PHYSICS 112
Homework 1
Due in class, Tuesday January 17.

Note: I will be out of town on Thursday January 19, and the lecture on that day will be given by Professor Josh Deutsch.

In the questions I use conventional units for temperature T and entropy S. If you prefer, you can use Kittel's temperature and entropy, τ and σ respectively, which are related to the conventional ones by $\tau = k_B T, \sigma = S/k_B$, where k_B is Boltzmann's constant.

1. **Entropy and Temperature**
   Suppose the number of states of energy U is given by $g(U) = CU^{3N/2}$, where C is a constant and N is the number of particles.

   (a) Show that $U = \frac{3}{2}Nk_BT$.
   (b) Show that $(\partial^2 S/\partial U^2)_N$ is negative.

   Note: This expression for $g(U)$ actually applies to a classical ideal gas.

2. **Paramagnetism**
   Consider the toy model of N non-interacting "Ising" spins in a magnetic file B, discussed in class. Each spin has a magnetic moment $\mu_i = \mu S_i$, where S_i takes values ± 1. The *total* magnetization, M, is equal to $\sum_i \mu_i$, and the difference between the number of up spins and the number of down spins, i.e. $\sum_i S_i$, is equal to $2l$. The energy is given by

$$U = -MB = -B\sum_i \mu_i = -2\mu l B,$$

   Take the entropy, the logarithm of the multiplicity, to be that given in the book

$$\frac{S}{k_B} = \sigma(l) \simeq \log g(N,0) - 2l^2/N,$$

   for $l \ll N$.

   Using the result that the temperature is given by $1/T = (\partial S/\partial U)_N$, find the magnetization per spin m, where

$$m \equiv \frac{M}{N} = \frac{1}{N}\sum_{i=1}^{N} \mu_i,$$

   as a function of T and B.
   Note: The result you obtain, which is known as Curie's law, will be rederived later in the course in a simpler way which does not involve the multiplicity function.

3. **Quantum Harmonic Oscillator** Consider a set of N oscillators of frequency ω. Assume that the total quantum number is fixed to be n, i.e.

$$n_1 + n_2 + \cdots + n_N = n\,.$$

The multiplicity function, i.e. the number of (ordered) ways in which N non-negative integers can add up to n, is given by

$$g(N,n) = \frac{1}{n!}\,N(N+1)+\cdots(N+n-1) = \frac{(N+n-1)!}{n!\,(N-1)!}\,.$$

This result is not obvious but a derivation is given in Kittel and Kroemer p. 25.

(a) Using this expression for g, determine the entropy of the oscillators assuming N and n are large (so you can use Stirling's approximation $\log N! \simeq N\log N - N$, and also replace $N-1$ by N).

(b) Let U denote the total energy $n\hbar\omega$ of the oscillators. Express the entropy as $S(U,N)$. Using $1/T = (\partial S/\partial U)_N$ show that the total energy at temperature T is

$$U = N\,\frac{\hbar\omega}{\exp(\hbar\omega/k_BT)-1}\,.$$

Note: This is the Planck result which we will rederive later in the course by a more powerful method which is much simpler because it does not require us to find the multiplicity function.

4. **The meaning of "never"**

It has been said that "six monkeys set to strum unintelligently on typewriters for millions of years would be bound in time to produce all the books in the British Museum." This statement is nonsense, for it gives a misleading conclusion about very, very large numbers. Could all the monkeys in the world have typed out even a single specified book in the age of the universe?

Suppose that 10^{10} monkeys have been seated at typewriters for the age of the universe, 10^{18} s. Suppose that a monkey can hit 10 keys per second on the typewriter per second. Assume the typewriter has 44 keys (we won't distinguish between upper and lower case). Assume that Shakespeare's *Hamlet* has 10^5 characters.

Will the monkeys hit upon *Hamlet*?

(a) Show that the probability that any given sequence of 10^5 characters typed at random will come out in the correct sequence (the sequence of *Hamlet*) is

$$\left(\frac{1}{44}\right)^{100{,}000} \simeq 10^{-164{,}345},$$

where we used that $\log_{10} 44 = 1.64345$.

(b) Hence show that the probability that one of the monkeys will type Hamlet in the age of the universe is about $10^{-164321}$.

Note that multiplying by the number of sequences that are typed during the age of the universe makes only a small difference to the power of 10 in the probablity. The probability is *zero* in any operational sense, so the statement made at the beginning of this question is nonsense: one book, much less an entire library, will *never* occur in the total literary production of the monkeys.

(c) Would the monkeys even produce "to be or not to be" (18 characters including spaces)?