PHYSICS 112
Homework 2
Due in class, Thursday January 26

Note:

- The due day is Thursday. (This is because we are a little behind with the material. I expect to be on a Tuesday homework schedule soon.)
- Typed homework solutions will be available on the class website after the work has been handed it. This site is password protected. The username is 112. I will announce the password in class.
- I am intending to have the midterm in class on Thursday Feb. 9. If there is a potential conflict please let me know.

1. **Free energy of a 2-state system**
   Consider system with just two states, which is the simplest model one can study in statistical physics. Take the two states to have energies 0 and ϵ.
   (a) Find the free energy as a function of temperature.
   (b) From the free energy determine the energy and entropy.
   (c) Determine the heat capacity (also known as the specific heat) at constant volume and sketch it as a function of temperature.

2. **Magnetization**
   Consider the model system of N non-interacting magnetic moments discussed in class and in the book. The energy of magnetic moment i is $-\mu_i B = \pm\mu B$ where $\mu_i = \pm\mu$ and μ is the size of the magnetic moment.
   (a) Use the partition function to determine the magnetization M *exactly* as a function of temperature and magnetic field.
   (b) For $\mu B \ll k_B T$ show that
   $$M = N\mu^2 \frac{B}{k_B T} \, .$$
   Note: You derived this by a more complicated method, using results appropriate for a closed system rather than a system in thermal contact with a reservoir as here, in Qu. 2, HW 1.

3. **Free energy of a simple harmonic oscillator**
   A simple harmonic oscillator has energies $\epsilon_n = n\hbar\omega$, where $n = 0, 1, 2, \cdots$ (neglecting the zero-point energy of $(1/2)\hbar\omega$ which is not important here).
   (a) Show that the free energy is equal to
   $$F = k_B T \log\left[1 - \exp(-\hbar\omega/k_B T)\right] \, .$$
   (b) From this expression, show that the entropy is given by
   $$\sigma \equiv \frac{S}{k_B} = \frac{(\hbar\omega/k_B T)}{\exp(\hbar\omega/k_B T) - 1} - \log\left[1 - \exp(-\hbar\omega/k_B T)\right] \, .$$
   There are plots of the entropy and specific heat in Kittel and Kroemer, Figs. 3.13 and 3.14.

4. **Energy fluctuations**
Consider a system in thermal contact with a reservoir at temperature T. Show that the mean square fluctuation in the energy of the system is given by

$$(\Delta E)^2 \equiv \langle (E - \langle E \rangle)^2 \rangle = \langle E^2 \rangle - \langle E \rangle^2 = k_B T^2 \left(\frac{\partial U}{\partial T} \right)_V , \tag{1}$$

where U means the average energy, i.e. $\langle E \rangle$.
Hint: It is simplest to start off with an expression for U involving sums over states, and show that, differentiating this with respect to T, the right hand side of Eq. (1) gives the energy fluctuation.

5. **Rotation of diatomic molecules**
Diatomic molecules can rotate which gives a contribution to the energy in addition to that from translational motion. According to quantum mechanics the energy is quantized and given by

$$\epsilon_j = j(j+1)\epsilon_0$$

where $j = 0, 1, 2, \cdots$ is the angular momentum quantum number, and ϵ_0 is related to the moment of inertia and Planck's constant. The multiplicity (degeneracy) of each level is $2j + 1$.

   (a) Write down an expression for the partition function as a sum over j.
   Note: It is important to understand that the sum in the partition function is over all *states*, not energies, and the degeneracy matters.
   (b) Evaluate the partition function approximately for high temperatures, $k_B T \gg \epsilon_0$, by replacing the sum by an integral.
   (c) Evaluate the partition function approximately for low temperatures, $k_B T \ll \epsilon_0$, by just considering the first two terms in the sum.
   (d) Determine the energy U and heat capacity C in these two limits.
   Note: You should find that C approaches k_B at high temperatures (unity in Kittel's units).
   (e) Sketch the behavior of U and C as a function of temperature, indicating the limiting behaviors for $T \to 0$ and $T \to \infty$.

6. **Zipper problem**
A zipper has N links. Each link is either open with energy ϵ or closed with energy 0. We require, however, that the zipper can only open from the left end, and that link l can only unzip if all links to the left $(1, 2, \cdots, l-1)$ are already open.

   (a) Show that the partition function can be summed to give

$$Z = \frac{1 - \exp\left[-(N+1)\epsilon/k_B T\right]}{1 - \exp(-\epsilon/k_B T)} .$$

   (b) Find the average number of open links in the limit $\epsilon \gg k_B T$.

The model is a very simplified model of the unwinding of two-stranded DNA molecules—see C. Kittel, Amer. J. Physics, 917 (1969).

7. **Partition function and free energy of two independent systems**

   (a) Show that the total partition function Z_{tot} of two independent systems, "1" and "2", with partition functions Z_1 and Z_2, is given by the *product*

$$Z_{\text{tot}} = Z_1 \, Z_2 .$$

   (b) Show that the total free energy is the *sum* of the free energies of the two independent systems.