

**PHYSICS 112**  
**Homework 7 Solutions**

1. Since

$$k = \frac{n\pi}{L}$$

where  $n = 1, 2, 3, \dots$  the number of states with wavevector between  $k$  and  $k + dk$  is  $(L/\pi)dk$ . If the corresponding (infinitesimal) range of energy is  $d\epsilon$  then we have

$$\rho(\epsilon) d\epsilon = 2 \frac{L}{\pi} dk, \tag{1}$$

where the factor of 2 comes because there are 2 electron spin states for each wavevector, and  $\epsilon = \hbar^2 k^2 / (2m)$ . Now we have

$$\frac{d\epsilon}{dk} = \frac{\hbar^2 k}{m} = \frac{\hbar^2}{m} \sqrt{\frac{2m\epsilon}{\hbar^2}} = \sqrt{\frac{2\hbar^2 \epsilon}{m}},$$

and substituting into Eq. (1) gives

$$\rho(\epsilon) = \frac{2L}{\pi} \sqrt{\frac{m}{2\hbar^2}} \epsilon^{-1/2} = \boxed{\frac{L}{\pi} \sqrt{\frac{2m}{\hbar^2}} \epsilon^{-1/2}}.$$

2. The question is asking why  $\mu(T)$  *decreases* with  $T$  in  $d = 3$  whereas it initially *increases* with  $T$  in  $d = 1$ , as shown in Fig. 7.7 in the book. The difference is related to the fact that the density of states *increases* with  $\epsilon$  in  $d = 3$  ( $\rho(\epsilon) \sim \epsilon^{1/2}$ ) whereas it *decreases* in  $d = 1$  ( $\rho(\epsilon) \sim \epsilon^{-1/2}$ ).

Now at  $T = 0$  we have

$$N = \int_0^{\epsilon_F} \rho(\epsilon) d\epsilon, \tag{2}$$

where  $\epsilon_F \equiv \mu(T = 0)$ . Furthermore, at finite- $T$  we have

$$N = \int_0^{\infty} \rho(\epsilon) f(\epsilon - \mu) d\epsilon = \int_0^{\infty} \frac{\rho(\epsilon)}{e^{(\epsilon - \mu(T))/k_B T} + 1} d\epsilon, \tag{3}$$

where

$$f(x) = \frac{1}{e^{\beta x} + 1},$$

is the Fermi-Dirac distribution. The values of  $N$  in Eqs. (2) and (3) are, of course, equal. Note that

$$f(-x) = 1 - f(x), \tag{4}$$

which is easy to verify. In going from  $T = 0$  to small but non-zero  $T$  electrons are moved from states just below  $\epsilon_F$  to states just above  $\epsilon_F$ .

Consider, for example, the case of  $\boxed{d = 1}$  for which  $\rho(\epsilon)$  is a decreasing function. Then, from Eq. (4), if  $\mu(T)$  were maintained equal to  $\epsilon_F$  we would remove more electrons from below  $\epsilon_F$  than we would put into states above  $\epsilon_F$  (simply because there are more states below  $\epsilon_F$ ). This would decrease the total number of electrons. In order to maintain the number of electrons at the fixed value of  $N$  we therefore have to  $\boxed{\text{increase } \mu}$ .

Similarly for  $\boxed{d = 3}$  where  $\rho(\epsilon)$  is a decreasing function we have to  $\boxed{\text{decrease } \mu}$  to keep the number of particles constant.

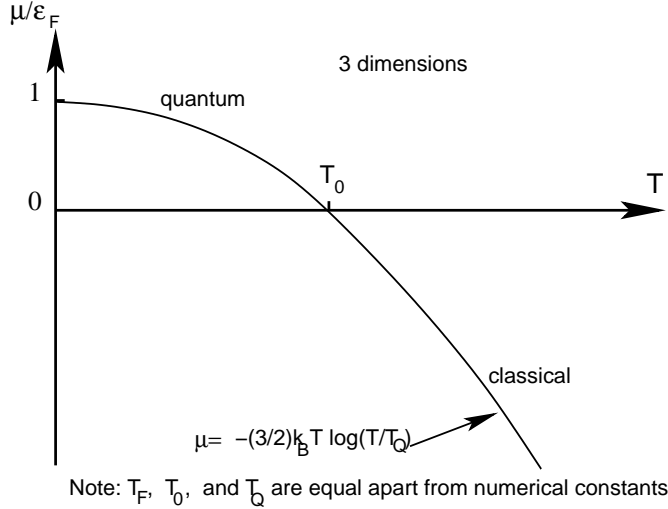


Figure 1: Sketch of  $\mu(T)$  with  $T$  in three dimensions.

A sketch of  $\mu(T)/\epsilon_F$  against  $T$  is given in Fig. 1.

Note that  $\epsilon_F = \mu(T = 0)$  and  $T_F = \epsilon_F/k_B$ . The temperature where  $\mu = 0$  is denoted by  $T_0$ . We have also showed in class that, in the high temperature, i.e. classical, limit,  $\mu(T) = -\frac{3}{2}k_B T \log T/T_Q$ . All these temperatures,  $T_F, T_Q$ , and  $T_0$  are of the form

$$T_i = \frac{c_i}{k_B} \frac{\hbar^2}{m} n^{2/3}, \quad (3-d),$$

where  $n = N/V$  and the  $c_i$  are numerical constants of order unity, which are different for  $i = F, Q$  and 0.

In one dimension, the variation of  $\mu(T)$  with  $T$  is fairly similar except that, as found in this question,  $\mu$  initially increases with  $T$  before decreasing. This is shown in Fig. (2).

3. (a) The Fermi energy  $\epsilon_F$  is determined from

$$N = \int_0^{\epsilon_F} \rho(\epsilon) d\epsilon$$

i.e.

$$N = A \frac{m}{\pi \hbar^2} \int_0^{\epsilon_F} d\epsilon = A \frac{m}{\pi \hbar^2} \epsilon_F, \quad (5)$$

which can be written

$$\boxed{\epsilon_F = n \frac{\pi \hbar^2}{m}}, \quad (6)$$

where  $n = N/A$ .

- (b) At finite- $T$ ,  $N$  is given by

$$N = \int_0^{\infty} \frac{\rho(\epsilon)}{e^{\beta(\epsilon-\mu)} + 1} d\epsilon = A \frac{m}{\pi \hbar^2} \int_0^{\infty} \frac{d\epsilon}{e^{\beta(\epsilon-\mu)} + 1},$$

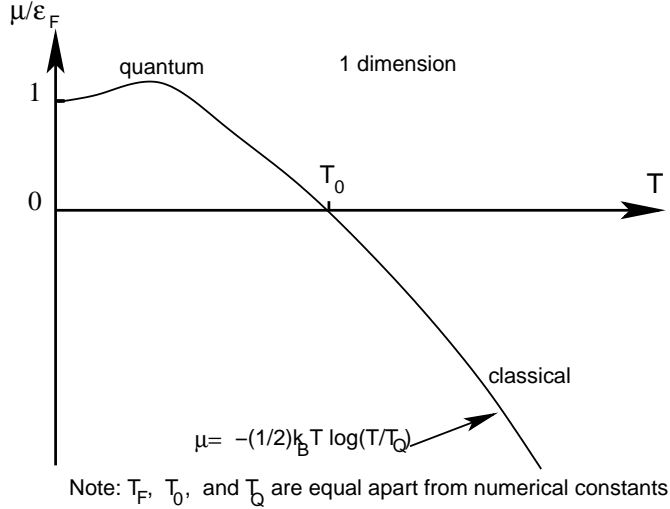


Figure 2: Sketch of  $\mu(T)$  with  $T$  in one dimension.

which, using Eq. (5), can be expressed as

$$\begin{aligned}
 \epsilon_F &= \int_0^\infty \frac{d\epsilon}{e^{\beta(\epsilon-\mu)} + 1} \\
 &= \int_0^\infty \frac{e^{\beta(\mu-\epsilon)} d\epsilon}{1 + e^{\beta(\mu-\epsilon)}} \\
 &= -\frac{1}{\beta} \left[ \ln \left( 1 + e^{\beta(\mu-\epsilon)} \right) \right]_0^\infty \\
 &= \frac{1}{\beta} \ln \left( 1 + e^{\beta\mu} \right).
 \end{aligned}$$

Simplifying this gives

$$e^{\beta\mu} = e^{\beta\epsilon_F} - 1,$$

which can be written as

$$\begin{aligned}
 \frac{\mu(T)}{\epsilon_F} &= \frac{k_B T}{\epsilon_F} \ln \left( e^{\epsilon_F/k_B T} - 1 \right) \\
 &= \boxed{t \ln \left( e^{1/t} - 1 \right)}, \tag{7}
 \end{aligned}$$

where  $t = k_B T / \epsilon_F = T / T_F$ .

A plot of Eq. (7) is given in Fig. 3. Note that whereas Figs. 1 and 2 are *sketches*, Fig. 3 is a *precise plot*.

(c) For  $t \ll 1$ ,  $e^{1/t} \gg 1$ , so  $t \ln(e^{1/t} - 1) \simeq t \ln e^{1/t} = 1$ , and hence

$$\boxed{\mu(T \rightarrow 0) = \epsilon_F.}$$

(d) For  $t \gg 1$  we expand  $e^{1/t} = 1 + (1/t) + \dots$  and so

$$\mu(T) = \epsilon_F \frac{k_B T}{\epsilon_F} \ln \left( \frac{T_F}{T} \right) = -k_B T \ln \left( \frac{T}{T_F} \right). \tag{8}$$

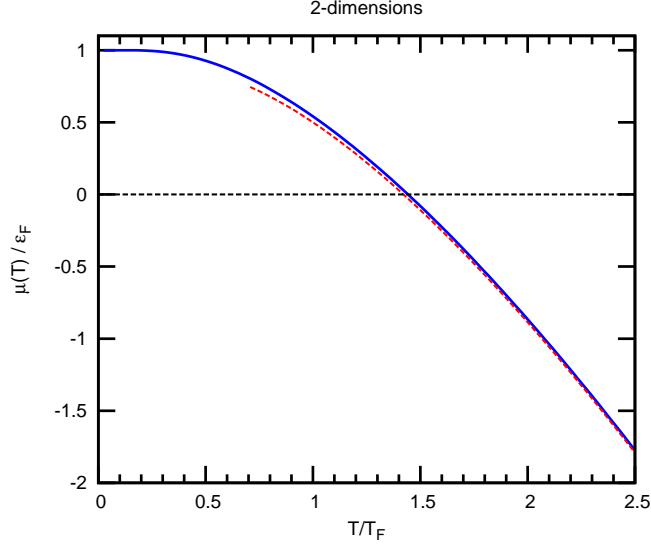


Figure 3: Plot of  $\mu(T)/\epsilon_F$  against  $T/T_F$  in two dimensions, according to Eq. (7). The dashed line is the high temperature (classical) limit  $\mu(T)/\epsilon_F = -t \log t + \frac{1}{2}$  with  $t = T/T_F$ . (Strictly speaking,  $\frac{1}{2}$  is a sub-leading term going beyond the classical limit.) The chemical potential vanishes at  $T = T_0$  where  $T_0/T_F = 1/\log(2) = 1.443$ . In the quantum limit,  $T \ll T_F$ ,  $\mu(T)$  tends to a constant which is called the Fermi energy,  $\epsilon_F$ .

This function, including the next term in the expansion (which is  $\frac{1}{2}$ ), is shown as the dashed line in Fig. 3. Since

$$\frac{T}{T_F} = \frac{k_B T}{\epsilon_F} = \frac{m k_B T}{n \pi \hbar^2},$$

where we used Eq. (6), Eq. (8) can be written

$$\boxed{\mu(T) = -k_B T \ln \left[ \frac{1}{n} \left( \frac{m k_B T}{\pi \hbar^2} \right) \right]},$$

as required.

4. In the limit  $\mu \rightarrow 0^-$ , the number of particles that can be put in the  $\epsilon > 0$  states is given by

$$N_e(T) = \int_0^\infty \frac{\rho(\epsilon)}{e^{\epsilon/k_B T} - 1} d\epsilon,$$

where  $\rho(\epsilon)$ , the single-particle density of states, is given in  $d = 1$  by

$$\rho(\epsilon) = \frac{L}{\pi} \sqrt{\frac{2m}{\hbar^2 \epsilon}}.$$

To control the integral we put a small but non zero lower limit,  $\delta$  and so

$$\begin{aligned} N_e(T) &= \frac{L}{\pi} \sqrt{\frac{2m}{\hbar^2}} \lim_{\delta \rightarrow 0} \int_\delta^\infty \frac{1}{\epsilon^{1/2}} \frac{1}{e^{\epsilon/k_B T} - 1} d\epsilon \\ &= \frac{L}{\pi} \sqrt{\frac{2m k_B T}{\hbar^2}} \lim_{\delta \rightarrow 0} \int_{\delta/k_B T}^\infty \frac{1}{x^{1/2}} \frac{1}{e^x - 1} dx \end{aligned}$$

The integrand diverges as  $1/x^{3/2}$  for  $x \rightarrow 0$  and hence the integral diverges as  $[1/x^{1/2}]_{x=\delta/k_B T}$ , and so, for  $\delta \rightarrow 0$  we have

$$N_e(T) = \frac{L}{\pi} \sqrt{\frac{2m}{\hbar^2}} \frac{k_B T}{\sqrt{\delta}},$$

which diverges for  $\delta \rightarrow 0$  at any non-zero temperature.

*Note:* This means that we can put as many particles as we need to in the  $\epsilon > 0$  states, so there will be no macroscopic occupancy of the  $\epsilon = 0$  state, i.e. no Bose-Einstein condensation.

5. In the Bose condensed state, the chemical potential is essentially zero. A finite fraction of the particles is in the  $\epsilon = 0$  state, but these don't contribute to the energy or specific heat. (One might worry that they could contribute to the entropy, but a calculation shows that they only give a *total* contribution  $\ln N_0$ , where  $N_0$  is the number of particles in the condensate, and so the entropy per spin vanishes in the thermodynamic limit.)

Using the expression for the density of states discussed in class, we have

$$U = \frac{V}{4\pi^2} \left( \frac{2m}{\hbar^2} \right)^{3/2} \int_0^\infty \frac{\epsilon \epsilon^{1/2}}{e^{\beta\epsilon} - 1} d\epsilon$$

so

$$\boxed{\frac{U}{V} = \frac{I}{4\pi^2} \left( \frac{2m}{\hbar^2} \right)^{3/2} (k_B T)^{5/2}}, \quad (9)$$

where

$$I = \int_0^\infty \frac{x^{3/2}}{e^x - 1} dx.$$

The numerical value of  $I$  is  $I = 1.78329\dots$ , but you don't need to show this.

The specific heat at constant volume is obtained by differentiating Eq. (9) with respect  $T$ :

$$\boxed{\frac{C_V}{V k_B} = \frac{5I}{8\pi^2} \left( \frac{2m k_B T}{\hbar^2} \right)^{3/2}}. \quad (10)$$

The entropy is obtained from

$$S(T) - S(0) = \int_0^T \frac{C_V(T')}{T'} dT'.$$

Using Eq. (10) and the third law ( $S(0) = 0$ ) one gets

$$\boxed{\frac{S}{V k_B} = \frac{5I}{12\pi^2} \left( \frac{2m k_B T}{\hbar^2} \right)^{3/2}}.$$

6. (a) For a single fermion state we have

$$\langle n \rangle = \frac{0 + 1 \cdot e^{\beta(\mu - \epsilon)}}{1 + e^{\beta(\mu - \epsilon)}} = \frac{1}{e^{\beta(\epsilon - \mu)} + 1},$$

and similarly

$$\langle n^2 \rangle = \frac{0^2 + 1^2 \cdot e^{\beta(\mu - \epsilon)}}{1 + e^{\beta(\mu - \epsilon)}} = \frac{1}{e^{\beta(\epsilon - \mu)} + 1} = \langle n \rangle.$$

Consequently

$$\boxed{\langle (\Delta n)^2 \rangle \equiv \langle n^2 \rangle - \langle n \rangle^2 = \langle n \rangle (1 - \langle n \rangle) .}$$

(b) For a single boson state the grand partition function is given by

$$Z = \frac{1}{1 - e^x} ,$$

where  $x = \beta(\mu - \epsilon)$ . Now

$$\begin{aligned} \langle n \rangle &= \frac{1}{Z} \sum_{n=0}^{\infty} n e^{nx} \\ &= \frac{1}{Z} \frac{\partial Z}{\partial x} \\ &= \frac{1}{e^{-x} - 1} = \frac{1}{e^{\beta(\epsilon - \mu)} - 1} , \end{aligned}$$

and similarly

$$\begin{aligned} \langle n^2 \rangle &= \frac{1}{Z} \sum_{n=0}^{\infty} n^2 e^{nx} \\ &= \frac{1}{Z} \frac{\partial^2 Z}{\partial x^2} \\ &= (1 - e^x) \frac{e^x(1 + e^x)}{(1 - e^x)^3} = \frac{1 + e^{-x}}{(e^{-x} - 1)^2} \\ &= 2\langle n \rangle^2 + \langle n \rangle . \end{aligned}$$

Hence we have

$$\boxed{\langle (\Delta n)^2 \rangle \equiv \langle n^2 \rangle - \langle n \rangle^2 = \langle n \rangle (\langle n \rangle + 1) .}$$

7. In the heat pump, heat  $Q_{\text{low}}$  is absorbed from outside at temperature  $T_{\text{low}}$  and greater amount of heat  $Q_{\text{high}}$  is emitted into the inside of the building at a higher temperature  $T_{\text{high}}$ . The difference is the work done

$$W = Q_{\text{high}} - Q_{\text{low}} .$$

Now in a Carnot cycle

$$\frac{Q_{\text{high}}}{T_{\text{high}}} = \frac{Q_{\text{low}}}{T_{\text{low}}} ,$$

and so

$$\boxed{\frac{W}{Q_{\text{high}}} = 1 - \frac{Q_{\text{low}}}{Q_{\text{high}}} = 1 - \frac{T_{\text{low}}}{T_{\text{high}}} .}$$