

PHYSICS 112
Final Exam, 2012, solutions

1. [20 points]

(a) There are four states whose energies are

$$\begin{aligned} ++, & \quad \boxed{E = -J}, \\ +- , & \quad \boxed{E = +J}, \\ -+ , & \quad \boxed{E = +J}, \\ -- , & \quad \boxed{E = -J}. \end{aligned}$$

(b) The partition function is given by $Z = 2(e^{\beta J} + e^{-\beta J}) = \boxed{4 \cosh \beta J}$. The free energy is given by $F = -k_B T \ln Z = \boxed{-k_B T \ln(4 \cosh \beta J)}$.

(c)

$$\langle S_1 S_2 \rangle = (+1)(+1)P_{++} + (+1)(-1)P_{+-} + (-1)(+1)P_{-+} + (-1)(-1)P_{--},$$

where the probability that the both spins are +1 is given by

$$P_{++} = \frac{e^{\beta J}}{Z} = \frac{e^{\beta J}}{4 \cosh \beta J},$$

and similarly for the other states. Hence we get

$$\langle S_1 S_2 \rangle = \frac{(+1)e^{\beta J} + (-1)e^{-\beta J} + (-1)e^{-\beta J} + (+1)e^{\beta J}}{4 \cosh \beta J} = \frac{2e^{\beta J} - 2e^{-\beta J}}{4 \cosh \beta J} = \frac{4 \sinh \beta J}{4 \cosh \beta J} = \boxed{\tanh \beta J}.$$

(d)

$$\boxed{\langle S_1 \rangle = 0.}$$

This follows from symmetry since the energy is invariant under flipping both the spins. Of course, it can also be obtained by doing the Boltzmann average over the four states as in Qu. 1c.

2. [18 points]

(a) The grand partition function is

$$\mathcal{Z} = 1 + 2e^{\beta(\mu+I)}.$$

Hence the probability that the atom is ionized is

$$\boxed{P_{\text{ionized}} = \frac{1}{1 + 2e^{\beta(\mu+I)}}.}$$

(b) Now there is an additional state with energy $-2I+U$, so the grand partition function becomes

$$1 + 2e^{\beta(\mu+I)} + e^{\beta(2\mu+2I-U)}.$$

The probability that the atom binds two electrons is

$$P_{\text{bind } 2} = \frac{e^{\beta(2\mu+2I-U)}}{1 + 2e^{\beta(\mu+I)} + e^{\beta(2\mu+2I-U)}} \quad (1)$$

$$= \frac{1}{1 + 2e^{\beta(-I+U-\mu)} + e^{\beta(-2I+U-2\mu)}}. \quad (2)$$

(c)

$$P_{\text{bind } 2} = \frac{1}{[1 + e^{\beta(-I-\mu)}]^2}.$$

This result is the square of the Fermi-Dirac distribution for energy $-I$. The reason for this result is because, for $U = 0$, there are two bound states, each of which can be occupied independently of the other. The FD distribution gives the probability that one of the states is occupied, so the probability that they are both occupied is the square of this.

3. [12 points]

(a) Since

$$f = \left(\frac{\partial F}{\partial l}\right)_T, \quad S = -\left(\frac{\partial F}{\partial T}\right)_l,$$

and

$$\frac{\partial^2 F}{\partial l \partial T} = \frac{\partial^2 F}{\partial T \partial l}$$

we get the Maxwell relation

$$\left(\frac{\partial f}{\partial T}\right)_l = -\left(\frac{\partial S}{\partial l}\right)_T.$$

(b) According to the third law of thermodynamics, $S = 0$ at $T = 0$ and hence $(\partial S/\partial l)_T$ vanishes as $T \rightarrow 0$. Hence, using the Maxwell relation

$$\left(\frac{\partial f}{\partial T}\right)_l \rightarrow 0 \text{ as } T \rightarrow 0.$$

4. [25 points]

We are given that

$$\rho(\epsilon) = A \frac{m}{2\pi\hbar^2},$$

where A is the area of the system so the number of particles is given by

$$N = \int_0^\infty \frac{\rho(\epsilon)}{e^{\beta(\epsilon-\mu)} - 1} d\epsilon. \quad (3)$$

(a) Eq. (3) can be written as

$$N = A \frac{m}{2\pi\hbar^2} \int_0^\infty \frac{d\epsilon}{e^{\beta(\epsilon-\mu)} - 1}.$$

Making the substitution $x = \beta(\epsilon - \mu)$ gives

$$n \equiv \frac{N}{A} = n_Q \int_{-\beta\mu}^\infty \frac{dx}{e^x - 1},$$

where

$$n_Q = \frac{mk_B T}{2\pi\hbar^2}$$

is the *two-dimensional* quantum concentration.

(b) Writing

$$\frac{1}{e^x - 1} = \frac{e^{-x}}{1 - e^{-x}}$$

we get

$$n = n_Q \int_{-\beta\mu}^{\infty} \frac{e^{-x}}{1 - e^{-x}} dx = n_Q [\ln(1 - e^{-x})]_{-\beta\mu}^{\infty} = -n_Q \ln(1 - e^{\beta\mu}),$$

so

$$e^{-n/n_Q} = 1 - e^{\beta\mu}.$$

Hence the chemical potential is given by

$$\boxed{\mu = k_B T \ln[1 - e^{-n/n_Q}]}.$$

(c) As T decreases n_Q decreases, and so e^{-n/n_Q} decreases. However, e^{-n/n_Q} never goes negative, so μ is always negative even down to $T = 0$. Hence there is *no* Bose-Einstein condensation in two dimensions for the Bose gas.

For $T \rightarrow 0$ we have $n_Q \rightarrow 0$ and so $e^{-n/n_Q} \rightarrow 0$. Hence

$$\frac{\mu}{k_B T} \simeq -e^{-n/n_Q} = \boxed{-\exp\left[-\frac{2\pi\hbar^2 n}{mk_B T}\right]}.$$

Hence μ becomes exponentially small in $1/T$ at low- T (but is always non-zero at any non-zero T).

(d) For large T , n_Q is large and so $e^{-n/n_Q} \simeq 1 - n/n_Q$ which gives the result for the classical ideal gas in two dimensions:

$$\boxed{\mu = -k_B T \ln\left(\frac{n_Q}{n}\right)}.$$

5. [25 points]

(a) The values of the energy for the states with $S = +2, +1, 0, -1$ and -2 are $-2B, -B, 0, +B$ and $+2B$, and so

$$\langle S \rangle = \frac{2 \cdot e^{2\beta B} + 1 \cdot e^{\beta B} + 0 \cdot e^0 + (-1) \cdot e^{-\beta B} + (-2) \cdot e^{-2\beta B}}{e^{2\beta B} + e^{\beta B} + e^0 + e^{-\beta B} + e^{-2\beta B}} = \frac{2 \sinh(\beta B) + 4 \sinh(2\beta B)}{1 + 2 \cosh(\beta B) + 2 \cosh(2\beta B)}.$$

This can be expressed as

$$\langle S \rangle = f\left(\frac{B}{k_B T}\right)$$

where

$$\boxed{f(x) = \frac{2 \sinh(x) + 4 \sinh(2x)}{1 + 2 \cosh(x) + 2 \cosh(2x)}}. \quad (4)$$

(b) For small x we use $\sinh(x) = x + \dots$, $\cosh(x) = 1 + \dots$ to get

$$\boxed{f(x) = \frac{2x + 8x + \dots}{5 + \dots} = 2x + \dots}.$$

- (c) Now each spin interacts with z neighbors S_j . In the mean field approximation this means that each spin feels a mean field of strength equal to zJm where $m \equiv \langle S_j \rangle$. Hence m is given by the same expression as above but with $B = zJm$, i.e.

$$\boxed{m = f\left(\frac{zJm}{k_B T}\right)}, \quad (5)$$

where $f(x)$ is given by Eq. (4).

- (d) Assuming that the transition is second order, the transition temperature T_c is where the $m \rightarrow 0$ and hence is where coefficients of m on both sides of Eq. (5) are equal. Using the earlier parts of the question gives

$$1 = 2\frac{zJ}{k_B T_c}$$

which gives

$$\boxed{k_B T_c = 2zJ.}$$