Quantum Integrability in Systems with Finite Number of Levels

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Abstract We consider the problem of defining quantum integrability in systems with finite number of energy levels starting from commuting matrices and construct new general classes of such matrix models with a given number of commuting partners. We argue that if the matrices depend on a (real) parameter, one can define quantum integrability from this feature alone, leading to specific results such as exact solvability, Poissonian energy level statistics and to level crossings.

1 Introduction

The field of quantum integrable systems, originally a somewhat abstruse topic, has received a great deal of attention in recent years with the realization that many typical models are realizable in atomic, mesoscopic and macroscopic many-body systems. In describing molecular systems such as benzene, we deal with Hamiltonians defined on a finite-dimensional Hilbert space. These arise from a lattice of finite size, with spins, fermions or bosons populating the sites. In condensed matter systems too, one often studies finite size systems as a prelude to taking the thermodynamic limit or to obtain finite size corrections. Our concern in this work is with the construction of general quantum integrable models, their characterization

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and definition of quantum integrability in a finite dimensional context,[1] a task that is considerably more delicate than the corresponding classical case.

We may view typical condensed matter problems as models with a finite number of discrete single-particle energy levels, and sometimes (but not always) with a fixed number of particles. The Hamiltonian is then 'just' an $N \times N$ Hermitian matrix, with a suitable N. This description is, by design, far removed from its parentage in the space of many body models. Given such a Hamiltonian matrix, can we say whether it is integrable or not? What is the precise notion of quantum integrability in this case; can we separate $N \times N$ Hermitian matrices into two clearly distinct classes—integrable and otherwise? If yes, what are the consequences of quantum integrability that can be *derived from its definition*, i.e. the characteristic properties of such 'integrable' matrices?

In this paper, we present a coherent view point that emerges from our recent exploration of such questions. We present a summary of our previous [1–5] and new results and list some open questions that remain unanswered. To put this enquiry into context, note that in the thermodynamic limit $N \to \infty$, impressive alternative approaches are available. For example Baxter's work on the spectra of infinite dimensional corner transfer matrices [6] culminates in the realization of extra symmetries and structure arising in that limit, formalized by the Yangian approach [7, 8]. Another viewpoint is of geometric origin due to Sutherland [9] and emphasizes non-diffractive scattering as a true hallmark of quantum integrability. Much as we admire these powerful viewpoints, our chosen task of understanding finite dimensional matrix systems takes us in quite a different direction.

In classical mechanics a system with n degrees of freedom is said to be integrable if it has *n functionally independent* integrals of motion that Poisson commute [10], i.e. are in involution. An exact solution of equations of motion follows from this definition, and it further can be shown that the motion is confined to invariant tori cut out in the phase space by the conservation laws. Unfortunately, a straightforward import of this elegant notion of integrability into quantum mechanics is problematic.

The difficulties are at least two fold: firstly what is a degree of freedom? In quantum models with a fixed number of particles, this number seems closest to the classical notion of a degree of freedom. However, in addition to the particle number there are other integer parameters to take into account such as the number of sites or the magnitude of spins. For example, in the Hubbard model with L sites do we count the number of electrons n_e or the number of holes $2L - n_e$? Or does the Heisenberg model for L spin-$\frac{1}{2}$ particles have the same number of degrees of freedom as that for L spin-5 particles?

The second difficulty is in defining what constitutes a nontrivial integral of motion. From an elementary theorem in algebra [11], we know that an arbitrary $N \times N$ Hermitian matrix commutes with N other Hermitian matrices that may be chosen as the projection operators along each component in the diagonal representation. We could equivalently express the matrix and all its commuting partners as a power series in some chosen non degenerate matrix. A similar statement can be proved for general commuting Hermitian operators with discrete or continuous spectra [12–14]. If now we are given a special Hamiltonian matrix, and told that it is quantum integrable (in some specific sense that is yet to be defined), possessing n equally special commuting partner matrices; we might (rather stubbornly) ignore this extra

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[1]Imagine that we are given a specific Hermitean 6×6 matrix, can we determine if it is the descendant of a quantum integrable model in a particular number-symmetry subsector? An explicit example might be from the numerical study in Ref. [17], where a number of matrices are found by restricting the Hubbard model to a particular number sector with all space time symmetries factored out. We provide a systematic method of looking at such questions in this work.

information and compute the N elementary commuting partners of the given matrix as per the earlier prescription. Don't we then have too many partners? What, if any, is the distinction between these two sets of commuting partners? From such questions and considerations one seems to be forced to call integrable either all or none Hamiltonians with finite Hilbert spaces.

Our goal here is to attempt a resolution of these difficulties and provide a practically useful, simple and at the same time rigorous definition of quantum integrability for Hamiltonian matrices. Our interest is of course not in the definition per se but mostly in a systematic construction of new quantum integrable models and a clear delineation and derivation of their properties starting from the definition. Characteristics normally attributed to integrable models include: exact solution for their spectra, Poisson level statistics, crossings of levels of the same symmetry in parameter-dependent Hamiltonians etc. However, they are never derived from one another or within some unified framework, but each property has to be established independently and on a model by model basis.

Consider, for example, the 1d Hubbard model. Shortly after the model was exactly solved via Bethe's Ansatz [15, 16], a separate rigorous study by Heilmann and Lieb [17] of its parameter independent symmetries revealed numerous level crossings violating the Wigner von Neuman non-crossing rule for eigenvalues in generic systems. These findings can be rationalized by the notion that copious level crossings are a hallmark of quantum integrability, and hint at parameter dependent conservation laws in such a model. Furthermore, by solving the Bethe ansatz equations [18], it was shown that some degenerate energy levels that can not be explained in terms of U-independent symmetries have distinct values with respect to higher conserved quantities. Much later, one of us [19, 20] found an embedding of the 1d Hubbard model into the Yang–Baxter scheme, thereby displaying an explicit set of parameter dependent conservation laws. Not surprisingly, a numerical analysis [21] of level spacings finds Poisson statistics that crosses over to GOE when 'integrability' is destroyed by adding an off-site Coulomb repulsion. Further work [1] by us on making a causal link between the conservation laws and the level crossings in the 1d Hubbard model has been a fruitful source of insights that we have subsequently explored and report here.

Often, one of the above characteristics is singled out and adopted as a definition (see e.g. Ref. [22] for examples). It seems to us that while level statistics and crossings are useful, and even powerful tests of integrability, these are hardly suitable as *definitions*. These criteria arguably encompass a broader class of systems than those normally thought integrable.[2] It is also difficult to see how, starting from either of them, one could derive other properties, e.g. an exact solution. Universal statistics emerges only in the limit of large matrices, $N \to \infty$, or for an ensemble of matrices, while here we are looking for a notion that also works for a stand-alone Hamiltonian matrix with fixed N. In addition, there are exceptional points in parameter space of systems usually recognized as integrable, where the statistics is non-Poissonian, e.g. for Gaudin magnets [23]. Starting from a well-defined notion of integrability that we propose below, we construct broad classes of new integrable models where such deviations occur more generally and explain their origin. Similarly, we find examples that fail the level crossing test—such as 'accidental' degeneracies in non-integrable systems and (rare) instances of integrable systems without crossings.

It turns out that such a simple and yet well-defined notion can in fact be formulated as we recognized in a series of papers [1–5]. The main distinguishing feature of our approach

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[2]For example, the Anderson localization of electrons in a random potential is often viewed as a transition from Poisson (localized) to GOE (extended) statistics. In this context it is not obvious if or how the concept of quantum integrability plays a role.

is that it leads in a unified fashion to a general construction of new quantum integrable models, their exact solution as well as allows for a systematic study of various properties such as level crossings, level statistics etc. The main idea is to consider the dependence of commuting operators on a real parameter, which we denote u. This is typically an interaction constant or an applied field, e.g. Coulomb interaction constant in the Hubbard model, magnetic field in Gaudin magnets [24, 25], pairing strength in the BCS and anisotropy in the XXZ Heisenberg models, etc. Besides usual space-time and internal space symmetries, which are parameter independent, these models have u-dependent conservation laws (alternatively termed dynamical symmetries or conserved currents) as discussed above. For the BCS and Gaudin models the Hamiltonian and all conserved currents are linear in the parameter [24–26]. In the XXZ and Hubbard models the Hamiltonian and the first dynamical symmetry are linear in u; the rest are polynomials in it of order two and higher [19, 27–32].

The rest of the paper is organized as follows. In Sects. 2 through 6 we present a comprehensive summary of our previous results. For a detailed discussion and relevant derivations we refer the reader to Refs. [2–5]. Sections 7 and 9 are devoted to new results. In Sect. 7 we provide a link between our notion of quantum integrability and the conventional approach based on the Yang–Baxter equation. Section 9 summarizes our upcoming publication [33] on level statistics in models that are integrable under our definition.

2 Definition

In several examples of quantum integrable models of interest, such as the anisotropic Heisenberg model and the Hubbard model, there is a set of commuting operators linear or polynomial in an interaction type parameter, termed u here. This set of course includes the Hamiltonian, which is typically linear in u—e.g. the interaction constant U for the Hubbard model. This seems to be the most common situation in parameter dependent integrable models and leads us [2–5] to consider operators of the form $H(u) = T + uV$, where T and V are $N \times N$ Hermitian matrices. A key observation is that fixed u-dependence implies a natural well-defined notion of a nontrivial integral of motion. A typical (e.g. randomly generated) $H(u)$ commutes only with $(a + bu)\mathbf{1} + cH(u)$, a trivial operator linear in u. Here a, b, and c are real numbers and $\mathbf{1}$ is the identity operator.

The requirement that there exist a nontrivial commuting partner linear in u severely constrains the matrix elements of $H(u)$. In fact, as we will see below, a real symmetric $H(u)$ that has such a partner is fixed by less than $(N-1)(N+8)/2$ real parameters. In contrast, $N(N+1)$ real parameters are necessary to specify the matrix elements of T and V for a generic $H(u)$, indicating that matrices with fixed parameter-dependence split into two non-overlapping sets—those with nontrivial commuting partners and those without.

Moreover, there is a natural classification of integrable families according to the number n of linearly independent commuting operators. Specifically, we define an *integrable family* as a vector space of n linearly independent $N \times N$ Hermitian matrices

$$H^k(u) = T^k + uV^k \quad \text{such that,}$$

$$\left[H^i(u), H^j(u)\right] = 0 \quad \text{for all } u \text{ and } i, j = 1, \ldots, n. \tag{1}$$

In addition, we impose an (optional) condition that $H^i(u)$ have no common u-independent symmetries—there is no constant matrix Ω ($\neq a\mathbf{1}$) such that $[\Omega, H^i(u)] = 0$ for all u and i.

If there are such symmetries, $H^i(u)$ are simultaneously block-diagonal and Eq. (1) reduces to that for smaller matrices (blocks) without u-independent symmetries.[3]

Linear independence means that $\sum_i a_i H^i(u) = (a + bu)\mathbf{1}$ with real a_i if and only if all $a_i = 0$ and $a = b = 0$. Note that, for convenience, we chose to separate multiples of identity from our list of nontrivial commuting matrices $H^k(u)$. Finally, n is defined as the maximum number of nontrivial independent commuting matrices in a given family, i.e. any $H(u) = T + uV$ that commutes with all $H^i(u)$ can be written as

$$H(u) = \sum_{i=1}^{n} a_k H^k(u) + (a + bu)\mathbf{1}. \tag{2}$$

Thus $H^i(u)$ act as basis vectors in the n-dimensional vector space $\mathcal{V}_n$ of commuting matrices, defined up to a multiple of identity matrix $\mathbf{1}$.

We propose the following classification of integrable matrices linear in a parameter into *types*. The maximum possible number of linearly independent $N \times N$ commuting Hermitian matrices is $n = N - 1$, not counting multiples of the identity matrix. We call this a *type 1* or maximally commuting family of matrices. Similarly, families with $n = N - 2$ independent commuting $H^i(u)$ are termed *type 2*, and a general *type M* family is defined through $n = N - M$. The maximum value of M is $M = N - 2$ when there are *two* nontrivial members in the family *in addition* to $(a + bu)\mathbf{1}$. Note that $M = N - 1$ means $n = 1$, i.e. an arbitrary *nonintegrable* matrix that has no commuting partners besides itself and the identity. Since we study integrable matrices in this paper, everywhere below $M \geq N - 2$.

We argue that the following properties are direct consequences of the above definition of quantum integrability: (1) an exact solution for the eigenspectra of $H^i(u)$ in terms of roots of a *single* algebraic equation, (2) $H^i(u)$ satisfy Yang–Baxter equation, (3) eigenvalues of $H^i(u)$ typically (but not always) cross as functions of u; the number of crossings depends on both N and M, (4) eigenvalues have Poisson statistics in $N \to \infty$ limit except for some special cases of certain measure zero in the space of all integrable families. Below we *prove* statements (1) through (3) for type 1 families (in this case crossings are always present) and comment on similar results of [5] for other types. Poisson statistics will be demonstrated numerically in a separate paper; here we briefly discuss some of its main findings.

Equation (1) can be solved for matrix elements of $H^i(u)$ at least for some types of integrable families. All type 1 families were constructed in [2, 3]. In [5] all type 2, 3 and some type M for arbitrary M were obtained. To facilitate further discussion, let us first cast Eq. (1) into a different form. Using $H^k(u) = T^k + uV^k$ and equating to zero terms at all orders of u, we obtain

$$\left[T^i, V^j \right] = \left[T^j, V^i \right], \qquad \left[T^i, T^j \right] = \left[V^i, V^j \right] = 0. \tag{3}$$

It is convenient to choose the basis in the target Hilbert space to be the common eigenbasis of the mutually commuting matrices V^i. The first commutation relation in Eq. (3) in this

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[3]For example, in the 1d Hubbard model $H^i(u)$ represent blocks of the Hamiltonian and linear in u conserved currents corresponding to a certain complete set of u-independent symmetry quantum numbers, see [5] for more details.

basis reads[4] $T^i_{km}/(d^i_k - d^i_m) = T^j_{km}/(d^j_k - d^j_m) \equiv S_{km}$, where d^i_k are the diagonal elements of V^i. This implies that T^i can be written as

$$T^i = W^i + [V^i, S] \tag{4}$$

where W^i is a diagonal matrix. Note that the antihermitian matrix S is the same for all members of the family and is therefore independent of the basis in $\mathcal{V}_n$. Now the commutation relations $[V^i, V^j] = 0$ and $[T^i, V^j] = [T^j, V^i]$ are satisfied. The remaining equation, $[T^i, T^j] = 0$, takes the form

$$[[V^i, S], [V^j, S]] = [[V^j, S], W^i] - [[V^i, S], W^j]. \tag{5}$$

Before we proceed with type 1, let us discuss the number of parameters involved in constructing a generic real symmetric integrable family.

We can generate integrable families of various types by solving Eq. (5) numerically [5]. The algorithm is as follows. First, we arbitrarily specify some diagonal matrices V^i, V^j, W^i and W^j ($4N$ real inputs) and solve for the antisymmetric matrix S. This yields a discrete set of solutions for S and two commuting matrices $H^{i,j}(u) = W^{i,j} + [V^{i,j}, S] + uV^{i,j}$ for each S. To determine, the remaining basic matrices $H^k(u)$ in the family with a given S, we take $j = k$ in Eq. (5) (with V^i, W^i and S obtained before) and solve this equation for V^k and W^k.

Some choices of the $4N$ inputs in the above procedure produce the same commuting family. In fact, as seen from Eq. (2), there is $2(n + 2)$-parametric freedom (values of a_k, a and b) in picking two matrices within the family. This means that we can fix $2(n + 2)$ out of $4N$ parameters by taking linear combinations within the family, i.e. by going to a different basis in the vector space of commuting matrices. In addition, Eqs. (5) and (4) are invariant with respect to rescaling $V^i \to \alpha V^i$ and $S \to S/\alpha$, which fixes one more parameter in V^i. This leaves $4N - 2(n + 2) - 1 = 2N + 2M - 5$ real parameters to specify a generic type M family.

To further select a particular matrix $H(u) = T + uV$ within the family, one needs to pick $n + 2$ coefficients in Eq. (2), in addition to $2N + M - 5$ parameters that specify the commuting family, i.e. a general type M matrix involves $4N - n - 3 = 3N + M - 3$ real parameters. Since $M \leq N - 2$ the maximum number is $4N - 5$. This counting has been done in the common eigenbasis of V^i. Going to an arbitrary basis in the target Hilbert space adds another $N(N - 1)/2$ real parameters for a general orthogonal transformation. We see that the total number of parameters is less than $(N + 8)(N - 1)/2$, about half of $N(N + 1)$ for a non-integrable real-symmetric matrix of the form $A + uB$.

3 Type 1

The 'master' equation (5) is simplest for type 1. In this case there are N linearly independent $H^i(u)$ (including the identity) and consequently N linearly independent diagonal matrices V^i. By taking linear combinations we can go to a basis in the vector space $\mathcal{V}_{N-1}$ such that

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[4]More precisely it reads $T^i_{km}(d^j_k - d^j_m) = T^j_{km}(d^i_k - d^i_m)$. If $d^i_k = d^i_m$ for some, but not all i, S_{km} is defined through the non-degenerate eigenvalues. If $d^i_k = d^i_m$ for all i, matrices V^i share a common 2×2 identity block. Then with a u-independent unitary transform one can go to a basis where $T^i_{km} = 0$ for all i. In either case Eq. (4) still holds.

$V_{kk}^i \equiv d_k = \delta_{ik}$. Equation (5) reads $f_{jk}^i \equiv W_{jj}^i - W_{kk}^i = -S_{ij} S_{ik}/S_{jk}$. These equations are consistent when $f_{jk}^i + f_{kl}^i + f_{lj}^i = 0$, yielding the following four index relation [2]:

$$S_{ij} S_{jk} S_{kl} S_{li} + S_{ik} S_{kl} S_{lj} S_{ji} + S_{il} S_{lj} S_{jk} S_{ki} = 0. \tag{6}$$

The most general solution of this equation is [3, 4]

$$S_{jk} = \frac{\gamma_j \gamma_k^*}{\varepsilon_j - \varepsilon_k}, \tag{7}$$

where real ε_i and complex γ_i are unrestricted parameters that fix the commuting family.

Next, using Eq. (6) and $V_{kj}^i = \delta_{kj} \delta_{ik}$, we determine W^i from Eq. (5), which is linear in W^i, and T^i from Eq. (4). The most general member of a type 1 family, $H(u) = \sum_{i=1}^{N} d_i H^i(u)$ with arbitrary real d_i, is

$$\left[H(u)\right]_{mn} = \gamma_m \gamma_n^* \left(\frac{d_m - d_n}{\varepsilon_m - \varepsilon_n}\right), \quad m \neq n,$$

$$\left[H(u)\right]_{mm} = u\, d_m - \sum_{k \neq m} |\gamma_k|^2 \left(\frac{d_m - d_k}{\varepsilon_m - \varepsilon_k}\right). \tag{8}$$

Note that d_m are eigenvalues of V by design. In particular, for basic operators $H^i(u)$ we have $d_k = \delta_{ik}$, i.e. their nonzero matrix elements are

$$\left[H^i(u)\right]_{ij} = \frac{\gamma_i \gamma_j^*}{\varepsilon_i - \varepsilon_j}, \quad j \neq i,$$

$$\left[H^i(u)\right]_{jj} = u\, \delta_{ij} - \sum_{k \neq j} |\gamma_k|^2 \left(\frac{\delta_{ij} - \delta_{ik}}{\varepsilon_m - \varepsilon_k}\right). \tag{9}$$

4 Type $M > 1$

A similar construction is possible for real symmetric integrable families of arbitrary type M [5], though the expressions for the matrix elements are somewhat more involved. Specifically, we have (see Ref. [5] for the derivation and more details)

$$S_{kl} = \frac{1}{2} \frac{\gamma_k \gamma_l}{\varepsilon_k - \varepsilon_l} (\Gamma_k + \Gamma_l), \tag{10}$$

where

$$\Gamma_m \equiv \Gamma(\varepsilon_m), \qquad \Gamma(\sigma) = \pm \sqrt{\frac{\prod_{j=1}^{M} (\phi_j - \sigma)}{\prod_{j=1}^{M} (\lambda_j - \sigma)}}, \tag{11}$$

γ_i, ε_i are arbitrary real parameters playing the same role as in Type 1. The sign in Eq. (11) can be chosen at will individually for each ε_m. In addition, type M features M new real parameters ϕ_i that need to be chosen so that the radicand in Eq. (11) is real (see below).

Finally, quantities λ_i in Eq. (11) are by construction solutions of the following equation with arbitrary real B:

$$f(\lambda_i) \equiv \sum_{j=1}^{N} \frac{\gamma_j^2}{\lambda_i - \varepsilon_j} = B. \tag{12}$$

All N roots λ_i of this equation are real. Indeed, $f(\lambda) \to +\infty$ as $\lambda \to \varepsilon_k^+$ and $f(\lambda) \to -\infty$ as $\lambda \to \varepsilon_{k+1}^-$, where ε_k are ordered, $\varepsilon_1 < \varepsilon_2 < \cdots < \varepsilon_N$. It follows that $f(\lambda) = u$ has a real solution between ε_k and ε_{k+1} for any k, i.e. $\varepsilon_m < \lambda_m < \varepsilon_{m+1}$. One more root is located above ε_N, $\lambda_N > \varepsilon_N$, for $B > 0$ and below ε_1, $\lambda_1 < \varepsilon_1$, for $B < 0$, where we ordered λ_k so that $\lambda_1 < \lambda_2 < \cdots < \lambda_N$.

To ensure the reality of $\Gamma(\varepsilon_m)$ for any ε_m it is sufficient (though not necessary) to choose parameters ϕ_i so that $\varepsilon_i < \phi_i < \lambda_i$ for $B > 0$ and $\lambda_i < \phi_i < \varepsilon_i$ for $B < 0$. This is the only restriction on ϕ_i.

The most general member of this type M commuting family is

$$\left[H(u)\right]_{mn} = \gamma_m \gamma_n \left(\frac{d_m - d_n}{\varepsilon_m - \varepsilon_n}\right) \frac{\Gamma_m + \Gamma_n}{2}, \quad m \neq n,$$

$$\left[H(u)\right]_{mm} = u\, d_m - \sum_{j \neq m} \gamma_j^2 \left(\frac{d_m - d_j}{\varepsilon_m - \varepsilon_j}\right) \frac{1}{2} \frac{(\Gamma_m + \Gamma_j)(\Gamma_j + 1)}{\Gamma_m + 1},$$

where, unlike the Type 1 case, d_m are not arbitrary, but are given by

$$d_m = g_0 + \sum_{j=1}^{N-M} \frac{g_j}{\lambda_{j+2} - \varepsilon_m} \tag{13}$$

and g_j are arbitrary real numbers.

Commuting families obtained by the above prescription (termed *ansatz type M* families in Ref. [5]) contain $2N + M + 1$ arbitrary parameters—$2N$ of γ_i's and ε_i's, M of P_i's and the parameter B. As discussed in detail in Ref. [5], there are certain gauge transformations, such as a uniform scaling of γ_i and ε_i or a uniform shift of ε_i, that leave the commuting family invariant. This allows to fix three of the parameters meaning that the number of parameters needed to uniquely specify a type M commuting family produced by this construction is $2N + M - 2$.

On the other hand, we argued in Sect. 2 based on numerical evidence and other considerations that a *general* type $M \geq 3$ family is uniquely specified by $2N + 2M - 5$ parameters. This suggests that our construction can produce *all* real symmetric commuting families only for $M = 1, 2, 3$, while for $M > 3$ it yields only a subset of such families. The completeness for $M = 1, 2$ was explicitly demonstrated in Refs. [3, 5], while for type $M = 3$ it is supported by numerical tests.

In fact, a correspondence between real symmetric type M families and compact Riemann surfaces of genus $g \geq M - 1$ was conjectured in Ref. [5]. It turns out that the above formulas produce families that correspond to hyperelliptic Riemann surfaces of genus $g = M - 1$. All Riemann surfaces of genus 0, 1, and 2 are hyperelliptic which explains the completeness of the construction for $M = 1, 2, 3$.

5 Exact Solution

The exact spectra of type 1 matrices were obtained in [3]. The components of an eigenvector (column) $\vec{v}_m(u)$ of $H(u)$ given by (8) are

$$\left[\vec{v}_m(u)\right]_j = \frac{\gamma_j}{\lambda_m - \varepsilon_j}, \tag{14}$$

with respective eigenvalue

$$E_m(u) = \sum_{k=1}^{N} \frac{d_k |\gamma_k|^2}{\lambda_m - \varepsilon_k}, \tag{15}$$

where the λ_i, $i = 1, \ldots, N$ are determined from a single algebraic equation

$$f(\lambda_m) \equiv \sum_{j=1}^{N} \frac{|\gamma_j|^2}{\lambda_m - \varepsilon_j} = u. \tag{16}$$

As discussed in the previous section all N roots λ_i of this equation are real.

The above equations can be verified directly by evaluating $\sum_j [H(u)]_{ij} [\vec{v}_m(u)]_j$. Ansatz type M families of Sect. 3 have an exact solution in terms of a single equation similar to Eq. (16)

$$\sum_{j=1}^{N} \frac{1}{2} \frac{\gamma_j^2}{\sigma - \varepsilon_j} \big(\Gamma(\sigma) + \Gamma_j \big) - \frac{B}{2} \big(\Gamma(\sigma) - 1 \big) = u. \tag{17}$$

Having solved this equation for σ, we obtain the eigenvalues

$$E_\sigma(u) = \sum_{k,j} \frac{d_k}{\lambda_k - \varepsilon_j} \frac{\gamma_j^2}{\sigma - \varepsilon_j} \frac{1}{2} \big(\Gamma_j + \Gamma(\sigma) \big), \tag{18}$$

and the corresponding eigenvectors

$$\big[\vec{v}_\sigma(u) \big]_j = \frac{1}{2} \frac{\gamma_j}{\sigma - \varepsilon_j} \big(\Gamma(\sigma) + \Gamma_j \big). \tag{19}$$
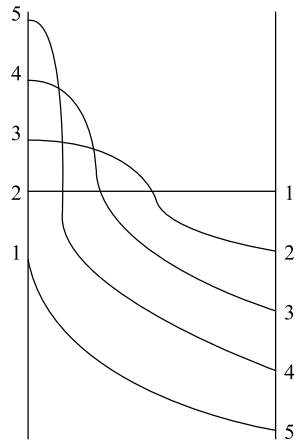
6 Level Crossings

First, we show that any type 1 matrix $H(u) = T + uV$ has at least one level crossing [3]. To this end, let us analyze the evolution of eigenvalues $E_m(u)$ with u. We observe that $\lambda_m \to \varepsilon_m$ as $u \to +\infty$. In this limit the main contribution to Eqs. (15) and (16) comes from the $j = m$ term, yielding $E_m \to |u| d_m$. Similarly, we obtain $x_m \to \varepsilon_{m+1}$ and $E_m \to -|u| d_{m+1}$ for $u \to -\infty$. It is not surprising that eigenvalues of $H(u)$ tend to $\pm|u| d_k$ since d_k are eigenvalues of V and uV dominates $H(u)$ for large u. What is important however is that we know to which particular $u d_k$ a given $E_m(u)$ tends in both limits. Symbolically, we can write $k \to k - 1 \pmod{N}$ meaning the eigenvalue goes from $-|u| d_k$ on the left ($u \to -\infty$) to $|u| d_{k-1}$ on the right ($u \to +\infty$).

The presence of levels crossings can now be proved by contradiction. Suppose there are no crossings. Since eigenvalues are continuous functions of u, this implies that their ordering must be the same at all u. The top level must connect the largest eigenvalue at $u \to -\infty$ to the largest eigenvalue at $u \to +\infty$, the bottom level goes from the lowest eigenvalue at $u \to -\infty$ to that at $u \to +\infty$, etc. Let d_k be ordered as[5] $d_i < d_j < \cdots < d_m$. Then, the largest (lowest) eigenvalue at $u \to -\infty$ is $-|u| d_i$ $(-|u| d_m)$ and the largest (lowest)

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[5]Degenerate d_k are regarded as level crossings at $u = \infty$. That such definition is necessary is seen e.g. by redefining the parameter $u \to 1/x$ and multiplying the matrices by x. We have the same commuting family now linear in x, but with the crossing moved to $x = 0$. Note also that we cannot fix the order of d_k without loss of generality, because we have already chose our indexing so that ε_k are ordered.

Fig. 1 A schematic energy level diagram for $N = 5$ demonstrating maximum number of crossings $(N - 1)(N - 2)/2$. The diagram is for the ordering $d_1 > d_2 > \cdots > d_N$ and uses the rule $k \to k - 1$ derived in the text

eigenvalue at $u \to +\infty$ is $|u|d_m$ ($|u|d_i$), i.e. we have $i \to m$ and $m \to i$ for the top and bottom levels, respectively. On the other hand, according to the $k \to k - 1$ rule established above this implies $m = i - 1 \pmod N$ and at the same time $i = m - 1 \pmod N$. We obtain $0 = 2 \pmod N$, which does not hold for any $N \geq 3$, i.e. the above assumption that levels do not cross cannot be true. Thus, at least one level crossing is inevitable.

Allowed values of the total number of crossings $n_\times$ in a type 1 matrix can be determined by analyzing the representation of an arbitrary type 1 matrix $\widetilde{H}(u)$ in terms of powers of any other nontrivial $H(u)$ that belongs to the same commuting family. One finds [3] that this expansion is necessarily of the form

$$\widetilde{H}(u) = \sum_{m=0}^{N-1} \frac{Q^m(u)}{P^H(u)} \left[H(u) \right]^m, \tag{20}$$

where $P^H(u)$ is a polynomial in u of degree $(N - 1)(N - 2)/2$ with real coefficients that depend on matrix elements of $H(u)$ only. $Q^m(u)$ are polynomials in u of order $m - 1$ lower than $P^H(u)$. This expansion breaks down at a given value of $u = u_\times$ only when $H(u)$ has a crossing at $u = u_\times$ and $P^H(u_\times) = 0$. Thus, crossings of $H(u)$ occur at the roots of $P^H(u)$ (see [3] for a detailed proof). We conclude that the maximum number of crossings in a type 1 matrix is

$$n_\times^{\max} = \frac{(N - 1)(N - 2)}{2}. \tag{21}$$

This upper bound is realized e.g. for matrices (8) such that $d_1 > d_2 > \cdots > d_N$, see Fig. 1. $n_\times$ has a definite parity, that of $n_\times^{\max}$, because the coefficients of $P^H(u)$ are real and its complex roots therefore come in conjugate pairs. For example, 4×4 type 1 matrices have either 1 or 3 crossings, $n_\times = 2, 4, 6$ for 5×5 etc.

Crossings are also ubiquitous in type $M > 1$ integrable families. It is possible however to deliberately engineer 4×4 type 2 matrices *without* crossings [3]. Other than that, we do not have rigorous results for higher types. Empirically, one finds [5] that the expansion (20) still holds, but the degree of $P^H(u)$ and therefore the maximum number of crossings is reduced to $(N - 1)(N - 2)/2 - g$, where $g \geq M - 1$ is the genus of the corresponding Riemann surface, typically $g = M - 1$ (see the end of Sect. 3).

Interestingly, one can also construct 4×4 real symmetric matrices $A + uB$ that have crossings but no u-independent symmetry and no commuting partners linear in u (this is

impossible for $N = 3$), see [1, 3] for details. This indicates that either 'accidental' degeneracies are possible or one can meaningfully introduce nonlinear in u conserved currents in such cases.

7 Yang–Baxter Equation for the Matrix Model

We next show how the Type-1 matrices can be fit into the Yang–Baxter formulation of integrable systems, by displaying matrix objects S, Eq. (24), that play the role of scattering amplitudes.[6] Let us note that this construction has not yet been realized for general Type-M matrices. Thus we consider a purely matrix model in N dimensions with states $|i\rangle$, $1 \le i \le N$, the projection operators π_{ij} and the identity matrix $\mathbf{1}$:

$$\pi_{ij} = |i\rangle\langle j|$$
$$\mathbf{1} = \sum_i \pi_{ii}$$

With $i \ne j$, let us define a 'dressed' permutation operator

$$\Pi_{ij} = \frac{|\gamma_i|^2 + |\gamma_j|^2}{2}\mathbf{1} + \gamma_i\gamma_j^*(\pi_{ij} + \pi_{ji}) - |\gamma_i|^2\pi_{jj} - |\gamma_j|^2\pi_{ii}.$$

If we set $\gamma_i \to \gamma_j$, Π reduces to the permutation operator

$$\Pi_{ij} \to |\gamma_j|^2 P_{ij}, \tag{22}$$

where P_{ij} acts as:

$$P_{ij}|k\rangle = \delta_{ik}|j\rangle + \delta_{jk}|i\rangle + (1 - \delta_{ik})(1 - \delta_{jk})|k\rangle. \tag{23}$$

Let us introduce g as a coupling parameter and also the composite parameter $x = (\varepsilon, \gamma)$ so that the scattering operator $\mathcal{S}_{ij}$ is defined as:

$$\mathcal{S}_{ij}(x_j|x_i) \equiv \mathcal{S}_{ij} = \frac{(\varepsilon_j - \varepsilon_i)\mathbf{1} + 2g\,\Pi_{ij}}{(\varepsilon_j - \varepsilon_i) + g\,(|\gamma_i|^2 + |\gamma_j|^2)} \tag{24}$$

The action of a particular scattering matrix $\mathcal{S}_{nm}$ on the relevant states is given by

$$\mathcal{S}_{nm}|k\rangle = |k\rangle \quad (k \ne n, m)$$
$$\mathcal{S}_{nm}|n\rangle = t_{nm}^+|n\rangle + r_{nm}|m\rangle \tag{25}$$
$$\mathcal{S}_{nm}|m\rangle = t_{nm}^-|m\rangle + r_{nm}|n\rangle$$

with the reflection (r) and transmission (t) amplitudes defined by

$$r_{nm} = \frac{2g\gamma_n\gamma_m^*}{\varepsilon_m - \varepsilon_n + g(|\gamma_m|^2 + |\gamma_n|^2)},$$
$$t_{nm}^+ = \frac{\varepsilon_m - \varepsilon_n + g(|\gamma_n|^2 - |\gamma_m|^2)}{\varepsilon_m - \varepsilon_n + g(|\gamma_m|^2 + |\gamma_n|^2)}, \tag{26}$$
$$t_{nm}^- = \frac{\varepsilon_m - \varepsilon_n + g(|\gamma_m|^2 - |\gamma_n|^2)}{\varepsilon_m - \varepsilon_n + g(|\gamma_m|^2 + |\gamma_n|^2)}.$$

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[6]If we set $\gamma_i = 1$, this construction is a simple extension of the permutation operator for $SU(N)$ in Yang's well known work [34] into matrix space, but seems to be new for the general case $\gamma_i \ne 1$.

We state the two operator relations that are needed, and readily verified, for any three indices 1, 2, 3:

$$\Pi_{12}\Pi_{32}\Pi_{31} = \Pi_{31}\Pi_{32}\Pi_{12}, \tag{27}$$

$$\left[\Pi_{12}, (\Pi_{31} + \Pi_{32})\right] = 0. \tag{28}$$

With C_{ij} as constants we can now verify the condition

$$(C_{12}\mathbf{1} + g\Pi_{12})(C_{32}\mathbf{1} + g\Pi_{32})(C_{31}\mathbf{1} + g\Pi_{31})$$
$$= (C_{31}\mathbf{1} + g\Pi_{31})(C_{32}\mathbf{1} + g\Pi_{32})(C_{12}\mathbf{1} + g\Pi_{12}), \tag{29}$$

provided C's satisfy the triangle law:

$$C_{32} = C_{31} + C_{12}. \tag{30}$$

Dividing by a suitable constant and consulting Eq. (25) we therefore verify the Yang–Baxter equation for the $\mathcal{S}$

$$\mathcal{S}_{ik}\mathcal{S}_{jk}\mathcal{S}_{ij} = \mathcal{S}_{ij}\mathcal{S}_{jk}\mathcal{S}_{ik}, \tag{31}$$

and the initializing condition

$$\lim_{x_j \to x_i} \mathcal{S}_{ij}(x_j|x_i) \to P_{ij}. \tag{32}$$

We consider the N sites (indices) and add an auxiliary index α (that is outside the original space of N states) so that $\pi_{i\alpha}, \pi_{\alpha i}, \pi_{\alpha\alpha}$ are added to the list of operators. Now define a (monodromy) matrix

$$\mathcal{T}_\alpha = B_\alpha(x_\alpha)\mathcal{S}_{N\alpha}\mathcal{S}_{N-1\,\alpha}\cdots\mathcal{S}_{1\alpha}. \tag{33}$$

This is a function of all the indicated variables:

$$\mathcal{T}_\alpha \equiv \mathcal{T}_\alpha\big(x_\alpha|\{x_1, x_2, \ldots, x_N\}\big), \tag{34}$$

and also a boundary field term $B_\alpha(x_\alpha)$. For the boundary term (twist) to give commuting operators, this term must be chosen to satisfy the condition [35]

$$[\mathcal{S}_{\alpha\beta}, B_\alpha B_\beta] = 0, \tag{35}$$

whereby we choose

$$B_\alpha(x_\alpha) = \mathbf{1} + \frac{g}{u}\pi_{\alpha\alpha}, \tag{36}$$

with the freedom of an arbitrary parameter u.

The transfer matrix is obtained by tracing over the auxiliary index α

$$\mathbf{T}\big(x_\alpha|\{x_1, x_2, \ldots, x_N\}\big) = \sum_\alpha \langle\alpha|\mathcal{T}_\alpha|\alpha\rangle. \tag{37}$$

Using Baxter's classic proof [36] for commutation of transfer matrices, slightly generalized to the case of twisted boundary conditions [24, 25, 35], we conclude that

$$\big[\mathbf{T}(x_\alpha), \mathbf{T}(x_\beta)\big] = 0, \tag{38}$$

for arbitrary x_α and x_β while holding $\{x_j\}g, u$ fixed. We note that Eq. (38) is also valid as $x_\alpha \to x_i$ and hence conclude $[\mathbf{T}_i, \mathbf{T}_j] = 0$, where

$$\mathbf{T}_j = \lim_{x_\alpha \to x_j} \mathbf{T}\big(x_\alpha | \{x_1, x_2, \ldots, x_N\}\big)$$

$$= \mathcal{S}_{j-1,j}(x_j, x_{j-1}) \ldots \mathcal{S}_{1,j}(x_j, x_1).B_j(x_j)\mathcal{S}_{N,j}(x_j, x_N) \ldots \mathcal{S}_{j+1,j}(x_j, x_{j+1}). \quad (39)$$

An expansion in powers of the interaction strength g produces the currents:

$$\mathbf{T}_j = \mathbf{1} + \frac{g}{u} H^j(u) + O\big(g^2\big), \quad (40)$$

with

$$H^i(u) = \pi_{ii} + u \sum_j \frac{\gamma_i \gamma_j^*(\pi_{ij} + \pi_{ji}) - |\gamma_i|^2 \pi_{jj} - |\gamma_j|^2 \pi_{ii}}{\varepsilon_i - \varepsilon_j}.$$

Considering terms of order $O(g)$ in $[\mathbf{T}_i, \mathbf{T}_j]$, we conclude

$$\big[H^i(u), H^j(u)\big] = 0. \quad (41)$$

Note that $H^i(u)$ are the exactly the basis operators of Eq. (9) written in terms of projection operators [4].

8 Links to Various Models

First, note that one can choose an arbitrary Hermitian matrix V and still have $H(u) = T + uV$ to be a member of any given type 1 family. Indeed, Eq. (8) is written in the eigenbasis of V. Both this basis and the eigenvalues of V—parameters d_k—can be chosen arbitrarily. By symmetry one can instead choose T at will, though this is not apparent from Eq. (8). But as soon as e.g. V is fixed, T is severely constrained—one is left with only $2N$ parameters γ_i, ε_i to specify its matrix elements. This ability to choose either V or T arbitrarily means in particular that any u-independent Hamiltonian can be 'embedded' into a type 1 family in many different ways. For example, one can choose V to be the isotropic Heisenberg model, or the Haldane-Shastry model, and find T so that $T + uV$ belongs to a given type 1 family.

Type 1 integrable families are closely related [3] to Gaudin magnets [24, 25] $\hat{h}_i = B\hat{s}_i^z + \sum_{j \neq i} \hat{\mathbf{s}}_i \hat{\mathbf{s}}_j (\varepsilon_i - \varepsilon_j)^{-1}$, where $\hat{\mathbf{s}}_j$ are quantum spins of arbitrary length s_j. In the sector with (conserved) $\hat{S}^z = \sum_j \hat{s}_j^z$ equal to its maximum (minimum) possible value less (plus) one, $\hat{h}_i$ are N commuting $N \times N$ matrices which form a type 1 family with $u = B$ and $\gamma_j^2 = s_j$. The BCS model is obtained [26] as $\sum_j \varepsilon_j \hat{h}_j$ for $\gamma_j^2 = s_j = 1/2$ and a replacement $u = B \to 1/g$, where g is the dimensionful BCS coupling constant.

Some blocks of the 1D Hubbard model characterized by a complete set of u-independent symmetry quantum numbers are type 1 matrices, though most blocks are type $M > 1$ [5]. A similar typology can be developed for e.g. the 1D XXZ Hamiltonian and other sectors of Gaudin and BCS models using the method of [5] for determining the type of parameter-dependent matrices. Interestingly, this implies that at least in some blocks there is an exact solution in terms of a single algebraic equation—Eq. (16) or a similar equation for higher types [5]—a vast simplification as compared to Bethe's Ansatz.

One can also construct fermionic (bosonic) Hamiltonians [4] out of type 1 matrices as

$$\hat{H} = \sum_{mn} [H(u)]_{mn} a_m^\dagger a_n, \quad (42)$$

where a_n are the usual fermionic (bosonic) destruction operators. $[\hat{H}_1, \hat{H}_2] = 0$ as long as the corresponding matrices $H_1(u)$ and $H_2(u)$ commute, i.e. belong to the same family.

9 Level Statistics

We have performed an extensive numerical study of level statistics of type 1 and higher type matrices [33] for various choices of parameters. Almost in all cases the statistics is Poissonian for $N \gg 1$ with high accuracy, even when we deliberately attempt to adjust the parameters to get a different statistics. Let us briefly describe the main results e.g. for the level-spacing distribution.

It is convenient to redefine the parameter $u \to 1/x$ and replace $T + uV \to V + xT$. To get a proper large N limit one has to make sure that the scaling of parameters $d_k, \varepsilon_k, \gamma_k$ in Eq. (8) and x with N is such that the eigenvalues of V and xT scale in the same way for large N. As discussed above, the matrix V is arbitrary, so at $x = 0$ one can have any admissible level statistics. Consider, for example, three representative cases: (a) V is a random real symmetric matrix with independent identically distributed matrix elements V_{jk} for $j \leq k$, (b) eigenvalues d_k of V are independent uniformly distributed random numbers, and (c) d_k display level *attraction*, $P(s) = as^\omega \exp(-bs^{1+\omega})$ with $-1 < \omega < 0$. The level-spacing distribution $P(s)$ for V is Wigner-Dyson $P(s) = 2ase^{-as^2}$ in (a) and Poissonian $P(s) = e^{-s}$ in (b). As soon as V is chosen, T is no longer arbitrary and we find that it has Poissonian $P(s)$ in all three cases for all choices of parameters γ_k and ε_k we considered as long as ε_k and d_k are uncorrelated.

Specifically, motivated in part by the BCS and Gaudin examples discussed in the previous section we took: (1) ε_k that are also eigenvalues of a random matrix and $\gamma_k = \text{const}$ independent of k, (2) same as (1) but with random uncorrelated γ_k and (3) independent uniformly distributed ε_k and $\gamma_k = \text{const}$. For all these choices the level-spacing distribution for T is very well approximated by Poissonian $P(s) = e^{-s}$, where s is the level-spacing in units of the mean level-spacing. In case (a) above the level statistics of $H(x) = V + xT$ at $x = 0$ is Wigner-Dyson, but we find that it crosses over to Poisson at $|x| \approx 1/N$ and remains Poisson for larger $|x|$. Case (c) is analogous to (a)—a crossover to Poisson behavior at $|x| \approx 1/N$. In case (b) the statistics is Poissonian for all x. Similar behavior is found in spectral rigidity. We conclude that one can arrange for any statistics at a given value of the parameter $x = x_0$, but this becomes an isolated point in $N \to \infty$ limit, while for $x \neq x_0$ integrability as defined in Sect. 2 enforces Poisson statistics.

The only exception to Poisson statistics other than at an isolated value of x we were able to identify is when parameters d_k and ε_k are correlated, so that $d_k = f(\varepsilon_k)$, where $f(\varepsilon)$ is a smooth function of ε in $N \to \infty$ limit[7] and $\gamma_k = \text{const}$. This is the case in e.g. the BCS model where $d_k = \varepsilon_k$ (see above). In such cases the statistics is distinctly non-Poissonian and, moreover, in case (a) above, for example, $P(s)$ crosses over at $x = O(N^0) \equiv O(1)$ from the Wigner-Dyson $P(s) = 2ase^{-as^2}$ to a more repulsive distribution $P(s) \propto s^4$ for small s. The repulsion is softened by randomizing γ_k. More importantly, the statistics quickly becomes Poissonian when the correlation between d_k and ε_k is destroyed, $d_k = f(\varepsilon_k)(1 + \eta_k)$, where η_k are random. We find Poisson distribution already for $\eta_k = O(1/N)$ at $x = O(1)$, see also [23] for a similar study of Gaudin model. $d_k = f(\varepsilon_k)$ define exceptional 'surfaces' of certain measure zero in parameter space. This seems analogous to the harmonic oscillator exceptions to the Poisson distribution in classical integrable systems [37]. There too one finds increased level repulsion for oscillators instead of Poisson $P(s)$.

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[7]To obtain $f(\varepsilon)$ in this limit, we scale ε_k and d_k with N so that they fill finite intervals when $N \to \infty$.

Some of these numerical observations can be understood using perturbation theory. Energies to the first order in x are given by the second equation in (8), where we set $|\gamma_j|^2 = 1/N$ to achieve proper scaling for large N as discussed above.[8] We have

$$E_m(x) \approx d_m - \frac{x}{N} \sum_{j \neq m} \left(\frac{d_m - d_j}{\varepsilon_m - \varepsilon_j} \right). \tag{43}$$

The first term comes from V, which we take to have Wigner-Dyson $P(s)$, the second—from T, which is determined by the integrability condition (3) and whose level statistics we do not control. Let us estimate x at which the two terms in Eq. (43) become comparable. Without loss of generality we can take $d_k = O(N^0) = O(1)$ and we must also take $\varepsilon_k = O(1)$ so that T and xV scale in the same way for large N. Suppose ε_k are ordered as $\varepsilon_1 < \varepsilon_2 < \cdots < \varepsilon_N$. When d_k and ε_k are uncorrelated $d_m - d_j$ is $O(1)$ when j is close to m, i.e. when $(\varepsilon_m - \varepsilon_j) = O(1/N)$. The second term in Eq. (43) is then $xc_m \ln N$, where $c_m = O(1)$ is a random number only weakly correlated with d_m.

If we now order d_m, c_m in general will not be ordered, i.e. if $d_{m+1} > d_m$ is the closest level to d_m and therefore $(d_{m+1} - d_m) = O(1/N)$, the corresponding difference $(c_{m+1} - c_m) = O(1)$. The contributions to level-spacings from the two terms in Eq. (43) become comparable for $x = x_c \approx 1/(N \ln N)$. It makes sense that the second term introduces a trend towards Poisson distribution because it is a (nonlinear) superposition of ε_k and d_k—eigenvalues of two uncorrelated random matrices. Thus, we expect a crossover from Wigner-Dyson to Poisson distribution at $x = x_c$.

This argument breaks down when $d_k = f(\varepsilon_k)$, since in this case $(d_m - d_j) = O(1/N)$ when $(\varepsilon_m - \varepsilon_j) = O(1/N)$. The two terms in Eq. (43) become comparable only at $x = O(1)$ in agreement with the numerics for this case. Moreover, the second term no longer trends towards Poisson statistics. Relaxing the correlation between d_k and ε_k with $d_k = f(\varepsilon_k)(1 + \eta_k)$ and going through the same argument, one expects a crossover to Poisson statistics at $x = O(1)$ for $\eta_k = O(1/N)$.

10 Discussion

A distinct feature of the notion of quantum integrability proposed in this paper is the parameter (u) dependence. This is in contrast to the classical notion that does not require any such dependence. We however find it necessary to be able to quantize the classical definition in a meaningful way. In this sense our definition is more demanding than its classical counterpart.

On the other hand, there is another distinction from the classical notion that makes our definition seem less stringent—even a single nontrivial integral of motion linear in u is sufficient to declare the model integrable. This was originally motivated by the absence of a well-defined analog of the number of degrees of freedom in quantum mechanics. There is evidence however that the presence of a single such integral is actually much more consequential in the quantum case. For example, for any $H(u) = T + uV$ with a single nontrivial commuting partner $I(u) = K + uW$[9] one also finds numerous (about N linearly independent) integrals of motion quadratic in u that commute with both $H(u)$ and $I(u)$ [38]. We

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[8]Equivalently, one can redefine $x \to x/N$ as in the BCS model for which $x = g = \lambda \delta$, where λ is the dimensionless coupling and $\delta \propto 1/N$ is the mean level spacing in ε_k.

[9]Such examples are available analytically (type $N - 2$ matrices) or can be generated numerically by solving Eq. (5), see [5] for details.

note that a generic matrix of the form $A + uB$ has no quadratic integrals other than its own square and a multiple of identity. One of the implications of this is that, for example, currents of higher order in the parameter in the 1D Hubbard and XXZ models might follow from the linear ones, i.e. be in some sense trivial given the linear integral. These are however open questions that require further research.

One can also consider e.g. a situation when a Hamiltonian of the form $H(u) = T + uV$ or $T + uV + u^2 W$ has no nontrivial integrals linear in u, but a number of quadratic ones. Such systems exist and can also be classified and, at least some of them, explicitly parametrized. Nevertheless, this 'higher order in the parameter' integrability seems less relevant as in most physical examples of parameter-dependent integrable models one is able to identify a parameter such that the Hamiltonian and at least one of the currents are linear in it.

It is worth commenting on the relationship between our viewpoint and the usual set of 'beliefs' based on model integrable systems. Two related points emerge (i) the belief that the number of constants of motion is $\sim L \propto \log(\mathcal{N})$ where L is the number of sites and $\mathcal{N}$ the size of the total Hilbert space and (ii) the so called 'rule of three', i.e. the belief that any many body lattice model in 1-dimension with a fixed number of particles reveals its integrability only in the three particle sector [39, 40],[10] since the one particle sector and two particle sectors have as many constants of motion as the particle number (total energy and momentum).

With regard to (i), we distinguish between the much larger $\mathcal{N}$ and N of this work. By a process of block diagonalizing the Hamiltonian operator into different sectors, one arrives at a direct sum representation of the full Hamiltonian. Each sub block is 'irreducible' in the sense that the space-time and internal space (parameter-independent) symmetries have been extracted out, and our considerations revolve around such sub blocks with a smaller and variable dimension N. Our point is that any such sub block must be special in the sense discussed here. Stitching back the irreducible blocks to reconstruct the full Hamiltonian matrix requires a detailed knowledge of the symmetries used in the first place. While possible in principle, we regard this process as of secondary importance as compared to the one undertaken here, namely the characterization of the sub blocks themselves.

With regard to (ii) our studies of two typical examples give some insight into this question. Firstly, the Gaudin magnets show that the two and higher particle number sectors yield matrices that have non linear conservations laws in addition to the linear ones discussed here. Secondly, we can study the fermionic representation of type 1 matrices, Eq. (42). Here $n = L = N$, the sector $n_e = 1$ is isomorphic to type 1 matrix family (8), while other sectors are much larger matrices of high types $M = \binom{N}{n_e} - N + 1$,[11] i.e. they are apparently distinct and much more complicated integrable matrix families. With the hindsight of Eq. (42) it is evident that they all are different manifestations of the same type 1 model and are in some sense equivalent, but can this be formulated generally so that all such similar matrix integrable models are naturally grouped together and recognized to be related to each other as different representations of the same underlying structure? Further work is needed

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[10]The "rule of three", a term coined by McGuire in [40] originates from the fact that for three particles in one dimension, we have only two constants of motion generically. These are the total energy and the total momentum, and thus exact solvability is not to be expected. If the three particle case is solvable in some standard form, such as Bethe's *Ansatz*, the rule empirically suggests the existence of sufficient further conservation laws and hence of exact solvability with arbitrary number of particles.

[11]Individual sectors do not have additional sector-specific u-dependent integrals. This is the case for the fermion (42) and Gaudin models, but not for the Hubbard model where there are numerous such sector-specific integrals [5, 38].

to obtain clarity on these questions. Recent work [41] throws light on further symmetries in higher particle number sectors that are linear in the parameter u, and explores their non linear relationship to the operators in Eq. (42).

Other open questions include: a general construction of type $M > 3$ integrable families, analytical results for crossings in types $M > 1$ and for level statistics in all types, the relationship between the exact solution for types $M > 1$ through a single algebraic equation and Bethe's Ansatz.

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