PHYSICS 112
Final Exam, 2011, solutions

1. [**15 points**]

(a) The partition is given by

$$\boxed{Z = g_1 + g_2 e^{-\beta\Delta}\,,}$$

(note the degeneracy factors).

(b) The free energy is given by $F = -k_B T \log Z$, i.e.

$$\boxed{F = -k_B T \log(g_1 + g_2 e^{-\beta\Delta})\,.}$$

The energy is related to F by $U = (\partial/\partial\beta)(\beta F) = -(\partial/\partial\beta)\log Z$, so

$$U = -\frac{\partial}{\partial\beta}\log(g_1 + g_2 e^{-\beta\Delta}) = \boxed{\frac{\Delta g_2 e^{-\beta\Delta}}{g_1 + g_2 e^{-\beta\Delta}}\,.}$$

The specific heat is given by

$$\begin{aligned} C &= \frac{\partial U}{\partial T} \\ &= \frac{\Delta^2}{k_B T^2}\left[\frac{g_2 e^{-\beta\Delta}}{g_1 + g_2 e^{-\beta\Delta}} - \frac{g_2^2 e^{-2\beta\Delta}}{(g_1 + g_2 e^{-\beta\Delta})^2}\right] \\ &= \boxed{\frac{\Delta^2}{k_B T^2}\frac{g_1 g_2 e^{-\beta\Delta}}{(g_1 + g_2 e^{-\beta\Delta})^2}\,.} \end{aligned} \quad (1)$$

2. [**15 points**]
We are given that

$$E_n = n\epsilon + Un(n-1)\,,$$

(a) According to the Gibbs distribution we have

$$\boxed{\langle n\rangle = \frac{\sum_{n=0}^{\infty} n \exp\left[\beta\left\{n(\mu-\epsilon) - Un(n-1)\right\}\right]}{\sum_{n=0}^{\infty} \exp\left[\beta\left\{n(\mu-\epsilon) - Un(n-1)\right\}\right]}\,.}$$

(b) For $U = 0$ the denominator is a geometric series which sums to $1/(1-\exp(x))$ where $x = \beta(\mu-\epsilon)$. The numerator is the derivative of this with respect to x, i.e. $\exp(x)/(1-\exp(x))^2$. Hence

$$\langle n\rangle = \frac{\exp(x)}{1-\exp(x)} = \boxed{\frac{1}{\exp[\beta(\epsilon-\mu)] - 1}\,,}$$

the Bose-Einstein distribution.

(c) For $U \to \infty$ only the $n = 0$ and $n = 1$ terms contribute so we have

$$\langle n\rangle = \frac{\exp\left[\beta(\mu-\epsilon)\right]}{1 + \exp\left[\beta(\mu-\epsilon)\right]} = \boxed{\frac{1}{\exp[\beta(\epsilon-\mu)] + 1}\,,}$$

which is the same as the Fermi-Dirac distribution.

3. [**25 points**]
We are given that the density of states is

$$\rho(\epsilon) = V\,\frac{2S+1}{4\pi^2}\left(\frac{2m}{\hbar^2}\right)^{3/2}\epsilon^{1/2}\,.$$

(a) The Fermi energy ϵ_F is determined from

$$N = \int_0^{\epsilon_F}\rho(\epsilon)\,d\epsilon = V\,\frac{2S+1}{4\pi^2}\left(\frac{2m}{\hbar^2}\right)^{3/2}\int_0^{\epsilon_F}\epsilon^{1/2}\,d\epsilon = V\,\frac{2S+1}{6\pi^2}\left(\frac{2m}{\hbar^2}\right)^{3/2}\epsilon_F^{3/2},$$

which gives

$$\boxed{\epsilon_F = \left(\frac{\hbar^2}{2m}\right)\left(\frac{6\pi^2 n}{2S+1}\right)^{2/3},}$$

where $n = N/V$ is the particle density.

(b) The energy at $T = 0$ is given by

$$\begin{aligned} U &= \int_0^{\epsilon_F}\epsilon\rho(\epsilon)\,d\epsilon = V\,\frac{2S+1}{4\pi^2}\left(\frac{2m}{\hbar^2}\right)^{3/2}\int_0^{\epsilon_F}\epsilon^{3/2}\,d\epsilon = V\,\frac{2S+1}{10\pi^2}\left(\frac{2m}{\hbar^2}\right)^{3/2}\epsilon_F^{5/2}, \\ &= V\,\frac{2S+1}{10\pi^2}\left(\frac{\hbar^2}{2m}\right)\left(\frac{6\pi^2 n}{2S+1}\right)^{5/3} = \boxed{\frac{3}{10}\left(\frac{6\pi^2}{2S+1}\right)^{2/3}\left(\frac{\hbar^2}{m}\right)\frac{N^{5/3}}{V^{2/3}}\,.} \end{aligned}$$

(c) The pressure at $T = 0$ is given by

$$P = -(\partial U/\partial V)_{N,T} = \frac{2}{3}\frac{3}{10}\left(\frac{6\pi^2}{2S+1}\right)^{2/3}\left(\frac{\hbar^2}{m}\right)\left(\frac{N}{V}\right)^{5/3} = \boxed{\frac{1}{5}\left(\frac{6\pi^2}{2S+1}\right)^{2/3}\left(\frac{\hbar^2}{m}\right)\left(\frac{N}{V}\right)^{5/3}.}$$

(d) The pressures are equal in equilibrium, so

$$\frac{n_1^{5/3}}{2^{2/3}} = \frac{n_2^{5/3}}{4^{2/3}} \tag{2}$$

where n_1 is the density of particles in compartment 1 which has spin-1/2 particles, and n_2 is the density of particles in compartment 2 which has spin-3/2 particles. Taking the third power of Eq. (2) gives

$$\left(\frac{n_1}{n_2}\right)^5 = \left(\frac{1}{2}\right)^2,$$

or

$$\boxed{\frac{n_1}{n_2} = \left(\frac{1}{2}\right)^{2/5}} = 0.758\cdots\,.$$

(Since calculators are not allowed you are not required to give the numerical value.)
Note: As stated in the question the result that the densities are different is a *quantum* effect.

4. [**20 points**]
The law of mass action, given at the beginning of the exam, states that

$$\left(\frac{n_A}{n_{QA}}\right)^2 = \frac{n_B}{Z_B n_{QB}},$$

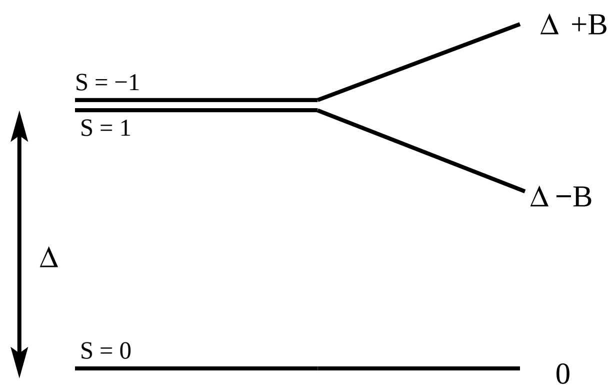
where $Z_B = e^{\beta \Delta E}$ is the partition function of the molecule B including just its lowest state which has energy $-\Delta E$. From the expression given for n_{Qj} in the exam, we see that n_{QA} and n_{QB} differ only because B has twice the mass of A. Hence $n_{QB} = 2^{3/2} n_{QA}$. We therefore have

$$\frac{n_A^2}{n_{QA}} = n_B \frac{e^{-\beta \Delta E}}{2^{3/2}}.$$

Hence, the condition that $n_A = n_B$ is

$$\boxed{\frac{n_A}{n_{QA}} = \frac{1}{2^{3/2}} e^{-\beta \Delta E}.}$$

5. [**25 points**]
The energy levels of a single spin are shown in the figure.



(a) The average value of S is

$$m \equiv \langle S_i \rangle = \frac{e^{\beta(-\Delta+B)} + 0 - e^{\beta(-\Delta-B)}}{e^{\beta(-\Delta+B)} + 1 + e^{\beta(-\Delta-B}} = \boxed{\frac{2e^{-\beta\Delta} \sinh \beta B}{1 + 2e^{-\beta\Delta} \cosh \beta B}.} \tag{3}$$

(b) Different spins interact through an *additional* energy

$$-J \sum_{\langle i,j \rangle} S_i S_j \,.$$

The terms involving spin i are $-S_i J \sum_j S_j$, and taking the average value of the spins on the neighboring sites j gives $-B^{MF} S_i$ where

$$\boxed{B^{MF} = zJm \,,} \tag{4}$$

where z is the number of neighbors.

(c) Substituting $B = B^{MF}$ from Eq. (4) into the equation for m in Eq. (3) gives

$$\boxed{m = \frac{2e^{-\beta\Delta} \sinh \beta zJm}{1 + 2e^{-\beta\Delta} \cosh \beta zJm}.}$$

(d) Assuming that the transition is continuous (second order) one can locate $\beta_c \equiv 1/k_B T_c$ by looking for a solution with m non-zero but infinitesimally small. Expanding the RHS of the last equation and just including the first order term, and setting $\beta = \beta_c$, gives

$$m = \frac{2\beta_c \, zJ \, e^{-\beta_c \Delta}}{1 + 2e^{-\beta_c \Delta}} \, m \,,$$

which is satisfied when the coefficient of m on the RHS is unity, i.e.

$$\boxed{k_B T_c = \frac{2e^{-\Delta/k_B T_c}}{1 + 2e^{-\Delta/k_B T_c}} \, zJ \,.}$$

(e) For the limit $\Delta \to 0$ the exponential factors tend to unity so

$$\boxed{k_B T_c = \frac{2}{3} \, zJ \,.}$$