PHYSICS 112
Homework 6
Due in class, Tuesday February 21

1. Consider a process taking place in a classical ideal gas at constant entropy.

   (a) Show that during this process
   $$C_V dT + P dV = 0.$$

   (b) Hence show that
   $$\frac{dT}{T} + (\gamma - 1)\frac{dV}{V} = 0,$$

   where $\gamma = C_P/C_V$.
   Note: We showed in class that $C_P - C_V = Nk_B$.

   (c) Hence show that
   $$TV^{\gamma-1} = \text{const.}$$

   even if there are internal degrees of freedom which lead to C_V being different from $(3/2)Nk_B$. Also show that
   $$PV^\gamma = \text{const.}$$

   (d) The isentropic and isothermal bulk moduli are defined by

   $$B_S = -V\left(\frac{\partial P}{\partial V}\right)_S ; \qquad B_T = -V\left(\frac{\partial P}{\partial V}\right)_T .$$

   Show that for a classical ideal gas

   $$B_S = \gamma P ; \qquad B_T = P .$$

   Note: The velocity of sound in a gas is given by $v = (B_S/\rho)^{1/2}$ (where ρ is the density). The *adiabatic* bulk modulus appears because there is not enough time during one cycle of the wave, for heat to enter or leave a region of the size of a wavelength. If the molecules have mass M, we have $P = \rho k_B T/M$, so the speed of sound of an ideal gas is $v = (\gamma k_B T/M)^{1/2}$. Apart from the numerical factor this is also the characteristic speed of *single molecules* in the gas.

2. **Classical ideal gas in two dimensions**
   Consider a classical ideal gas in two dimensions. There are N spinless atoms in area A at temperature T.

   (a) Determine the chemical potential.
   (b) Determine the energy U.
   (c) Determine the entropy.

   Note: The density of states of a particle in two dimensions was worked out in Qu. 1 of HW3. (Assume here that the particles are spinless so divide the result of that question by 2.)

3. **Ideal Gas Calculations**
   Consider one mole of an ideal monatomic gas at 300 K and 1 atm.

(a) First let the gas expand isothermally and reversibly to twice the initial volume. Follow this by an isentropic expansion from twice to four times the initial volume. How much heat (in Joules) is added to the gas in each of these two processes?

(b) What is the temperature at the end of the second process?

(c) Now suppose that first process is replaced by an irreversible expansion into the vacuum, to a volume of twice the initial volume. What is the increase in entropy in the irreversible expansion in joules per kelvin?

4. **Mean speeds in a Maxwellian distribution**

The Maxwell distribution for the speed of molecules in a gas is

$$P(v) = 4\pi \left(\frac{m}{2\pi k_B T} \right)^{3/2} v^2 \exp\left(-\frac{mv^2}{2k_B T} \right),$$

where m is the mass of the molecules.

(a) Show that the root mean square velocity, $v_{\rm rms} \equiv \langle v^2 \rangle^{1/2}$, is given by

$$v_{\rm rms} = \left(\frac{3k_B T}{m} \right)^{1/2},$$

(b) Show that the most probable speed is given by

$$v_{\rm mp} = \left(\frac{2k_B T}{m} \right)^{1/2}.$$

(c) Show that the mean speed is given by

$$\langle v \rangle = \left(\frac{8k_B T}{\pi m} \right)^{1/2}.$$

(d) We showed in class that the distribution of each component of the velocity is given by

$$P_z(v_z) = \left(\frac{m}{2\pi k_B T} \right)^{1/2} \exp\left(-\frac{mv_z^2}{2k_B T} \right).$$

Show that the average of the magnitude of v_z is given by

$$\langle |v_z| \rangle = \left(\frac{2k_B T}{\pi m} \right)^{1/2}.$$

5. **Energy of a relativistic Fermi gas**

For electrons with energy ϵ much greater than mc^2, where m is the rest mass, the energy is given by $\epsilon = pc = \hbar ck$, where p is the momentum and k the wavevector. Since this is the same dispersion relation as for photons, the density of states is the same as for photons.

(a) Show that in this extreme relativistic limit the value of the Fermi energy (the chemical potential in the limit $T \to 0$) is given by

$$\epsilon_F = \pi^{2/3} \hbar c \, (3n)^{1/3},$$

where $n = N/V$.

(b) Show that the ground state energy is given by

$$U_0 = \tfrac{3}{4} N \epsilon_F .$$

6. **Pressure of a degenerate Fermi gas**

   Show that the pressure of a (non-relativistic) Fermi gas at $T = 0$ is given by

$$P = \frac{(3\pi^2)^{2/3}}{5} \frac{\hbar^2}{m} n^{5/3} ,$$

   where $n = N/V$.