Physics 112
An Integral

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In class we showed that the determination of the specific heat of a degenerate Fermi gas involved the following integral

$$\boxed{I = \int_{-\infty}^{\infty} \frac{x^2\, e^x}{(e^x+1)^2}\, dx\,.} \tag{1}$$

The integrand can be written as

$$\left(\frac{x}{e^{x/2}+e^{-x/2}}\right)^2 , \tag{2}$$

which is clearly an even function of x and so we can write I as an integral involving only positive values of x,

$$I = 2\int_0^{\infty} \frac{x^2\, e^x}{(e^x+1)^2}\, dx\,. \tag{3}$$

This is not exactly of a standard type, but can be related to a more standard integral since

$$\boxed{I = -2\left.\frac{dJ(a)}{da}\right|_{a=1} ,} \tag{4}$$

where

$$\begin{aligned} J(a) &= \int_0^{\infty} \frac{x}{e^{ax}+1}\, dx \\ &= \frac{1}{a^2}\int_0^{\infty} \frac{t}{e^t+1}\, dt\,, \end{aligned} \tag{5}$$

where, in the last line, we made the substitution $ax = t$. Hence, from Eqs. (4) and (5), we have

$$\boxed{I = 4\int_0^{\infty} \frac{t}{e^t+1}\, dt\,,} \tag{6}$$

which is of a fairly standard type.

We determine it, initially as a series, as follows:

$$\begin{aligned} I &= 4\int_0^{\infty} \frac{t}{e^t+1}\, dt \\ &= 4\int_0^{\infty} t\,\frac{e^{-t}}{1+e^{-t}}\, dt \\ &= 4\int_0^{\infty} t\left[e^{-t}-e^{-2t}+e^{-3t}-e^{-4t}+\cdots\right] dt \\ &= 4\int_0^{\infty} te^{-t}\, dt\left[1-\frac{1}{2^2}+\frac{1}{3^2}-\frac{1}{4^2}+\cdots\right], \end{aligned} \tag{7}$$

where, to get the last line, we made the substitution $t \to t/2$ in the second term, $t \to t/3$ in the third term, and so on. The integral $\int_0^\infty te^{-t}\,dt$ is equal to $1!\,(=1)$, and so

$$I = 4\left\{\left[1 + \frac{1}{3^2} + \frac{1}{5^2} + \cdots\right] - \left[\frac{1}{2^2} + \frac{1}{4^2} + \frac{1}{6^2} + \cdots\right]\right\}. \tag{8}$$

In the first set of rectangular brackets we include the missing terms (which involve even integers) and then subtract them back out in the second term, i.e.

$$\begin{aligned} I &= 4\left\{\left[1 + \frac{1}{2^2} + \frac{1}{3^2} + \frac{1}{4^2} + \cdots\right] - 2\left[\frac{1}{2^2} + \frac{1}{4^2} + \frac{1}{6^2} + \cdots\right]\right\} \\ &= 4\left\{\left[1 + \frac{1}{2^2} + \frac{1}{3^2} + \frac{1}{4^2} + \cdots\right] - \frac{1}{2}\left[1 + \frac{1}{2^2} + \frac{1}{3^2} + \cdots\right]\right\} \\ &= 4\left(1 - \frac{1}{2}\right)\left[1 + \frac{1}{2^2} + \frac{1}{3^2} + \frac{1}{4^2} + \cdots\right] \\ &= 2\,\zeta(2)\,, \end{aligned} \tag{9}$$

where

$$\boxed{\zeta(2) \equiv 1 + \frac{1}{2^2} + \frac{1}{3^2} + \frac{1}{4^2} + \cdots} \tag{10}$$

is a zeta function and has value $\pi^2/6$. (This will have been shown in 116C.) Hence, from Eqs. (9) and (10), we have

$$\boxed{I = \frac{\pi^2}{3}\,,} \tag{11}$$

as stated in class.