PHYSICS 112
Homework 9

Due Thursday, March 15 (the last class). Note the non-standard day of the week, which is to give you the maximum amount of time to do the work. Solutions to this homework will be posted on the class web site `http://physics.ucsc.edu/~peter/112` after the class.
The last class will be a review section. Please let me know of any requests for topics that you would like me to cover.

The Final Exam is on Tuesday March 20, 12:00–3:00 pm.

- The exam will be **cumulative** with a *slight* emphasis on material since the midterm.
- I am trying to arrange an **overflow room, where some of you will take the final.** Details will be announced in class and on the website.
- The exam will be **closed book**, but you can bring in **one sheet of notes** which you have prepared yourself (both sides is OK).
- **No calculators.** Cell phones must be switched off.
- You **must explain your reasoning.** Simply copying a result from your crib sheet will not do.

1. **Energy and entropy of a van der Waals gas**
   We showed in class that the free energy of a van der Waals gas is

$$F = -Nk_BT\left\{\log\left[n_Q\frac{V-Nb}{N}\right]+1\right\} - N\,\frac{N}{V}\,a\,.$$

   (a) From this show that the entropy of the van der Waals gas is given by

$$\frac{S}{k_B} = N\left\{\log\left[n_Q\frac{V-Nb}{N}\right]+\frac{5}{2}\right\}\,.$$

   (b) Show that the energy is given by

$$U = \tfrac{3}{2}Nk_BT - N\left(\frac{N}{V}\right)a\,.$$

   Note that the first term is the usual kinetic energy, and the second term is the (negative) potential energy from the attractive part of the potential. (There is no contribution from the strong repulsive part of the potential, because it acts like to hard wall from which the particles simply recoil.)

2. **Gas-solid equilibrium**
   Consider a version of the example in Eqs. (26)–(32) of Ch. 10 of the second edition of the book (Eqs. (22)–(28) of Ch. 20 of the first edition) in which we let the oscillators in the solid move in *three* dimensions.

   (a) Show that in the high temperature region ($k_BT \gg \hbar\omega$) the vapor pressure is given by

$$P = \left(\frac{m}{2\pi}\right)^{3/2}\frac{\omega^3}{(k_BT)^{1/2}}\,\exp(-\epsilon_0/k_BT)\,,$$

   where m is the mass.

(b) Show that the latent heat is $\epsilon_0 - k_B T/2$.

(c) Explain this last result intuitively.

3. **Magnetization of a vector spin**
Consider a classical 3-component vector spin $\mathbf{S}$, of unit length, which is in a magnetic field $\mathbf{B}$ and so has the energy
$$E = -\boldsymbol{\mu} \cdot \mathbf{S} = -\gamma\, \mathbf{S} \cdot \mathbf{B}\,,$$
where γ is the magnitude of the magnetic moment $\boldsymbol{\mu}$.

   (a) Calculate the average magnetic moment $m = \gamma \langle S^z \rangle$ assuming that the field is along the z-direction.
   Note: For comparison, the corresponding result for an Ising spin is
   $$m = \gamma \tanh\left(\frac{\gamma B}{k_B T}\right) .$$

   (b) Does the magnitude of the moment vary if the direction of the field is changed? (This part requires only a little thought, no calculation.)

4. **Maxwell equation involving the magnetic field**
In the presence of an external magnetic field B, the thermodynamic identity for the free energy F is (assuming V is kept constant)
$$dF = -SdT - MdB,$$
where M is the total magnetic moment (magnetization times volume) of the system. Derive an appropriate Maxwell equation to show that
$$\left(\frac{\partial M}{\partial T}\right)_B = 0$$
at $T = 0$.

5. **Observation of an Ising simulation**
Go to the URL `http://physics.ucsc.edu/~peter/ising/ising.html` on my web site and observe the java applet of an Ising simulation on the square lattice. (You need java to be activated on your computer to see this.)

The up spins are white and the down spins are blue. You can change the temperature either by sliding the end of the red bar in the thermometer, or by typing a temperature into the box. You can start the spins with a random initial configuration "init warm", or a fully aligned initial configuration "init cold". (*Note:* the label of the box marked "init hot" is perhaps misleading; it sets the spins in an anti-parallel configuration which is the one with maximum energy. This actually corresponds to a negative temperature, see Kittel and Kroemer, Appendix E.)

It was shown first by Kramers and Wannier, and confirmed by Onsager who computed exactly the free energy in zero magnetic field, that the transition temperature is given by $T_c = 2/\ln(1+\sqrt{2}) = 2.269$. We set the (nearest-neighbor) interaction to be unity and also take $k_B = 1$ in this question. Note that the mean field transition temperature is $T_c^{MF} = 4$ (the number of neighbors), so the correlation effects, neglected in mean field theory, *reduce* the transition temperature. (This is a general result.)

Answer the following questions (qualitatively):

(a) Set the temperature to be well above T_c. Start with "cold", "warm", and "hot" spin configurations. Comment on what you see? Do you see large correlated "blobs" of the same color or only small blobs? Does the system very quickly settle down to a state independent of the initial spin state?

(b) Set the temperature to be equal to, or slightly greater than, T_c. Comment on the size of the correlated regions ("blobs" of the same color). Do the fluctuations in the system in the system happen more slowly than when the system is well above the transition temperature?

(c) Set the temperature below T_c and with a cold start. What do you see? Now set it to a warm start and comment on what you see.

6. **Critical exponents in mean field theory**

In the mean field theory for the Ising model, the magnetization m is determined self-consistently from the equation

$$m = \tanh\left(\frac{J_0 m + B}{k_B T}\right),$$

where we have set the spins to be of unit length, $S_i = \pm 1$, and $J_0 = zJ$ where $J\,(> 0)$ is the strength of the nearest neighbor interaction and z is the number of nearest neighbors. We showed in the class handout that the transition temperature T_c (i.e. the temperature where $m \to 0$ for $B \to 0$) is given by $k_B T_c = J_0$.

(a) Show that for $T > T_c$, the magnetic susceptibility, defined by

$$\chi = \lim_{B\to 0} \frac{\partial m}{\partial B},$$

is given by

$$\chi = \frac{1}{k_B(T - T_c)}.$$

Note: This shows that the susceptibility *diverges* at the (ferromagnetic) transition. It is conventional to define a "critical exponent" γ by

$$\chi \sim t^{-\gamma}$$

where $t = (T - T_c)/T_c$ is called the "reduced temperature". We see that in mean field theory $\gamma = 1$. In fact, γ is usually rather bigger than this: experimentally, values around 1.3 are typical in three dimensions, and the exact value for the two-dimensional Ising model is $\gamma = 7/4$.

(b) Show that at $T = T_c$,

$$m \sim B^{1/3}$$

for small B.

Note: One conventionally defines a critical exponent δ by $m \sim B^{1/\delta}$ at criticality. We see that $\delta = 3$ in mean field theory. In fact, δ is typically close to 5 in three dimensions.

7. **Spin-1 Ising model**

Consider a spin-1 Ising model in which the spin S_i take values $-1, 0$ and 1. The energy is, as usual

$$E = -J \sum_{\langle i,j\rangle} S_i S_j,$$

where the sum is over all nearest-neighbor pairs on the lattice. Assume that each site has z nearest-neighbors, so, for example, $z = 4$ for a square lattice. Calculate the transition temperature in the mean field approximation.

8. **First order phase transition in Landau Theory**
Consider the Landau free energy for an Ising-like order parameter, m, in the absence of an external magnetic field:

$$F(m) = \frac{1}{2}\, a(T)\, m^2 + \frac{1}{4}\, c\, m^4 + \frac{1}{6}\, d\, m^6 \,.$$

The parameter a varies linearly with T and goes through zero at some temperature T_0, i.e. $a(T) = \alpha(T - T_0)$. Show that the transition is continuous (second order) if $c > 0$ but that it is discontinuous (first order) if $c < 0$ (in which case we need the sixth order term with $d > 0$ for stability).
Note: You should include several sketches of $F(m)$ in your answer.