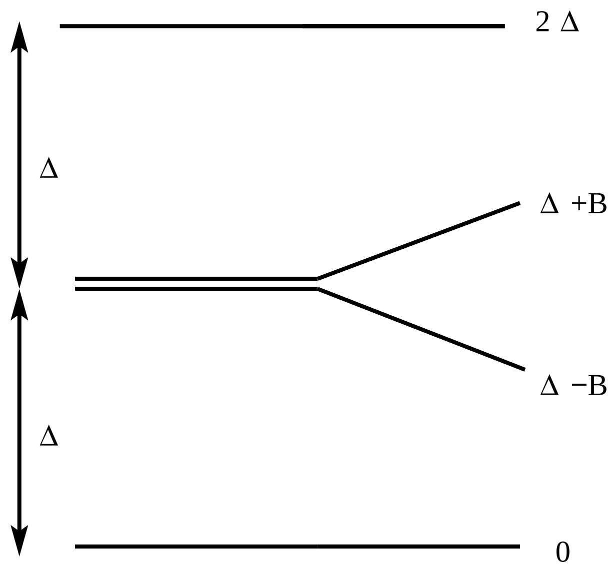
PHYSICS 112
Midterm, 2012: Solutions

1. The energy levels are as shown in the figure.



(a) In zero magnetic field we have

i.

$$\boxed{Z = 1 + 2e^{-\beta\Delta} + e^{-2\beta\Delta}\,.}$$

ii. The probability that the system is in the state with energy 0 is given by

$$P_0 = \frac{e^0}{Z} = \boxed{\frac{1}{1 + 2e^{-\beta\Delta} + e^{-2\beta\Delta}}\,,}$$

the probability that it has energy Δ is given by

$$P_\Delta = \frac{2e^{-\beta\Delta}}{Z} = \boxed{\frac{2e^{-\beta\Delta}}{1 + 2e^{-\beta\Delta} + e^{-2\beta\Delta}}\,,}$$

and the probability that it has energy 2Δ is given by

$$P_{2\Delta} = \frac{e^{-2\beta\Delta}}{Z} = \boxed{\frac{e^{-2\beta\Delta}}{1 + 2e^{-\beta\Delta} + e^{-2\beta\Delta}}\,,}$$

iii. The free energy is given by

$$F = -k_B T \ln Z = \boxed{-k_B T \ln(1 + 2e^{-\beta\Delta} + e^{-2\beta\Delta})\,.}$$

iv. The entropy is given by

$$S = -\frac{\partial F}{\partial T} = \boxed{k_B \ln(1 + 2e^{-\beta\Delta} + e^{-2\beta\Delta}) + \frac{2\Delta}{T}\,\frac{e^{-\beta\Delta} + e^{-2\beta\Delta}}{1 + 2e^{-\beta\Delta} + e^{-2\beta\Delta}}\,.}$$

v. The average energy is given by

$$U = \Delta P_\Delta + 2\Delta P_{2\Delta} = \boxed{2\Delta\frac{e^{-\beta\Delta} + e^{-2\beta\Delta}}{1 + 2e^{-\beta\Delta} + e^{-2\beta\Delta}}\,.}$$

vi. Denoting the state with energy Δ and spin 1 by Δ_1 and that with energy Δ and spin -1 by Δ_{-1}, we have

$$\langle S\rangle = 0\cdot P_0 + 1\cdot P_{\Delta_1} + (-1)\cdot P_{\Delta_{-1}} + 0\cdot P_{2\Delta} = \boxed{0\,,}$$

since $P_{\Delta_1} = P_{\Delta_{-1}}$. This could have been anticipated because, in the absence of a magnetic field, there is nothing in the system to prefer an up (positive) spin to a down spin. Hence $\langle S\rangle = 0$ *by symmetry.*

(b) In a magnetic field,

i.

$$\boxed{Z = 1 + 2e^{-\beta\Delta}\cosh(\beta B) + e^{-2\beta\Delta}\,.}$$

(Remember $2\cosh x = e^x + e^{-x}$).

ii. As in zero field

$$\langle S\rangle = 0\cdot P_0 + 1\cdot P_{\Delta_1} + (-1)\cdot P_{\Delta_{-1}} + 0\cdot P_{2\Delta}\,,$$

but the probabilities of the states are now changed so we get

$$\boxed{\langle S\rangle = \frac{2e^{-\beta\Delta}\sinh\beta B}{1 + 2e^{-\beta\Delta}\cosh(\beta B) + e^{-2\beta\Delta}}\,,}$$

rather than 0. (Remember that $2\sinh x = e^x - e^{-x}$).

2. We are given that the density of states is

$$\rho(\epsilon) = \frac{V}{\pi^2}\,\frac{1}{(\hbar c)^3}\,\epsilon^2\,. \qquad (1)$$

(a) To get the total number of photons we multiply $\rho(\epsilon)$ by the Planck distribution

$$n_P(\epsilon) = \frac{1}{e^{\beta\epsilon} - 1}\,,$$

and integrate, i.e.

$$\langle N\rangle = \int_0^\infty \frac{\rho(\epsilon)}{e^{\beta\epsilon} - 1}\,d\epsilon = V\,\frac{1}{\pi^2(\hbar c)^3}\int_0^\infty \frac{\epsilon^2}{e^{\beta\epsilon} - 1}\,d\epsilon\,.$$

In the integral we make the substitution $x = \beta\epsilon$ so

$$\boxed{\langle N\rangle = \frac{V}{\pi^2}\left(\frac{k_B T}{\hbar c}\right)^3 I_2\,,}$$

where

$$\boxed{I_2 = \int_0^\infty \frac{x^2}{e^x - 1}\,dx\,.}$$

Note: The numerical value of the integral I_2 (which you are not asked to determine) is equal to $2.40411\ldots$.

(b) To get the total energy we multiply $\epsilon\rho(\epsilon)$ by the Planck distribution and integrate, i.e.

$$U \equiv \langle E \rangle = \int_0^\infty \frac{\epsilon\rho(\epsilon)}{e^{\beta\epsilon}-1}\,d\epsilon = V\,\frac{1}{\pi^2(\hbar c)^3}\int_0^\infty \frac{\epsilon^3}{e^{\beta\epsilon}-1}\,d\epsilon\,.$$

In the integral we make the substitution $x = \beta\epsilon$ so

$$\boxed{U = \frac{V}{\pi^2}\,\frac{(k_B T)^4}{(\hbar c)^3}\,I_3\,,}$$

where

$$\boxed{I_3 = \int_0^\infty \frac{x^3}{e^x - 1}\,dx\,.}$$

Note: The value of the integral I_3 (which again you are not asked to determine) is $\pi^4/15 = 6.49394\ldots$.

3. (a) According to the Boltzmann distribution the mean number of particles is given by

$$\langle n \rangle = \frac{0\cdot 1 + 2\cdot 1\cdot e^{-\beta\epsilon} + 2\cdot e^{-\beta(2\epsilon+U)}}{1 + 2\cdot e^{-\beta\epsilon} + e^{-\beta(2\epsilon+U)}} = \boxed{2\,\frac{e^{-\beta\epsilon} + e^{-\beta(2\epsilon+U)}}{1 + 2e^{-\beta\epsilon} + e^{-\beta(2\epsilon+U)}}\,.}$$

(b) i. For $U = 0$ we have

$$\langle n \rangle = \frac{2x + 2x^2}{1 + 2x + x^2} = \frac{2x(1+x)}{(1+x)^2} = \frac{2}{x^{-1}+1} = \boxed{\frac{2}{e^{\beta\epsilon}+1}\,,}$$

where $x = e^{-\beta\epsilon}$.

ii. For $U \to \infty$ we have

$$\langle n \rangle = \frac{2x}{1+2x} = \frac{2}{x^{-1}+2} = \boxed{\frac{1}{\frac{1}{2}e^{\beta\epsilon}+1}\,,}$$