PHYSICS 112
Homework 7
Due in class, Tuesday Feb. 28.

1. **Density of States in one dimension**
   Show that the density of states for free electrons in one dimension is given by

$$\rho(\epsilon) = \frac{L}{\pi} \left(\frac{2m}{\hbar^2} \right)^{1/2} \epsilon^{-1/2} \, .$$

2. **Chemical Potential versus Temperature**
   Explain graphically why the initial curvature of μ versus T is upward for a Fermi gas in one dimension, but downward in three dimensions (see Fig. 7.7 in Kittel and Kroemer).
   Hint: The expressions for the single particle density of states in one and three dimensions are different. For one dimension, the result is given in the previous question; for three dimensions it has been extensively discussed in class.

   Also sketch the variation of $\mu(T)$ with T for three dimensions, and one dimension. You should show a full range of temperature including the classical regime. (You should recall what is the sign of $\mu(T)$, in any dimension, in the classical regime.)

3. **$\boldsymbol{\mu}$(T) for a 2-dimensional Fermi gas.**
   We have seen that in dealing with identical quantum mechanical particles we use the Gibbs distribution (Grand Canonical Ensemble) in which we introduce a chemical potential $\mu(T)$, which depends on T and which is adjusted to get the correct particle density $n \equiv N/V$.

   In general, it is quite complicated to determine the detailed temperature dependence of $\mu(T)$ and we usually have to resort to approximate expressions valid in limited ranges of T or numerical computation. For example, at very high temperatures we are in the classical limit where we showed that, for three-dimensions,

$$\mu(T) = -k_B T \ln \left[\frac{1}{n} \left(\frac{m k_B T}{2\pi\hbar^2} \right)^{3/2} \right] . \tag{1}$$

   This is for spinless particles. For particles of spin-S the argument of the log is multiplied by $2S+1$ as shown in class.

   A case where we can get an analytic expression for $\mu(T)$ over the *whole* range of T, including the low-T *quantum* regime is the *two-dimensional* Fermi gas. The reason that this is simpler than the three-dimensional case is that the single-particle density of states is a *constant.* In fact we showed in Qu. 1 of HW 3 that (for $S = 1/2$)

$$\rho(\epsilon) = A \, \frac{m}{\pi\hbar^2} \, ,$$

   where A is the area of the system. In this question you will **assume** this expression for $\rho(\epsilon)$ and use it to determine a closed form expression for $\mu(T)$ for electrons.

   (a) Consider $T = 0$. The value of μ at $T = 0$ is called the Fermi energy, ϵ_F. Show that ϵ_F is given by

$$\epsilon_F = n \, \frac{\pi\hbar^2}{m} \, , \tag{2}$$

where, since we are in two-dimensions, we define n to be the number of electrons per unit *area*:

$$n = \frac{N}{A}\,.$$

(b) Now consider finite-T. Show that the expression for the total number of particles

$$N = \int_0^\infty \rho(\epsilon) f(\epsilon)\, d\epsilon\,,$$

where $f(\epsilon)$ is the Fermi function, can be written as

$$\epsilon_F = \int_0^\infty \frac{d\epsilon}{e^{\beta(\epsilon-\mu)}+1}\,,$$

which is an *implicit* expression for $\mu(T)$. However, we can do the integral and thereby obtain an *explicit* expression for $\mu(T)$. Show that the integral is equal to

$$\frac{1}{\beta}\ln\left(1+e^{\beta\mu}\right)\,.$$

Hint: Write the integrand as $e^{-\beta(\mu-\epsilon)}/[1+e^{-\beta(\mu-\epsilon)}]$.
Hence show that

$$\mu(T) = k_B T \ln\left(e^{\epsilon_F/k_B T} - 1\right)\,,$$

and that this can be expressed as

$$\boxed{\frac{\mu(T)}{\epsilon_F} = t\ln\left(e^{1/t}-1\right)} \tag{3}$$

where

$$t = \frac{T}{T_F}\,,$$

in which $T_F = \epsilon_F/k_B$ is called the Fermi temperature.

(c) Show that for $t \ll 1$

$$\mu(T) = \epsilon_F\,,$$

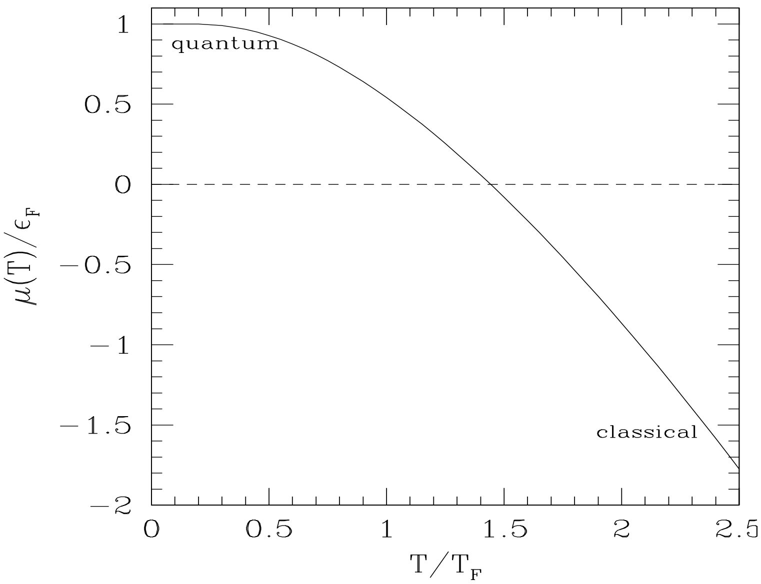
as expected.

(d) In Qu. 2 of HW 6 we showed that for the classical ideal gas in two-dimensions

$$\mu(T) = -k_B T \ln\left[\frac{1}{n}\left(\frac{mk_BT}{\pi\hbar^2}\right)\right]\,, \tag{4}$$

which is the 2-d version of Eq. (1), and we have included a factor of 2 in the log for the spin multiplicity of electrons. Show that Eq. (3) reduces to Eq. (4) at high temperatures.

A plot of $\mu(T)$ in two-dimensions according to Eq. (3) is shown below:



In three-dimensions, the variation of μ with T is qualitatively the same.

4. **Absence of Bose-Einstein condensation in one-dimension.**
We have seen that the maximum number of bosons , $N_e(T)$, that can be put in the $\epsilon > 0$ states in three-dimensions is *finite* and tends to zero for $T \to 0$. This implies that at low-T, there must be a *macroscopic* occupation of the $\epsilon = 0$ state, which is called Bose-Einstein condensation. $N_e(T)$ is obtained by setting $\mu \to 0^-$, i.e.

$$N_e(T) = \int_0^\infty \frac{\rho(\epsilon)}{e^{\epsilon/k_B T} - 1} \, d\epsilon \,, \tag{5}$$

where $\rho(\epsilon)$ is the single-particle density of states. At $T = T_{BE}$, the Bose-Einstein condensation temperature, $N_e(T_{BE}) = N$, the actual number of particles, and at higher temperatures, all the the particles can be accommodated without a macroscopic occupation of the $\epsilon = 0$ state. This requires $\mu(T) < 0$ for $T > T_{BE}$, as discussed in detail in a handout.

Now consider the case of one-dimension, for which we have showed that

$$\rho(\epsilon) = \frac{L}{\pi} \sqrt{\frac{2m}{\hbar^2}} \, \epsilon^{-1/2} \,. \tag{6}$$

This is for spin-0 particles and is half the expression given before for electrons.

Show that with $\rho(\epsilon)$ given by Eq. (6) $N_e(T)$ is *infinite.* To do this you should set the lower limit of the integral in Eq. (5) to be some small value $\epsilon_{\rm min}$ and consider the limit $\epsilon_{\rm min} \to 0$.

Note: This means that any number of particles can be placed in the $\epsilon \neq 0$ states at any T and hence *there is no Bose-Einstein condensation in a Bose gas in one-dimension.* One might ask whether interactions between the particles (neglected in our treatment of the ideal Bose gas) could change this result. It turns out that the answer is no. There is a rigorous result that there is no Bose-Einstein condensation in one-dimension.

5. **Energy, heat capacity, and entropy of a degenerate Bose gas**
Find expressions for the energy, heat capacity, and entropy as a function of T for N non-interacting bosons of spin zero in a volume V for temperatures below the Bose-Einstein condensation temperature T_E. Put the definite integrals in dimensionless form; they need not be evaluated.

6. **Number Fluctuations in Bose and Fermi gases**

   (a) For a single fermion state show that

   $$\langle (\Delta n)^2 \rangle = \langle n \rangle \left(1 - \langle n \rangle\right) .$$

   Note that the fluctuation vanishes both when the mean occupancy is 1 and 0.

   (b) For a single boson state show that

   $$\langle (\Delta n)^2 \rangle = \langle n \rangle \left(1 + \langle n \rangle\right) .$$

   Note that when the mean occupancy $\langle n \rangle$ is much greater than unity, the rms fluctuation, $\langle (\Delta n)^2 \rangle^{1/2}$ is approximately equal to $\langle n \rangle$, which is very large (and not of order $\langle n \rangle^{1/2}$ as might naively have been expected).

7. **Heat Pump**
   Consider a heat pump in which we heat a building by absorbing heat from the outside at temperature $T_{\rm low}$ and emitting it into the building at a higher temperature $T_{\rm high}$. Show that for a reversible heat pump the ratio of the amount of work, W, which must be supplied, to the heat extracted into the building, $Q_{\rm high}$, is given by the *Carnot efficiency*

   $$\frac{W}{Q_{\rm high}} = 1 - \frac{T_{\rm low}}{T_{\rm high}} .$$

   Note: In practice, $(T_{\rm high} - T_{\rm low})/T_{\rm high} \ll 1$, so the amount of heat extracted and pumped into the building can be much *greater* than the work which must be supplied, so potentially heat pumps could be very efficient for heating buildings, Unfortunately, they are currently not economical, see Kittel and Kroemer p. 236.