PHYSICS 112
Homework 6 Solutions

1. (a) The thermodynamic identity is
$$TdS = dU + PdV.$$
If the volume changes by a small amount dV, and the temperatures by dT at constant entropy we have $0 = dU + PdV$. Now $dU = C_V dT$ since the energy of an ideal gas only depends on T, not the volume, so
$$\boxed{C_V dT + PdV = 0\,.}$$
(b) Writing $P = Nk_BT/V$ and recalling that $Nk_B = C_P - C_V$ we get
$$C_V \frac{dT}{T} + (C_P - C_V)\frac{dV}{V} = 0.$$
Dividing by C_V gives the desired result
$$\boxed{\frac{dT}{T} + (\gamma - 1)\frac{dV}{V} = 0,}$$
where $\gamma = C_P/C_V$.
(c) Integrating the last expression gives
$$\ln T + (\gamma - 1)\ln V = \text{const.}$$
and exponentiating gives
$$\boxed{TV^{\gamma-1} = \text{const.}}$$
Substituting $T = PV/Nk_B$ into the result of the previous part gives
$$\frac{PV}{Nk_B}V^{\gamma-1} = \text{const.}$$
and so
$$\boxed{PV^\gamma = \text{const}'.}$$
where const$'$. is another constant. (Remember we are keeping N constant here.)
(d) At constant T, we have $PV = C$ where $C = Nk_BT$ is constant, and so
$$B_T = -V\left(\frac{\partial P}{\partial V}\right)_T = (-V)\left(-\frac{C}{V^2}\right) = \frac{C}{V} = \boxed{P\,.}$$
At constant entropy, we have $PV^\gamma = C$, and so
$$B_S = -V\left(\frac{\partial P}{\partial V}\right)_S = (-V)\left(-\frac{\gamma C}{V^{(1+\gamma)}}\right) = \frac{\gamma C}{V^\gamma} = \boxed{\gamma P\,.}$$

2. The density of states in two dimensions was worked out in HW 3, Qu. 1 and is
$$\rho(\epsilon) = A\frac{m}{2\pi\hbar^2}\,,$$
where we divided that expression by 2 since we have spin $= 0$ here. Note that this is *independent of* ϵ.

As in three dimensions we have

$$\Omega = -k_B T \lambda \mathrm{z}^{(1)}$$

where now

$$\mathrm{z}^{(1)} = \sum_l e^{-\beta\epsilon_l} = A\,\frac{m}{2\pi\hbar^2}\int_0^\infty e^{-\beta\epsilon}\,d\epsilon \;= A\,\frac{mk_BT}{2\pi\hbar^2} = \frac{A}{A_Q}\,,$$

where A_Q, the "quantum area", is given by

$$A_Q = \frac{2\pi\hbar^2}{mk_BT}\,.$$

Note that $A_Q = V_Q^{2/3}$.

Results for the free energies for classical ideal gas in three dimensions go over with V replaced by A and V_Q replaced by A_Q. In particular:

(a)

$$\boxed{\mu = k_BT\ln(nA_Q) = -k_BT\ln\left[\frac{1}{n}\left(\frac{mk_BT}{2\pi\hbar^2}\right)\right]\,,}$$

where $n = N/A$ is the areal density.

(b) Also

$$F = Nk_BT\left[\ln(nA_Q) - 1\right]\,.$$

We obtain U from $U = (\partial/\partial\beta)(\beta F)$ which gives

$$\boxed{U = Nk_BT\,,}$$

noting that $A_Q \sim T^{-1}$ (not $T^{-3/2}$ which is the result in three dimensions).

(c) In the same way, $S = -\partial F/\partial T$ gives

$$\boxed{S = Nk_B\left[2 - \ln(nA_Q)\right]\,.}$$

3. Let us define $V_1 = V, V_2 = 2V, V_3 = 4V$, where V is the initial volume, and similarly $T_1 = T\,(=300)$, $T_2 = T_1$, and the final temperature is T_3.

(a) $V_1 \to V_2$ is isothermal. As discussed in class the heat supplied is $Nk_BT\ln(V_2/V_1) = Nk_BT\ln 2$. $V_2 \to V_3$ is isentropic so no heat is added. Hence the total heat added is

$$Nk_BT\ln 2 = 6.02\times10^{23}\times0.693\times1.38\times10^{-23}\times300 = \boxed{1728\,\mathrm{J}\,.}$$

where we used that 1 mole contains Avogadro's number of molecules $N = 6.02\times10^{23}$.

(b) In the first process T is constant. In the second process we have $TV^{2/3} = \text{const.}$ and so

$$T_3 = T\left(\frac{1}{2}\right)^{2/3} = 0.63T = \boxed{189\,K\,.}$$

(c) As discussed in the book the increase in entropy is

$$\Delta S = Nk_B\ln(V_2/V_1) = Nk_B\ln 2 = \boxed{5.76\,\mathrm{J\,K^{-1}}\,.}$$

.

4. We are given that the distribution of speeds is

$$P(v) = \sqrt{\frac{2}{\pi}} \left(\frac{m}{k_B T}\right)^{3/2} v^2 \exp\left(-\frac{mv^2}{2k_B T}\right) . \tag{1}$$

To answer this question we will need certain results for Gaussian integrals mentioned in class

$$\int_0^\infty e^{-a^2x^2/2}\, dx = \sqrt{\frac{\pi}{2}} \frac{1}{a} , \tag{2}$$

$$\int_0^\infty x^2 e^{-a^2x^2/2}\, dx = \sqrt{\frac{\pi}{2}} \frac{1}{a^3} , \tag{3}$$

$$\int_0^\infty x^4 e^{-a^2x^2/2}\, dx = 3\sqrt{\frac{\pi}{2}} \frac{1}{a^5} . \tag{4}$$

Note that Eq. (3) shows that the distribution in Eq. (1) is correctly normalized, i.e. $\int_0^\infty P(v)\, dv = 1$. We will also need

$$\int_0^\infty x e^{-a^2x^2/2}\, dx = \frac{1}{a^2} , \tag{5}$$

which is easy because indefinite integral is $-(1/a^2)e^{-a^2x^2/2}$, and

$$\int_0^\infty x^3 e^{-a^2x^2/2}\, dx = \frac{2}{a^4} , \tag{6}$$

which is done by integrating by parts to make it look like Eq. (5).

(a) Using Eq. (4) with $a^2 = m/k_BT$ we get

$$\langle v^2 \rangle = \int_0^\infty v^2 P(v)\, dv = \sqrt{\frac{2}{\pi}} \left(\frac{m}{k_B T}\right)^{3/2} 3\sqrt{\frac{\pi}{2}} \left(\frac{k_B T}{m}\right)^{5/2} = \frac{3k_B T}{m} ,$$

and so the rms velocity is given by

$$\boxed{v_{\rm rms} \equiv \langle v^2 \rangle^{1/2} = \sqrt{\frac{3k_B T}{m}} .}$$

(b) The most probable value of the speed, $v_{\rm mp}$, is where $P(v)$ in Eq. (1) has a maximum, i.e.

$$\left(2v_{\rm mp} - v_{\rm mp}^2 \frac{mv_{\rm mp}}{k_B T}\right) \exp\left(-\frac{mv_{\rm mp}^2}{2k_B T}\right) = 0,$$

i.e.

$$\boxed{v_{\rm mp} = \sqrt{\frac{2k_B T}{m}} .}$$

(c) From Eq. (3), the mean speed is given by

$$\langle v \rangle = \int_0^\infty v P(v)\, dv = \sqrt{\frac{2}{\pi}} \left(\frac{m}{k_B T}\right)^{3/2} \int_0^\infty v^3 \exp\left(-\frac{mv^2}{2k_B T}\right) dv = \sqrt{\frac{2}{\pi}}\, 2\sqrt{\frac{k_B T}{m}} = \boxed{\sqrt{\frac{8k_B T}{\pi m}}.}$$

(d) We are also given that the probability for a single component of velocity is

$$P_z(v_z) = \sqrt{\frac{m}{2\pi k_B T}} \exp\left(-\frac{mv_z^2}{2k_B T}\right) .$$

Noting that v_z can have either sign we get

$$\langle |v_z| \rangle = \int_{-\infty}^{\infty} |v_z| P_z(v_z)\, dv_z = 2\int_0^{\infty} v_z P_z(v_z)\, dv_z = 2\sqrt{\frac{m}{2\pi k_B T}} \int_0^{\infty} v_z \exp\left(-\frac{mv_z^2}{2k_B T}\right) = \boxed{\sqrt{\frac{2k_B T}{\pi m}},}$$

where we used Eq. (5) to get the final result. Note that $\langle |v_z| \rangle = \frac{1}{2}\langle v \rangle$.

5. Following the discussion in class, the number of states in which the magnitude of the wavevector lies between k and $k + dk$ is

$$2\left(\frac{L}{\pi}\right)^3 \frac{4\pi k^2\, dk}{8} .$$

We write this as $\rho(\epsilon)\, d\epsilon = \rho(\epsilon)\,(\partial\epsilon/\partial k)\, dk$, and so

$$\rho(\epsilon) = \frac{V}{\pi^2} k^2 \frac{1}{(\partial\epsilon/\partial k)} = \frac{V}{\pi^2} k^2 \frac{1}{\hbar c} = \frac{V}{\pi^2} \frac{\epsilon^2}{(\hbar c)^3} .$$

(a) At $T = 0$ we fill up all the states up to ϵ_F, i.e.

$$N = \frac{V}{\pi^2 (\hbar c)^3} \int_0^{\epsilon_F} \epsilon^2\, d\epsilon = \frac{V}{3\pi^2}\left(\frac{\epsilon_F}{\hbar c}\right)^3 . \tag{7}$$

This can be rearranged as

$$\boxed{\epsilon_F = \pi^{2/3} \hbar c\, (3n)^{1/3} .}$$

(b) The energy is given by

$$U = \int_0^{\epsilon_F} \epsilon\rho(\epsilon)\, d\epsilon = \frac{V}{4\pi^2} \frac{\epsilon_F^4}{(\hbar c)^3} = \boxed{\frac{3}{4} N\epsilon_F\, ,}$$

where we used Eq. (7).

6. (a) As shown in the book, the energy of an ideal Fermi gas at $T = 0$ is

$$U = \int_0^{\epsilon_F} \epsilon\rho(\epsilon)\, d\epsilon = \frac{3}{5} N\epsilon_F = \frac{3}{10}\, N\, \frac{\hbar^2}{m} \left(\frac{3\pi^2 N}{V}\right)^{2/3} . \tag{8}$$

From the thermodynamic identity

$$dU = TdS - PdV$$

we have

$$P = -\left(\frac{\partial U}{\partial V}\right)_S .$$

However at $T = 0$ the entropy is zero, (Third Law) and so constant S is equivalent to constant T. Hence we can obtain the pressure by differentiating Eq. (8) with respect to V, i.e.

$$P = -\frac{\partial U}{\partial V} = \boxed{\frac{(3\pi^2)^{2/3}}{5} \frac{\hbar^2}{m} n^{5/3} .}$$