PHYSICS 112
Homework 1 Solutions

1. (a) From $g(U) = CU^{3N/2}$ we have, with $S = k_B \ln g$,

$$\frac{1}{T} = \left(\frac{\partial S}{\partial U}\right)_N = k_B \frac{3N}{2}\frac{1}{U}\,, \tag{1}$$

and hence

$$\boxed{U = \frac{3}{2} N k_B T\,.}$$

This is the form of the energy of an ideal gas, as we will discuss later in the course.

(b) Differentiating Eq. (1) again gives

$$\boxed{\left(\frac{\partial^2 S}{\partial U^2}\right)_N = -k_B \frac{3N}{2}\frac{1}{U^2}\,,}$$

which is negative.

2. We are given that

$$\sigma(l) \simeq \ln g(N,0) - \frac{2l^2}{N}\,,$$

and we also know that

$$U = -2l\mu B \quad (= -MB)\,, \tag{2}$$

so

$$\sigma(U) \simeq \ln g(N,0) - \frac{U^2}{2\mu^2 B^2 N}\,.$$

Differentiating with respect to U gives

$$\frac{1}{T} = k_B \left(\frac{\partial \sigma}{\partial U}\right)_N = -k_B \frac{U}{N\mu^2 B^2}\,.$$

Rearranging gives

$$U = -\frac{N\mu^2 B^2}{k_B T}\,,$$

and hence, from Eq. (2), the magnetization per site, $m \equiv M/N$, is given by

$$\boxed{m = \frac{\mu^2 B}{k_B T}\,.}$$

Note: We shall derive this result, which is known as Curie's law, by an easier method later in the class.

3. (a) If the energy of the i-th oscillator is $(n_i + 1/2)\hbar\omega$, then the constraint given in the question is

$$n_1 + n_2 + \cdots + n_N = n\,,$$

where n is fixed. The number of ways of dividing up the energy among the oscillators is given by (see Eq. (1.55) of Kittel and Kroemer)

$$g = \frac{(N+n-1)!}{n!(N-1)!}\,.$$

Taking logs, using Stirling's approximation $\ln N! \simeq N \ln N - N$, and replacing $N-1$ by N, the entropy $\sigma \equiv \ln g$ is given by

$$\boxed{\sigma = (N+n)\ln(N+n) - N\ln N - n\ln n\,.}$$

(b) Writing $n = U/(\hbar\omega)$ and differentiating with respect to U gives

$$\frac{1}{T} = \left(\frac{\partial S}{\partial U}\right)_N = k_B \left(\frac{\partial \sigma}{\partial U}\right)_N = k_B \frac{1}{\hbar\omega}\left[\ln(N+n) - \ln n\right] = k_B \frac{1}{\hbar\omega} \ln\left(\frac{1+n/N}{n/N}\right).$$

This gives

$$\boxed{\frac{n}{N}\ \left(\equiv \frac{U}{N\hbar\omega}\right) = \frac{1}{\exp(\hbar\omega/k_BT) - 1}\,.}$$

Note: This is called the Planck distribution.

4. (a) There are 44^{10^5} possible sequences of 10^5 characters. Hence the probability that the correct sequence will emerge is the inverse of this:

$$\left(\frac{1}{44}\right)^{100000} \simeq 10^{-164345}\,. \tag{3}$$

(b) We are given 10^{10} monkeys and 10^{18} seconds with 10^4 seconds per trial. Hence the number of trials is $10^{10} \times 10^{14} = 10^{24}$. Hence the probability of the correct sequence is

$$\boxed{10^{-164345} \times 10^{24} = 10^{-164321}\,.}$$

This probability is zero in any operational sense. Note that multiplying by the number of trials makes almost no difference to the power of 10 in Eq. (3).

(c) "To be or not to be" has 18 characters, including spaces. The probablity that this would be typed at random is $1/44^{18} \simeq 2.6 \times 10^{-30}$. We are given 10^{10} monkeys and 10^{18} seconds and lets say that it takes a monkey 1 second to type the 18 characters. The probability that one of the monkeys would type the phrase in the age of the universe is therefore $2.6 \times 10^{-30} \times 10^{28} = 0.026$. Hence the probability of the monkeys producing "To be or not to be" in the age of the universe is small, but not astronomically small.