PHYSICS 112
Homework 3 Solutions

1. As shown in class (see also handout) the allowed values of $\vec{k}$ in k-space form a regularly space grid with spacing π/L (where the system is of size L in each direction) and k_x, k_y etc. are taken to be positive.

   [*Note:* This is for "free" boundary conditions. Instead, one often uses "periodic" boundary conditions, where one includes negative as well as positive values of k_x etc. , and the spacing is $2\pi/L$. For a large system, one obtains the same results for the density of states for *any* boundary conditions.]

   Hence the density of points in k-space in two dimensions is $(L/\pi)^2 = A/\pi^2$ where A is the area the system. Hence the number of points in k-space which have magnitude of the wavevector between k and $k + dk$ is
$$\frac{A}{\pi^2}\, 2\pi k\, dk\, \frac{1}{4}\,, \tag{1}$$
   since the area of a thin ring of radius k and width dk is $2\pi k dk$, but we are only interested in the positive quadrant ($k_x > 0, k_y > 0$), so we divide by 4. Hence the density of states as a function of $k, \widetilde{\rho}(k)$, is given by
$$\widetilde{\rho}(k) = \frac{A}{2\pi} k\,. \tag{2}$$
   We now convert this to energy ϵ using
$$\widetilde{\rho}(k)\, dk = \rho(\epsilon)\, d\epsilon \tag{3}$$
   and the given dispersion relation.

   (a) For electrons, $\epsilon = \hbar^2 k^2/2m$ and so $kdk = m d\epsilon/\hbar^2$. We also need to multiply by 2 for the spin degeneracy. Hence Eqs. (2) and (3) gives
$$\boxed{\rho(\epsilon) = A\, \frac{m}{\pi\hbar^2}\,.}$$
   Note that this is a *constant* density of states.

   (b) For photons, $\epsilon = \hbar c k$ so $kdk = \epsilon d\epsilon/(\hbar c)^2$. We also need to multiply by 2 for the two photon polarizations, which gives
$$\boxed{\rho(\epsilon) = A\, \frac{1}{\pi(\hbar c)^2}\, \epsilon\,.}$$

2. The density of states for photons is
$$\rho(\epsilon) = \frac{V}{\pi^2} \frac{1}{(\hbar c)^3} \epsilon^2\,.$$
   Since the mean number of photons in a single mode is given by the Planck distribution, the total number of photons, summed over all modes, is given by
$$N = \frac{V}{\pi^2} \frac{1}{(\hbar c)^3} \int_0^\infty \frac{\epsilon^2}{e^{\beta\epsilon} - 1}\, d\epsilon\,.$$

Writing $\beta\epsilon = x$, this gives

$$\boxed{N = \frac{V}{\pi^2}\left(\frac{k_B T}{\hbar c}\right)^3 \int_0^\infty \frac{x^2}{e^x - 1}\, dx\,.}$$

As stated in the question, the value of the integral is $\boxed{2.404}$ and so we obtain the desired result.

3. (a) The total rate of energy generation of the sun is the "solar constant" times the area (in cm.^{2}) of a sphere whose radius is the earth-sun distance $= 1.5 \times 10^{13}$ cm, i.e. it is given by

$$0.136 \times 4\pi(1.5 \times 10^{13})^2 \simeq \boxed{4 \times 10^{26}\,\mathrm{J\,s^{-1}}\,.}$$

(b) This is equal to the area of the sun $[4\pi(7 \times 10^{10})^2]$, times the Stefan-Boltzmann constant $[5.67 \times 10^{-12}]$, times T^4 where T is the temperature of the surface of the sun, and so

$$4 \times 10^{26} = 4\pi(7 \times 10^{10})^2 \times 5.67 \times 10^{-12} \times T^4\,,$$

and so

$$\boxed{T = \left(\frac{4 \times 10^{26}}{4\pi(7 \times 10^{10})^2 \times 5.67 \times 10^{-12}}\right)^{1/4} \simeq 6000\,K\,.}$$

4. (a) As discussed in class

$$P = -\left(\frac{\partial U}{\partial V}\right)_S\,.$$

Now

$$U = \sum_j \hbar\omega_j n_j\,,$$

where j refers to a particular mode, and n_j is the mean photon occupation number which is given by the Planck distribution. The entropy is determined by the occupation numbers (see for example the Boltzmann definition of entropy discussed in class, and Qu. 7 in this assignment). Hence keeping the entropy constant is obtained by keeping the n_j constant. This gives

$$\boxed{P = -\sum_j \hbar\,\frac{\partial\omega_j}{\partial V}\,n_j\,.}$$

(b) Now

$$\omega_j = \hbar c k_j = \hbar c\,\frac{\pi}{L}\,(n_x, n_y, n_z)\,,$$

where the n_x etc. are integers specifying $\vec{k}$, and $V = L^3$. Hence $\omega_j = \text{const.}\,V^{-1/3}$ and so

$$\boxed{\frac{\partial\omega_j}{\partial V} = -\frac{\omega_j}{3V}\,.}$$

(c) From parts (a) and (b) we have

$$\boxed{P = \frac{1}{3V}\sum_j \hbar\omega_j = \frac{U}{3V}\,,}$$

so the pressure is $(1/3)\times$ (energy density).

5. (a) The total energy of the photon gas is given by

$$E = \sum_l \epsilon_l n_l \,,$$

where l denotes a photon mode and, n_l is the occupancy of the mode, i.e. $n_l = 0, 1, 2, \cdots$. (Note that here n_l is *not* the average occupancy (Planck distribution), which we will denote by $\langle n_l \rangle$.)
The sum over states clearly corresponds to summing over all the n_l. Hence the partition function is given by

$$Z = \sum_{\text{states}} \exp(-\beta E) = \sum_{\{n_l\}} \exp\left[-\beta \sum_l \epsilon_l n_l\right] \,, \tag{4}$$

where $\{n_l\}$ means sum over the *set* of n_l. Now the exponential of a sum is a product of exponentials, and the n_l can be summed over independently, so we can write Eq. (4) as

$$Z = \prod_l \left[\sum_{n_l=0}^{\infty} \exp(-\beta \epsilon_l n_l)\right] = \boxed{\prod_l z_l \,,}$$

where

$$\boxed{z_l = \sum_{n_l=0}^{\infty} \exp(-\beta \epsilon_l n_l) = [1 - \exp(-\beta \hbar \omega_l)]^{-1}}$$

is the partition function of mode l.

(b) Since the partition function is the product of partition functions for the individual modes, and since $F = -k_B T \ln Z$, the free energy is the sum of free energies of the modes, i.e.

$$F = k_B T \sum_l \ln\left[1 - \exp(-\beta \hbar \omega_l)\right] = \frac{V}{\pi^2} \frac{k_B T}{(\hbar c)^3} \int_0^\infty \epsilon^2 \ln\left[1 - e^{-\beta\epsilon}\right] d\epsilon \,,$$

where the second expression has used the density of photon states in a box. Writing $x = \beta\epsilon$ the last equation becomes

$$F = \frac{V}{\pi^2} \frac{(k_B T)^4}{(\hbar c)^3} \int_0^\infty x^2 \ln\left[1 - e^{-x}\right] dx \,. \tag{5}$$

Writing the integral as I and integrating by parts we have

$$\begin{aligned}
I &= \int_0^\infty x^2 \ln\left[1 - e^{-x}\right] dx = \int_0^\infty \frac{d}{dx}\left(\frac{x^3}{3}\right) \ln\left[1 - e^{-x}\right] dx \\
&= \left\{\frac{x^3}{3} \ln\left[1 - e^{-x}\right]\right\}_0^\infty - \frac{1}{3} \int_0^\infty \frac{x^3}{e^x - 1} dx \\
&= -\frac{1}{3} \int_0^\infty \frac{x^3}{e^x - 1} dx \\
&= -\frac{\pi^4}{45} \,,
\end{aligned}$$

where the last line uses the value of the integral given in the question. Substituting this value for I into Eq. (5) we have

$$\boxed{F = -V \frac{\pi^2 (k_B T)^4}{45 (\hbar c)^3} \,.}$$

6. (Due next week.)

7. From Qu. (5) the free energy of a single photon mode is given by

$$f = k_B T \ln \left[1 - e^{-\beta\hbar\omega} \right] ,$$

and hence its entropy is

$$\sigma \equiv \frac{s}{k_B} = -\frac{1}{k_B} \left(\frac{\partial f}{\partial T} \right) = -\ln \left[1 - e^{-\beta\hbar\omega} \right] + \frac{\hbar\omega}{k_B T} \frac{1}{e^{\beta\hbar\omega} - 1} .$$

(Note that I use the symbol n for the photon occupation and s for the entropy per mode.) Now we have

$$\begin{aligned} \langle n+1 \rangle &= 1 + \frac{1}{e^{\beta\hbar\omega} - 1} = \frac{1}{1 - e^{-\beta\hbar\omega}} , \\ \frac{\langle n+1 \rangle}{\langle n \rangle} &= e^{\beta\hbar\omega} , \end{aligned}$$

and so

$$\sigma = \ln\langle n+1 \rangle + \langle n \rangle \ln \left(\frac{\langle n+1 \rangle}{\langle n \rangle} \right) = \boxed{\langle n+1 \rangle \ln\langle n+1 \rangle - \langle n \rangle \ln\langle n \rangle .}$$

8. From Eq. (23) of Ch. 4 the entropy of the photon gas is given by

$$S = \text{const. } V T^3 , \tag{6}$$

and hence in a process at constant entropy we have

$$T V^{1/3} = \text{const.} \tag{7}$$

(Note that Eq. (6) can also be obtained by differentiating the final result of Qu. (5) with respect to T.)

   (a) The temperature of the cosmic microwave background is now $T_f = 2.73\,\mathrm{K}$. If it was originally $T_i = 3000\,\mathrm{K}$ and assuming the expansion takes place at constant entropy, we have from Eq. (7)

   $$T_i R_i = T_f R_f ,$$

   where R_i and R_f are the initial and final radii of the universe. (Note that the volume is proportional to the radius cubed.) This gives

   $$\boxed{\frac{R_i}{R_f} = \frac{T_f}{T_i} = \frac{2.73}{3000} \simeq 9 \times 10^{-4} .}$$

   (b) If the rate of expansion is constant, the time since the big bang is proportional to the radius and so

   $$\boxed{\frac{t_i}{t_f} = 9 \times 10^{-4} ,}$$

   i.e. the universe was about a thousandth of its present age.