PHYSICS 112

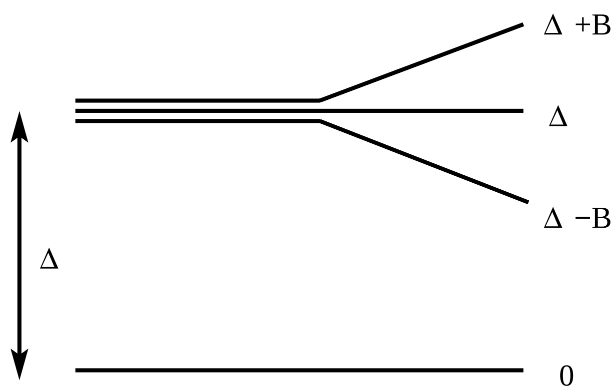
Practice Midterm, 2012 (including solutions). Time 1 hr. 10 mins.

Closed book. You may bring in one sheet of notes if you wish.
There are questions of both sides of the sheet.

1. [**40 points**]
An atom has a ground state with energy $E = 0$ and spin $S = 0$, and a "triplet" excited level with energy $E = \Delta$. The triplet consists of three states with spin $1, 0$ and -1. A magnetic field B splits the "degeneracy" of the triplet, and the energy levels become

$$\begin{aligned} E &= 0, & \text{spin} &= 0 \\ E &= \Delta - B, & \text{spin} &= 1 \\ E &= \Delta, & \text{spin} &= 0 \\ E &= \Delta + B, & \text{spin} &= -1\,, \end{aligned}$$

see the figure.



(a) For the case of *zero magnetic field* determine:
   i. the partition function
   ii. the probability that the system has energy 0, and the probability that it has energy Δ.
   iii. the free energy
   iv. the entropy
   v. the average energy
   vi. the expectation value of the spin, $\langle S \rangle$. Explain this result.

(b) For the case of a *magnetic field* B, determine
   i. the partition function
   ii. $\langle S \rangle$.

Note: Use the Boltzmann distribution.

2. [**35 points**]
Consider the fluctuations in the occupancy of simple harmonic oscillator (or equivalently the number of photons in a particular mode). As discussed in class, the probability of it being in the state with energy $(n + \frac{1}{2})\hbar\omega$ is

$$P(n) = \frac{e^{-nx}}{\displaystyle\sum_{m=0}^{\infty} e^{-mx}}\,,$$

where $x = \beta\hbar\omega$ with $\beta = 1/k_B T$. You may *assume* this.

(a) Obtain a closed expression for the partition function, Z where

$$Z = \sum_{n=0}^{\infty} e^{-nx} .$$

(b) Show that the mean number value of n, i.e. $\langle n \rangle \equiv \sum_n nP(n)$ is given by

$$\langle n \rangle = -\frac{\partial \ln Z}{\partial x}$$

and hence show that $\langle n \rangle$ is given by the Planck distribution

$$\langle n \rangle = \frac{1}{e^{\beta \hbar \omega} - 1} .$$

(c) Similarly show that $\langle \Delta n^2 \rangle$, the mean square fluctuation of n, is given by

$$\langle \Delta n^2 \rangle = \langle n^2 \rangle - \langle n \rangle^2 = -\frac{\partial \langle n \rangle}{\partial x} ,$$

where $\Delta n = n - \langle n \rangle$. Hence show that

$$\langle \Delta n^2 \rangle = \langle n \rangle (1 + \langle n \rangle).$$

This shows that, for large $\langle n \rangle$, the root mean square fluctuations, $\langle \Delta n^2 \rangle^{1/2}$, are actually proportional to $\langle n \rangle$ (not $\langle n \rangle^{1/2}$) and so are *large*.

3. [**25 points**]
Consider photons contained in a *two-dimensional* cavity of unit area. You are *given* that the density of states for the photons is

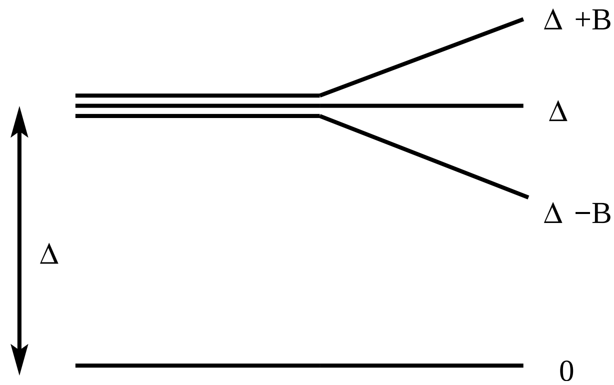
$$\rho(\epsilon) = \frac{1}{\pi} \frac{1}{\hbar^2 c^2} \epsilon .$$

Compute the total number of photons at temperature T.
Note: You are *given* that

$$\int_0^\infty \frac{x}{e^x - 1} \, dx = \frac{\pi^2}{6} .$$

Solutions

1. The energy levels are as shown in the figure.



(a) In zero magnetic field we have

i.

$$\boxed{Z = 1 + 3e^{-\beta\Delta}\,.}$$

ii. The probability that the system is in the state with energy 0 is given by

$$P_0 = \frac{e^0}{Z} = \boxed{\frac{1}{1 + 3e^{-\beta\Delta}}\,,}$$

and the probability that it has energy Δ is given by

$$P_\Delta = \frac{3e^{-\beta\Delta}}{Z} = \boxed{\frac{3e^{-\beta\Delta}}{1 + 3e^{-\beta\Delta}}\,.}$$

iii. The free energy is given by

$$F = -k_B T \ln Z = \boxed{-k_B T \ln(1 + 3e^{-\beta\Delta})\,.}$$

iv. The entropy is given by

$$S = -\frac{\partial F}{\partial T} = \boxed{k_B \ln(1 + 3e^{-\beta\Delta}) + \frac{3\Delta}{T}\,\frac{e^{-\beta\Delta}}{1 + 3e^{-\beta\Delta}}\,.}$$

v. The average energy is given by

$$U = \Delta P_\Delta = \boxed{3\Delta \frac{e^{-\beta\Delta}}{1 + 3e^{-\beta\Delta}}\,.}$$

vi. Denoting the states with energy Δ and spin $1, 0$, and -1 by Δ_1, Δ_0 and Δ_{-1} respectively, we have

$$\langle S \rangle = 0 \cdot P_0 + 1 \cdot P_{\Delta_1} + 0 \cdot P_{\Delta_0} + (-1) \cdot P_{\Delta_{-1}} = 0\,,$$

since $P_{\Delta_1} = P_{\Delta_{-1}}$. This could have been anticipated because, in the absence of a magnetic field, there is nothing in the system to prefer an up (positive) spin to a down spin. Hence $\langle S \rangle = 0$ *by symmetry*.

(b) In a magnetic field,

i.

$$\boxed{Z = 1 + e^{-\beta\Delta}\left[1 + 2\cosh(\beta B)\right]\ .}$$

(Remember $\cosh x = (e^x + e^{-x})/2$).

ii. As in zero field

$$\langle S\rangle = 0\cdot P_0 + 1\cdot P_{\Delta_1} + 0\cdot P_{\Delta_0} + (-1)\cdot P_{\Delta_{-1}}\,,$$

but the probabilities of the states are now changed so we get

$$\boxed{\langle S\rangle = \frac{2e^{-\beta\Delta}\sinh\beta B}{1 + e^{-\beta\Delta}\left[1 + 2\cosh(\beta B)\right]}\,.}$$

(Remember $\sinh x = (e^x - e^{-x})/2$).

2. (a)

$$Z = \sum_{n=0}^{\infty} e^{-nx}\,.$$

This is an infinite geometric series whose some is the first term divided by (1 - common ratio), i.e.

$$\boxed{Z = \frac{1}{1 - e^{-x}}\,.}$$

(b) The mean value of n is given by

$$\langle n\rangle = \sum_{n=0}^{\infty} nP(n) = \frac{\sum_{n=0}^{\infty} ne^{-nx}}{\sum_{n=0}^{\infty} e^{-nx}} = -\frac{\partial Z/\partial x}{Z} = \boxed{-\frac{\partial \ln Z}{\partial x}\,.}$$

Using the expression for Z from part (a) gives

$$\langle n\rangle = \frac{e^{-x}}{(1 - e^{-x})^2}\,(1 - e^{-x}) = \boxed{\frac{1}{e^{\beta\hbar\omega} - 1}\,,}$$

(with $x = \beta\hbar\omega$), the well-known Planck distribution.

(c) Firstly we note that

$$\langle \Delta n^2\rangle = \langle\,(n - \langle n\rangle)^2\,\rangle = \langle n^2\rangle - 2\langle\, n\,\langle n\rangle\,\rangle + \langle n\rangle^2 = \boxed{\langle n^2\rangle - \langle n\rangle^2\,.}$$

Using the definition of $\langle n\rangle$ we have

$$-\frac{\partial\langle n\rangle}{\partial x} = -\frac{\partial}{\partial x}\left[\frac{\sum_{n=0}^{\infty} ne^{-nx}}{\sum_{n=0}^{\infty} e^{-nx}}\right] = \frac{\sum_{n=0}^{\infty} n^2e^{-nx}}{\sum_{n=0}^{\infty} e^{-nx}} - \frac{\left[\sum_{n=0}^{\infty} ne^{-nx}\right]^2}{\left[\sum_{n=0}^{\infty} e^{-nx}\right]^2} = \langle n^2\rangle - \langle n\rangle^2 = \boxed{\langle \Delta n^2\rangle\,.}$$

Using the expression for $\langle n\rangle$ from part (b) gives

$$\langle \Delta n^2\rangle = -\frac{\partial\langle n\rangle}{\partial x} = \frac{e^x}{(e^x - 1)^2} = \frac{1}{e^x - 1} + \frac{1}{(e^x - 1)^2} = \boxed{\langle n\rangle(1 + \langle n\rangle)\,.}$$

If $\langle n\rangle$ is large we can neglect the factor of 1, and so $\langle \Delta n^2\rangle^{1/2} \propto \langle n\rangle$.

3. The mean number of photons in a state of energy ϵ is given by the Planck distribution

$$n(\epsilon) = \frac{1}{e^{\beta\epsilon} - 1} \, .$$

Hence the total number of photons is given by

$$\mathcal{N} = \int_0^\infty n(\epsilon)\rho(\epsilon)\, d\epsilon,$$

where $\rho(\epsilon)$ is the density of states given in the question.

Hence

$$\begin{aligned}
\mathcal{N} &= \frac{1}{\pi}\frac{1}{\hbar^2 c^2}\int_0^\infty \frac{\epsilon}{e^{\beta\epsilon} - 1}\, d\epsilon \\
&= \frac{1}{\pi}\left(\frac{k_B T}{\hbar c}\right)^2 \int_0^\infty \frac{x}{e^x - 1}\, dx \\
&= \boxed{\frac{\pi}{6}\left(\frac{k_B T}{\hbar c}\right)^2 ,}
\end{aligned}$$

using the value for the integral given in the question.