

PHYSICS 112

Homework 8

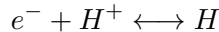
Due in class, Tuesday March 6

There will be one more homework.

Note that **the Final Exam** will be on Tuesday, March 20, 12:00–3:00 pm.

1. Ionization of the hydrogen atom

- (a) Consider the formation of atomic hydrogen from a proton and an electron, i.e. according to the reaction



Show that the equilibrium concentration of the reactants is

$$\frac{n_{e^-} n_{H^+}}{n_H} = n_{Q,e} \exp\left(-\frac{\Delta E}{k_B T}\right), \quad (1)$$

where

$$n_{Q,e} = \left(\frac{mk_B T}{2\pi\hbar^2}\right)^{3/2}$$

i.e. $n_{Q,e} \equiv 1/V_{Q,e}$ in which $V_{Q,e}$ is the quantum volume of the electrons, and ΔE is the ionization energy.

Note: Just consider the ground state of the hydrogen atom and the ionized state; ignore higher excited (bound) states of the atom. Neglect the spin of the particles, which does not affect the final answer. Equation (1) is called the **Saha equation**.

- (b) Note that if all the electrons and protons arise from the ionization of hydrogen atoms, we have $n_{e^-} = n_{H^+}$ and so

$$n_{e^-} = n_H^{1/2} n_{Q,e}^{1/2} \exp\left(-\frac{\Delta E}{2k_B T}\right). \quad (2)$$

Note that:

- i. The exponent in Eq. (2) involves $\Delta E/2$, not ΔE , so there's more to it than just the Boltzmann factor.
- ii. The electron concentration in Eq. (2) is proportional to the square root of the concentration of neutral hydrogen atoms.
- iii. If we add extra electrons, so they are no longer all due to the ionization of hydrogen atoms, then we have to go back to Eq. (1) which shows that the concentration of protons actually *goes down*.

Hence show that the temperature T_i at which half the hydrogen atoms are ionized is given by

$$\exp\left(-\frac{\Delta E}{k_B T_i}\right) = \frac{n_H}{n_{Q,e}}. \quad (3)$$

Note: You might have thought that the condition would be $e^{-\beta_i \Delta E} \simeq 1$, or equivalently $k_B T_i \simeq \Delta E$, but Eq. (3) also involves the density. Furthermore, if the density is low, i.e. $n_H/n_{Q,e} \ll 1$, then T_i is *much less* than $\Delta E/k_B$. The reason is that, at low densities, the ionized state has much more *entropy* than the bound atom, so the probability that the atom ionizes is much greater than the Boltzmann factor $\exp(-\beta \Delta E)$, which only includes

the effects of the *energy*. This result is important, for example, in the early universe where the temperature at which electrons and protons combined to form neutral hydrogen (and allow the thermal radiation, now observable as cosmic background radiation, to decouple from matter) is about 3000 K, far less than the ionization energy (13.6 eV) divided by k_B which is about 150,000 K.

2. Particle-antiparticle equilibrium

Consider the reaction of a particle with its anti-particle

$$A^+ + A^- = 0,$$

where A^+ and A^- may, for example be (i) electrons and positrons, or (ii) electrons and holes in a semiconductor. Denote the energy to create a particle-antiparticle pair by Δ . For the electron-position case, $\Delta = 2mc^2$, where m is the electron mass. For the case of case of electrons and holes in a semiconductor, Δ would be a property of the material. (For simplicity neglect the spin of the particles.)

- (a) Calculate the equilibrium density $n = n^+ = n^-$ (assuming that the densities of particles and antiparticles are equal).

Note: it is convenient to distribute the minimum energy of the particle-antiparticle pair (ΔE) equally between the particle and hole (i.e. half each).

- (b) Estimate the density of electrons and positrons produced thermally at the center of the sun, assuming that $T = 1.5 \times 10^7$ K.

Note: You are are given that the mass of the electron is 0.51 Mev, and $1 \text{ ev} \equiv 11,600 \text{ K}$.

3. Thermal expansion near absolute zero.

- (a) Prove the following three Maxwell relations

$$\left(\frac{\partial V}{\partial T}\right)_P = -\left(\frac{\partial S}{\partial P}\right)_T, \quad (4a)$$

$$\left(\frac{\partial V}{\partial N}\right)_P = +\left(\frac{\partial \mu}{\partial P}\right)_N, \quad (4b)$$

$$\left(\frac{\partial \mu}{\partial T}\right)_N = -\left(\frac{\partial S}{\partial N}\right)_T. \quad (4c)$$

Note that quantities which are kept constant on *both* sides of the equation are not explicitly indicated. For example, Eq. (4a), should really be written

$$\left(\frac{\partial V}{\partial T}\right)_{P,N} = -\left(\frac{\partial S}{\partial P}\right)_{T,N}.$$

- (b) Show from Eq. (4a) and the third law of thermodynamics that the volume coefficient of thermal expansion

$$\alpha = \frac{1}{V} \left(\frac{\partial V}{\partial T}\right)_P$$

approaches zero as $T \rightarrow 0$.

4. Calculation of dT/dP for water

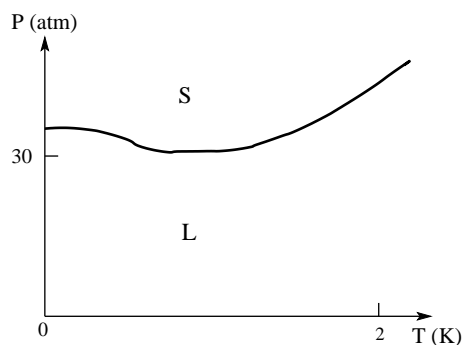
From the vapor pressure equation

$$\frac{1}{P} \frac{dP}{dT} = \frac{L}{k_B T^2},$$

which comes from the Clausius-Clapeyron equation with certain assumptions, calculate the value of dT/dP at $P = 1$ atm for the liquid-vapor equilibrium of water. The heat of vaporization at 100°C is 2260 J g^{-1} . Express your result in kelvin/atm.

5. Phase boundary between liquid and solid ^3He .

The figure below is a sketch of the coexistence line between the solid (S) and liquid (L) phases of ^3He at low temperature.



It has the following properties

- (a) For $T \rightarrow 0$, $dP/dT = 0$.
- (b) For T small but non-zero, $dP/dT < 0$.

Explain what these results tell you about the difference in entropy between the solid and liquid phases. Is the result for part (b) surprising?