PHYSICS 112
Homework 3

Due in class, Tuesday January 31. Note: we are now on a Tuesday schedule.

- **Date of MIDTERM:** Thursday Feb 9, in class. The topics to be included will be given later.
- **My office hours:** Mondays, 1:00 - 2:00, ISB 212, and at other times by appointment.
- **TA's office hours:** Fridays, 2:30 - 3:30, ISB 262.

1. **Density of States in Two-Dimensions**
   Consider the problem of finding the (single particle) density of states $\rho(\epsilon)$ (where ϵ is the energy of a single particle) of non-interacting particles in a box of size L in each direction. Consider, however, the case of two dimensions, rather than three dimensions discussed in class. Determine $\rho(\epsilon)$ in two dimensions for the following cases:
   (a) Electrons, where $\epsilon = \hbar^2 k^2/2m$, where k is the wavevector of the plane wave state. (Remember an electron has two spin states.)
   (b) Photons, where $\epsilon \equiv \hbar\omega = \hbar ck$. (Remember there are two polarizations for the photon.)

2. **Number of Thermal Photons**
   Show that the number of of photons, $\sum_j \langle n_j \rangle$, at equilibrium in a cavity of volume V at temperature T is
   $$\mathcal{N} = \frac{2.404}{\pi^2} V \left(\frac{k_B T}{\hbar c}\right)^3 .$$
   Hint: You will need the following integral:
   $$\int_0^\infty \frac{x^2}{e^x - 1}\, dx = 2.404\ldots$$
   Note: It is shown in the book (Eq. (23)) that the entropy depends on V and T in the same way. Hence the ratio $\sigma/\mathcal{N}$ (where σ is the dimensionless entropy discussed in the book) is just a number, given by
   $$\frac{\sigma}{\mathcal{N}} \simeq 3.602.$$
   The number of photons in the universe is about 10^8 times larger than the number of nucleons (protons and neutrons). The entropy per nucleon is also a number of order unity, and so the entropy of the universe is dominated by that of the photons.

3. **Surface Temperature of the Sun**
   The value of the total radiant energy flux density at the Earth from the Sun is called the *solar constant* of the Earth. The observed value is
   $$\text{solar constant} = 0.136\,\mathrm{J\,s^{-1}\,cm^{-2}}\,.$$
   Take the distance of the Earth from the Sun to be 1.5×10^{13}cm and the radius of the Sun to be 7×10^{10} cm.
   (a) Show that the total rate of energy generation of the Sun is $4 \times 10^{26}\,\mathrm{J\,s^{-1}}$.

(b) From this result, and the value of the Stefan-Boltzmann constant, $\sigma_B = 5.67 \times 10^{-12}\,\mathrm{J\,s^{-1}\,cm^{-2}\,K^{-4}}$, show that the temperature of the Sun treated as a black body, is $\simeq 6000\,\mathrm{K}$.
Note: The radiant energy flux, which is the rate of energy emission per unit area from a black body, is given by

$$J = \sigma_B T^4,$$

where

$$\sigma_B = \frac{\pi^2}{60}\frac{k_B^4}{\hbar^3 c^2}\,.$$

4. **Pressure of Thermal Radiation**

(a) For a photon gas show that the pressure can be written as

$$P = -\left(\frac{\partial U}{\partial V}\right)_S = -\sum_j n_j\,\hbar\,\frac{d\omega_j}{dV}\,,$$

where n_j is the number of photons in mode j.

(b) Show also that

$$\frac{d\omega_j}{dV} = -\frac{\omega_j}{3V}\,.$$

(c) Hence show that

$$P = \frac{U}{3V}\,.$$

5. **Free Energy of a Photon Gas**

(a) Show that the partition function of a photon gas is given by

$$Z = \prod_j \left[1 - \exp(-\hbar\omega_j/k_B T)\right]^{-1}\,, \tag{1}$$

where the product is over all the modes j.

(b) The free energy is found from Eq. (1) to be

$$F = k_B T \sum_j \log\left[1 - \exp(-\hbar\omega_j/k_B T)\right].$$

Transform the sum to an integral, integrate by parts, use the result that

$$\int_0^\infty \frac{x^3}{e^x - 1}\,dx = \frac{\pi^4}{15}\,,$$

and show that

$$F = -\frac{\pi^2}{45}\frac{V(k_B T)^4}{\hbar^3 c^3}\,.$$

6. **Heat capacity of photons and phonons**
Consider a solid with a Debye temperature equal to 100 K and with 10^{22} atoms cm^{-3}. Estimate the temperature at which the photon contribution to the heat capacity would equal to the phonon contribution at 1 K.

Note: Use the Debye approximation for the specific heat of the phonons. Note that $C \propto T^3$ for *both* photons and phonons because they have the same dispersion relation $\epsilon \equiv \hbar\omega = \hbar c k$, where c

is the speed of the wave. However, the speed of light is much greater than the speed of sound in a solid, which accounts for your answer being different from 1K. In fact Eq. (46), (in Kittel and Kroemer, 2nd Ed.), which gives the phonon energy in the Debye theory at low T, is equivalent to the Stefan-Boltzmann law, Eq. (20). This can be seen by using the relation between the Debye temperature θ_D, and the sound speed and atomic density, given in Eq. (44). The expression for phonons is 1.5 times as big because, for each wavevector, there are three polarizations for sound (one longitudinal and two transverse) whereas light only has two polarizations (there is no longitudinally polarized light wave).

7. **Entropy and Occupancy**
   For one photon mode of frequency ω show that the entropy can be expressed as a function of the occupancy, $\langle n\rangle = (\exp(\beta\hbar\omega) - 1)^{-1}$, as follows:
   $$\sigma \equiv \frac{s}{k_B} = [\langle n\rangle + 1] \log [\langle n\rangle + 1] - \langle n\rangle \log\langle n\rangle.$$
   Hint: Start from the partition function.

8. **Cosmic Microwave Background Radiation**
   Consider a gas of photons in volume V at temperature T. Let the volume expand adiabatically. From the expression for the entropy in the book
   $$S = \frac{4\pi^2}{45} V \left(\frac{k_B^4 T^3}{\hbar^3 c^3}\right),$$
   it follows that VT^3 stays constant during the expansion.

   The observed cosmic microwave background radiation has a black-body spectrum with a temperature 2.73 K. This arose because, in the early universe, radiation became thermally decoupled from the matter when both were at a temperature of about 3000 K.

   (a) What was the radius of the universe then compared with its value now?

   (b) If the radius has increased linearly with time (not quite correct because recent observations have shown that the expansion of the universe is accelerating), at what fraction of the present age of the universe did the decoupling take place?