, *Falkenhagen, Quantentheorie und Chemie, Leipziger Votrageer*, 1928, 95–111. The application of the $n = 3/2$ polytrope to obtain the properties of white dwarfs for the non-relativistic equation of state for degenerate electrons was first carried out in 1930, by E. A. Milne (see ref. 21 below), who referred to Thomas’s work, and a year later by Chandrasekhar (see ref. 18 below). At about the same time, Landau also derived the extension of the Thomas-Fermi equation for the extreme relativistic equation of state of a degenerate electron gas, corresponding to the Lane-Emden $n = 3$ polytrope, and obtained the white dwarf limit, *op. cit.* (ref. 29).
 11. E. C. Stoner, “The limiting density in white dwarfs”, *Philosophical magazine*, vii (1929), 63–70.
 12. Stoner’s method for obtaining the properties of white dwarfs was based on the concept that at equilibrium, the sum of the internal energy and the gravitational energy of the star should be a *minimum* for a fixed mass of the star. Fowler had assumed that the atoms in a white dwarf were

completely ionized, and that the internal energy and pressure was entirely due to a degenerate electron gas, while the ions mainly accounted for the mass of the star. Stoner understood that as the star contracts, the gravitational energy decreases, and since the density increases, the internal energy also increases. Hence, the total energy of the star either decreases or increases during the contraction of the star. By conservation of energy, when the total energy of the star *decreases*, radiation and/or other forms of energy must be emitted by the star. But without an external source of energy, the total energy of an isolated star cannot increase. Hence the contraction of the star must end if the total energy reaches a minimum, and then the star reaches an equilibrium.

13. About 35 years ago, without being aware of Stoner's seminal work, I applied the energy minimum principle to obtain the properties of white dwarfs in the uniform density approximation, with an approximate form of the relativistic equation of state similar to Stoner's. Now I find, as expected, that my results were similar to Stoner's. Compare, for example, the mass–radius relation shown here in Fig. 1 with the corresponding Fig. 1 in M. Nauenberg, "Analytic approximations for the mass–radius relation and energy of zero-temperature stars", *The astrophysical journal*, clxxv (1972), 417–30. At the time, I sent a pre-print of my article to Chandrasekhar with a cover letter asking for his comments, but unfortunately I did not receive a response; such a response would have alerted me a long time ago about Stoner's work.
14. W. Anderson, "Über die Grenzdichte der Materie und der Energie", *Zeitschrift für Physik*, liv (1929), 851–6. Even before the appearance of Stoner's paper (see below, ref. 16), Anderson attempted to introduce the effect of special relativity on the equation of state of a degenerate electron gas, and he also speculated that the necessary high density could occur in the interior of stars, apparently without being aware of Fowler's paper (*op. cit.* (ref. 7)), see W. Anderson, "Gewöhnliche Materie und strahlende Energie als verschiedene 'Phasen' eines und desselben Grundstoffes", *Zeitschrift für Physik*, liv (1929), 433–44.
15. This relativistic equation of state for a degenerate electron gas is often called the Anderson-Stoner equation. But Anderson's relativistic analysis and his formulation of this equation given in *opera cit.* (ref. 14) is incorrect.
16. E. C. Stoner, "The equilibrium of white dwarfs", *Philosophical magazine*, ix (1930), 944–63.
17. Stoner, *op. cit.* (ref. 11).
18. S. Chandrasekhar, "The density of white dwarfs", *Philosophical magazine*, xi (1931), 592–7.
19. Eddington, *op. cit.* (ref. 9).
20. E. C. Stoner and F. Tyler, "A note on condensed stars", *Philosophical magazine*, xi (1931), 986–93. In this paper the authors did not obtain the energy minimum for the extreme relativistic equation state by taking density distribution for the $n = 3$ polytropic solution. This calculation leads to the same value for the critical mass obtained by Chandrasekhar. But Stoner's condition that the derivative of the energy with respect to the central density is zero, is satisfied because the energy itself also vanishes in this limit. These subtle mathematical issues may have been the reason why the authors did not attempt to do this calculation.
21. E. A. Milne, "The analysis of stellar structure", *Monthly notices of the Royal Astronomical Society*, xci (1930), 4–55.
22. S. Chandrasekhar, "The Ritchmyer Memorial Lecture: Some historical notes", *American journal of physics*, xxxvii (1969), 577–84. Chandrasekhar recalls: "Soon after arriving in England, I showed these results to R. H. Fowler. Fowler drew my attention to two papers by Stoner, one of which had appeared earlier that summer. In these two papers Stoner had considered the energetics of homogeneous spheres on the assumption that the Fermi-Dirac statistics prevailed in them. While Stoner's result gave some valid inequalities for the problem, he had not derived the structure of the equilibrium configurations in which all the governing equations are satisfied. Fowler, of course, appreciated this difference, and he was satisfied with detailed results pertaining to the nonrelativistic configurations. But he appeared skeptical of my result on the critical mass, and so was E. A. Milne to whom he communicated it."

Two months after Chandrasekhar arrived in Cambridge, Milne sent him a letter concerning his paper on the maximum mass of a white dwarf which Fowler had forwarded to Milne. Contrary

to Chandrasekhar's recollection that Milne appeared sceptical of his result, Milne wrote:

I have been interested in your paper, it seems to me very useful. I have long been aware that [the] relativistic form of the equation of state, under degenerate conditions would be required, but it is hopeless to construct a new theory unless one goes a step at a time and works out each case fully first. So I have deliberately carried through the complete theory of centrally-condensed and collapsed stars with $p = K\rho^{5/3}$ for an equation of state.

When my paper appears you will be able to revise my estimate using methods similar to the ones I am developing. I think then your paper might well be accepted by R.A.S. for M.N. I believe there is already very great pressure in the November number, so your paper would have to wait a little in any case.

(Milne's letter, dated 2 Nov. 1930, is in Chandrasekhar's archive at the University of Chicago.)

23. According to a widely-quoted account described in Wali's biography, *Chandra* (ref. 42), 76, during Chandrasekhar's August 1930 voyage from India to England, "it suddenly occurred to him to ask the question: If the central density is so high, will the relativistic effects be important?" But in a letter to his father dated 30 August 1929, Chandrasekhar wrote: "As far my paper I had it nearly completed, writing it out, a paper by a German, Wilhelm Anderson, appeared discussing the same problem. Even mathematically this treatment was identically to mine. So the satisfaction is that I was able to do it independently. I do not intend sending it for publication." (Wali, private communication.) In his papers, Anderson, *op. cit.* (ref. 14), attempted to introduce the effect of special relativity on the equation of state of a degenerate electron gas. Hence, already a year before arriving in Cambridge, Chandrasekhar had become aware that special relativity changed this equation of state; but Anderson's analysis was faulty and Chandrasekhar apparently made the same error in his treatment.
24. S. Chandrasekhar, "The maximum mass of ideal white dwarfs", *Astrophysical journal*, lxxiv (1931), 81–82.
25. Fowler forwarded Chandrasekhar's result to Milne, who was an astrophysicist at Oxford University. Milne, while acknowledging that Chandrasekhar had worked out the relativistic degenerate star "most beautifully", wrote to him that "the flaw in your reasoning is that you cannot prove that the solution appropriate to the outer parts of the relativistic degenerate core is Emden's solution, it may be one of the others" (Wali, *op. cit.* (ref. 42), 121). For any polytropic solution, the density decreases uniformly from the centre of the star and vanishes at its boundary. Hence, for a sufficiently large central density the extreme relativistic equation of state is a valid approximation in the core, but it would fail near the boundary where the electrons become non-relativistic. Eventually, Chandrasekhar was led to the conclusion that a consistent solution for the critical mass required that the central density become infinite, as Stoner had shown earlier in the uniform density approximation, because in this limit the envelope, where the electrons would be non-relativistic, vanishes.
26. S. Weart, *Interview with Chandrasekhar*, Niels Bohr Library, American Institute of Physics, 1977.
27. S. Chandrasekhar, "The highly collapsed configurations of a stellar mass", *Monthly notices of the Royal Astronomical Society*, xci (1931), 456–66.
28. A. Livanova, *Landau: A great physicist and teacher*, transl. by J. B. Sykes (Oxford, 1980).
29. L. D. Landau, "On the theory of stars", *Physikalische Zeitschrift der Sowjetunion*, i (1932), 285–8. Reprinted in the *Collected papers of L. D. Landau*, ed. by D. T. Haar (New York, 1965), 60–62. In this paper, Landau does not explain how he arrived at his equation for the Fermi energy, but from his later writings it is clear that his approach was similar to that of Fermi and Thomas, *op. cit.* (ref. 10), who argued that this energy plus the potential at a given radial distance is a constant.
30. Thomas, *op. cit.* (ref. 10).
31. Landau, *op. cit.* (ref. 29).
32. On the basis of the assumption that the molecular weight of white dwarfs is 2, Landau obtained 1.5 solar mass for the value of the critical mass of a white dwarf. This value differs from the accepted

value of 1.4 solar mass only because Landau underestimated the mass of the Sun by 7%. Originally, Chandrasekhar's value for the critical mass was given as 0.91 solar mass, because he had taken for the molecular weight the value 2.5 assumed by astronomers at that time.

33. Landau's paper was submitted for publication on February 1931, *before* the appearance of Chandrasekhar's paper, which was published in the July 1931 issue of the *Astrophysical journal* (see ref. 24). But in his "Historical notes", *op. cit.* (ref. 22), 583, Chandrasekhar claimed that "Landau isolated the critical mass apparently without knowledge of my results published *two years earlier* [my italics]", giving 1933 as the publication date of Landau's article, although it had appeared a year earlier. In a footnote in his biography *Chandra*, *op. cit.* (ref. 42), 121, Wali "points out" that Landau's paper was published "a year later" than Chandrasekhar's, without clarifying that there was a year's delay due to the slow publication rate of the Soviet journal where Landau's paper appeared. Adding to this confusion, in his book *Empire of the stars*, A. Miller stated that Landau made his calculations after he had returned to the Soviet Union, and that "he was unaware of Chandra's earlier work because as a physicist Landau did not read the *Astrophysical journal*", *op. cit.* (ref. 42), 158.
34. Chandrasekhar, *op. cit.* (ref. 24).
35. During a later visit with Bohr in Copenhagen, on the same day that the news of Chadwick's discovery of the neutron appeared, Landau suggested the possibility of formation of neutron stars, where the source of internal pressure was due to degenerate neutrons rather than to electrons. But he did not publish his idea until six years later, in L. Landau, "The origin of stellar energy", *Nature*, cxli (1938), 333–4.
36. By applying minimum energy principle, Stoner obtained an analytic expression that gave the mass–density relation in *parametric* form, showing that the density is a function that increases monotonically, and more rapidly than the square of the star's mass. In particular, he obtained the fundamental result that the density approaches infinity for a finite mass. This is the limiting mass of white dwarfs, in which the mass scale is entirely determined by some of the fundamental constants of Nature.
37. S. Chandrasekhar, "The highly collapsed configurations of a stellar mass (second paper)", *Monthly notices of the Royal Astronomical Society*, xcv (1935), 207–25. Chandrasekhar's numerical results are given in Table I of his paper which is reproduced as Table 25 in his book *An introduction to the study of stellar structure* (Chicago, 1939). The curve given here in Fig. 1, which is based on Stoner's 1930 *analytic* calculation (see *op. cit.* (ref. 16)), is nearly identical to Fig. 2 in Chandrasekhar's paper, reproduced as Fig. 31 in his book.
38. Milne, *op. cit.* (ref. 21).
39. Chandrasekhar, *op. cit.* (ref. 27).
40. S. Chandrasekhar, *Eddington, the most distinguished astrophysicist of his time* (Cambridge, 1983).
41. E. C. Stoner, "The minimum pressure of a degenerate electron gas", *Monthly notices of the Royal Astronomical Society*, xcii (1932), 651–61, and "Upper limits for densities and temperatures in stars", *ibid.*, 662–76. In the last of his five papers on white dwarfs, Stoner followed Eddington's suggestion to apply his relativistic equation of state for a degenerate electron gas taking into account the effect of radiation pressure on the equilibrium state of dense stars.
42. K. C. Wali, *Chandra: A biography of S. Chandrasekha* (Chicago, 1991). A. Miller, *Empire of the stars: Obsession, friendship, and betrayal in the quest for black holes* (Boston, 2005).
43. S. Chandrasekhar, "On stars, their evolution and their stability", *Nobel lectures in physics 1981–1990* (Singapore, 1995), 142–64. This paper contains a complete list of Chandrasekhar's papers on white dwarfs.
44. At a meeting of an American Physical Society at Stanford University in December 1933, Walter Baade and Fritz Zwicky made the suggestion that the origin of supernova explosions was due to the collapse of massive stars into neutron stars. But apparently they were unaware of the existence of a white dwarf mass limit, because this limit was not mentioned in the abstract of their report which appeared as a letter to the editor, "Remarks on super-novae and cosmic rays", *Physical review*, xlvi (1934), 76–77.

45. In 1939 Chandrasekhar met Eddington during dinner at high table in Cambridge and asked him: "How much of your fundamental theory depends on your ideas on relativistic degeneracy?" Eddington replied, "Why, all of it". Miller, *op. cit.* (ref. 42), 131.
46. A. S. Eddington, "On relativistic degeneracy", *Monthly notices of the Royal Astronomical Society*, xciv (1935), 194–206; "Note on relativistic degeneracy", *Monthly notices of the Royal Astronomical Society*, xcvi (1935), 20–21; "The pressure of a degenerate electron gas and related problems", *Proceedings of the Royal Society*, A clii (1935), 253–72. Eddington's re-examination of relativistic degeneracy led him to some extreme statements such as: "The Stoner-Anderson modification is fallacious ... a rigorous treatment leads to the original [Fowler] equation of state", and "The Stoner-Anderson formula does not exist", quoted in L. Mestel, "Arthur Stanley Eddington: Pioneer of stellar structure theory", *Journal of astronomical history and heritage*, vii (2004), 65–73.
47. A detailed critique of Eddington's objections to Stoner's relativistic equation of state for a degenerate electron gas can be found in E. Schatzman, *White dwarfs* (Amsterdam, 1958), 68–73.
48. Pauli, whose opinion was also requested, responded sarcastically that "Eddington did not understand physics", Wali, *op. cit.* (ref. 42), 131.
49. On 23 January 1931, replying to the umpteenth letter from Chandrasekhar, Leon Rosenfeld wrote with respect to Eddington's objection to relativistic degeneracy, "Wouldn't it be a good policy to leave him alone, instead of losing one's time and temper in fruitless arguments?", Wali, *op. cit.* (ref. 42), 130.
50. C. Møller and S. Chandrasekhar, "Relativistic degeneracy", *Monthly notices of the Royal Astronomical Society*, xciv (1935), 673–6.
51. R. Peierls, "Note on the derivation of the equation of state for a degenerate relativistic gas", *Monthly notices of the Royal Astronomical Society*, xcvi (1936), 780–4.
52. Chandrasekar, *op. cit.* (ref. 37)
53. E. C. Stoner, "The minimum pressure of a degenerate electron gas", *Monthly notices of the Royal Astronomical Society*, xcii (1932), 651–61.
54. J. McDougall and E. C. Stoner, "Computation of Fermi-Dirac Functions", *Philosophical transactions of the Royal Society of London*, A cxxxvii (1938), 67–104. During the discussion period after his presentation at the 1939 conference on astrophysics in Paris, Chandrasekhar mentioned that "one of his pupils and Dr. Stoner are working on the problem of what happens in the region in which the equation of state of degenerate matter approaches the equation of state of a perfect gas", *op. cit.* (ref. 44, 2001), 106.
55. T. D. Lee, "Hydrogen content and energy-productivity mechanism of white dwarfs", *Astrophysical journal*, cxi (1950), 625L.
56. See Chandrasekar, *op. cit.* (ref. 22), 583.
57. S. Chandrasekar, *An introduction to the study of stellar structure* (New York, 1967).
58. Chandrasekar, *op. cit.* (ref. 57), 361.
59. Chandrasekar, *op. cit.* (ref. 37).
60. Chandrasekar, *op. cit.* (ref. 57), 421.
61. I have found several other occasions when Chandrasekar used the word "isolate", which may give a clue to its meaning in the present context. For example, in his book *Eddington, the most distinguished astrophysicist of his time*, Chandrasekar stated that when Eddington calculated the relation between mass and pressure in a star, he did not "isolate" its dependence on *natural constants*, "a surprising omission in view of his later preoccupations with natural constants", *op. cit.* (ref. 40), 14. Likewise, in his 1983 Nobel speech, Chandrasekar remarked that his inequality Eq. (14) "has isolated the combination of *natural constants* of the dimension of mass", *op. cit.* (ref. 43), 144. But it was Stoner who first isolated the natural constants that determine the limiting mass of a white dwarf (Stoner, *op. cit.* (ref. 16)).
62. Stoner, *op. cit.* (ref. 16), 949–51.
63. Chandrasekar, *op. cit.* (ref. 57), 451.
64. Stoner, *opera cit.* (refs 11 and 16).

65. Chandrasekhar, *op. cit.* (ref. 57), 451.
66. Wali, *op. cit.* (ref. 42).
67. Weart, *op. cit.* (ref. 26).
68. Miller, *op. cit.* (ref. 42), 14.
69. Miller, *op. cit.* (ref. 42), 133.
70. See, for example, *opera cit.* (refs 22, 26, 40 and 42).
71. Chandrasekhar, *op. cit.* (ref. 40), 50.
72. Chandrasekhar, *op. cit.* (ref. 22).
73. Milne, *op. cit.* (ref. 21).
74. By his own admission, initially Chandrasekhar was puzzled by his result, and he was not able to show until several months later that the critical mass was a maximum, and that in this limit the density was infinite. Moreover, he did not pursue the implications of this result, and for several years he assumed that at a certain value of the density, matter would become incompressible, an idea proposed earlier by Milne to avoid infinite density at the centre of his models of a star, *op. cit.* (ref. 21). Chandrasekhar formulated this idea as follows:

We are bound to assume therefore that a stage must come beyond which the equation of state $p = K\rho^{4/3}$ is not valid, for otherwise we are led to the physically inconceivable result that for $M = .92M_s$ [M_s = solar mass and $\mu = 2.5$], $r_1 = 0$, and $\rho = \infty$. As we do not know physically what the equation of state is that we are to take, we assume for definiteness the equation for the homogeneous material $\rho = \rho_{max}$, where ρ_{max} is the maximum density of which the material is capable... [*op. cit.* (ref. 37), 463].

For $M > .92M_s$ Chandrasekhar assumed that there was a homogeneous core with $\rho = \rho_{max}$ surrounded by a relativistic envelope. This required, however, an unrealistic model of the star, where the density must become discontinuous at an interface. It was not until 1934 that he dropped these crude models, after visiting Ambartsumian in Moscow, who suggested that he integrate directly the equations for gravitational equilibrium by applying the full relativistic equation of state for a degenerate electron gas at arbitrary densities; in other words, that he apply Stoner's equation of state.

75. Eddington, *op. cit.* (ref. 9), 114.
76. It is of interest to inquire what the relation is between the minimum energy principle use by Stoner, and the equation of gravitational equilibrium used by Chandrasekhar. Treating the minimum energy principle as a variational problem in which the total energy is a functional of the density, and this density is a function of the radial distance from the centre of the star, this variational approach leads to the equation of gravitational equilibrium. Its solution is the quantum mechanical ground state of an electron gas in the gravitational field of the ions, maintaining charge neutrality. This connection explains why Stoner and Chandrasekhar obtained the same relations for the density and mass of the star as functions of fundamental constants, but with somewhat different dimensionless quantities. I have not found any evidence that either Stoner or Chandrasekhar was aware of this connection, but it is implicit in Landau's work, *op. cit.* (ref. 29).
77. Landau, *op. cit.* (ref. 29). The mathematical details can be found at <http://physics.ucsc.edu/~michael>.
78. W. Israel, "Dark stars: The evolution of an idea", in *300 years of gravity*, ed. by S. Hawking and W. Israel (Cambridge, 1987), 199–276; and "Imploding stars, shifting continents, and the inconstancy of matter", *Foundations of physics*, xxvi (1996), 595–626. Also, *A source book in astronomy and astrophysics 1900–1975*, edited by K.R. Lang and O. Gingerich (Cambridge, MA, 1979), 433–44. Fred Hoyle credited Stoner with the discovery of the white dwarf mass limit (Wali, private communication). His former student, Leon Mestel, also mentioned Stoner and Anderson in connection with this limit, see L. Mestel, "The theory of white dwarfs", *Monthly notices of the Royal Astronomical Society*, cxii (1952), 583–97.
79. "Discussion of papers 4 and 5 by A. S. Eddington and E. A. Milne", *The observatory*, lviii (1935),

37–39.

80. Milne, *op. cit.* (ref. 21).
81. Chandrasekhar, *op. cit.* (ref. 27).
82. As late as 1934, Chandrasekhar still thought that the only “possible equations of state” for a degenerate electron gas were either the non-relativistic or the extreme relativistic forms of Stoner’s exact equation of state, see S. Chandrasekhar, “The physical state of matter in the interior of stars”, *The observatory*, lvii (1934), 93–99.
83. Stoner, “The minimum pressure ...” (ref. 41).
84. Stoner, “Upper limits ...” (ref. 41).
85. A. S. Eddington, “Upper limits to the central temperature and density of a star”, *Monthly notices of the Royal Astronomical Society*, xci (1931), 444–6.
86. For a given star mass M and central density ρ_c , Eddington had shown that the central pressure P must be *less* than the central pressure of a star with *uniform* density ρ_c , see *op. cit.* (ref. 85). Eddington’s theorem is the inequality
- $$P_c < \frac{1}{2}(4\pi/3)^{1/3}GM^{2/3}\rho_c^{4/3},$$
- which is mentioned in his letter to Stoner (see Fig. 2), but with the first factor on the right hand side of this inequality given incorrectly as $1/3$.