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Theory of extreme correlations using canonical Fermions and path integrals

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HIGHLIGHTS

- Spectral function of the Extremely Correlated Fermi Liquid theory at low energy.
- Electronic origin of low energy kinks in energy dispersion.
- Non Hermitian representation of Gutzwiller projected electrons.
- Analogy with Dyson–Maleev representation of spins.
- Path integral formulation of extremely correlated electrons.

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ABSTRACT

The t–J model is studied using a novel and rigorous mapping of the Gutzwiller projected electrons, in terms of canonical electrons. The mapping has considerable similarity to the Dyson–Maleev transformation relating spin operators to canonical Bosons. This representation gives rise to a non Hermitian quantum theory, characterized by minimal redundancies. A path integral representation of the canonical theory is given. Using it, the salient results of the extremely correlated Fermi liquid (ECFL) theory, including the previously found Schwinger equations of motion, are easily rederived. Further, a transparent physical interpretation of the previously introduced auxiliary Greens function and the 'caparison factor', is obtained.

The low energy electron spectral function in this theory, with a strong intrinsic asymmetry, is summarized in terms of a few expansion coefficients. These include an important emergent energy scale Δ_0 that shrinks to zero on approaching the insulating state, thereby making it difficult to access the underlying very low energy Fermi liquid behavior. The scaled low frequency ECFL spectral function, related simply to the Fano line shape, has a peculiar energy dependence unlike that of a Lorentzian. The resulting energy

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dispersion obtained by maximization is a hybrid of a massive and a massless Dirac spectrum $E^*_Q \sim \gamma\, Q - \sqrt{\Gamma_0^2 + Q^2}$, where the vanishing of Q, a momentum type variable, locates the kink minimum. Therefore the quasiparticle velocity interpolates between $(\gamma \mp 1)$ over a width Γ_0 on the two sides of $Q = 0$, implying a kink there that strongly resembles a prominent low energy feature seen in angle resolved photoemission spectra (ARPES) of cuprate materials. We also propose novel ways of analyzing the ARPES data to isolate the predicted asymmetry between particle and hole excitations.

1. Introduction

The intensely studied t–J model is often regarded as the effective low energy Hamiltonian for describing several observed phenomena in cuprate superconductors [1]. Here the $U \to \infty$ limit is presupposed, and hence the Hilbert space is restricted to a maximum of single occupancy at each site, i.e. Gutzwiller projected [2]. A few words on the choice of the t–J model are relevant here. The implied infinite U limit eliminates high energy (U scale) electronic states, known as the upper Hubbard band states. The residual low energy ($\lesssim$ 100 meV scale) excitations are probed by sensitive spectroscopies and transport phenomena, making the t–J model suitable for our task. At reasonably high U, say comparable to the band width in a Hubbard model, this elimination of the upper Hubbard band must already occur in part. Therefore the limit $U \to \infty$ must be regarded as a useful mathematical idealization of the very strong, or extreme correlation phenomenon. The resulting Gutzwiller projected electron operators, denoted by Hubbard's convenient notation of X operators [3], are rendered non canonical. The non-canonical nature of the electrons precludes the Wick's theorem underlying the Feynman diagram approach, whereby leading to the fundamental difficulty of the t–J model, namely the impossibility of a straightforward Feynman type perturbative expansion. This situation leads to enormous calculational difficulties, so that systematic and controlled analytical calculations with this model have been very difficult.

In a series of recent papers [4,6–8,5,9,10], we have shown that it is possible to overcome some of these difficulties by using alternate methods based on Schwinger's treatment of field theory with time dependent potentials. This idea yields exact equations of motion for the electron Greens function. These equations have the nature of functional differential equations, and provide a powerful launching pad for various approximations. The specific approximation pursued is a systematic expansions in a parameter λ related to double occupancy. Using this we have presented an analytical theory of the normal state of the t–J model termed the extremely correlated Fermi liquid (ECFL) theory. An interesting feature of the theory is that we find a non-Dysonian representation of the projected electron Greens function. This is a significant structural departure from the usual field theories, and arises in a most natural fashion. The Greens function is determined by a *pair* of self energies, denoted by $\Phi(\vec{k}, i\omega_n)$ and $\Psi(\vec{k}, i\omega_n)$, instead of the standard Dyson self energy $\Sigma(\vec{k}, i\omega_n)$ (see Eq. (21) below). The latter can be reconstructed from the pair by a simple inversion. Starting with rather simple pairs of self energies, it is found that non trivial complexity is introduced into the Dyson self energy through this inversion process. Explicit self consistent calculations in parameter λ have been carried out to $O(\lambda^2)$ so far, and yield reliable results for electron densities $0 \leq n \lesssim .7$. The detailed dynamical results of the ECFL theory have been benchmarked against independent theories in overlapping domains; e.g. against high temperature series results in Ref. [11]. The ECFL theory has been shown to have a momentum independent Dyson self energy in the limit of infinite dimensions [10]. This enables benchmarking against the dynamical mean field theory (DMFT) in Ref. [9]. Importantly, the results from the ECFL theory for the spectral function compare well with a large U Hubbard model solved by the DMFT method, and not just infinite U. The ECFL theory has also been benchmarked in Ref.

[12] against the exact solution of the asymmetric $U = \infty$ Anderson impurity model, obtained from the numerical renormalization group study of Krishnamurthy, Wilson and Wilkins [13]. In addition, a detailed comparison between the data on cuprate superconductors at optimal filling and the theoretical photoemission spectral lines of the ECFL theory has been carried out in Refs. [14,15], where excellent agreement is found. In all cases studied, the comparisons with ECFL are good, and seem to indicate the utility of this approach.

The ECFL formalism could initially seem somewhat unfamiliar, in view of its reliance on the analysis of the Schwinger equations of motion. This analysis was originally used to derive the main constituents of the theory, namely the auxiliary Greens function and the two self energies (detailed below). This type of analysis is somewhat removed from the toolkit of "standard" many body physics courses, and hence might obstruct a ready visualization of these objects. One goal of the present work is to show that these results are (A) minimal, i.e. having least redundancy, and (B) available more transparently. The latter follows from an important and novel *hat removal theorem*, leading to a compact mapping of the Hubbard operators to canonical Fermions. The mapping is given in Eq. (1) and described further in Section 3.2, leading to a path integral formulation (Section 7). It is possible that such a simplified presentation could lead to improved strategies for devising approximate methods, especially close to the insulating state.

This method rests on an exact replacement rule for the Hubbard X operators in terms of the canonical Fermi operators

$$X_i^{0\sigma} \to C_{i\sigma}, \qquad X_i^{\sigma 0} \to C_{i\sigma}^\dagger (1 - N_{i\bar{\sigma}}), \qquad X_i^{\sigma\sigma'} \to C_{i\sigma}^\dagger C_{i\sigma'}. \tag{1}$$

This replacement rule is shown to be exact when "right-operating" on states which satisfy the Gutzwiller constraint. This replacement is similar in spirit to the Dyson–Maleev representation [16, 17], where spin operators are expressed in terms of canonical Bosonic operators. With the advantage of this representation, most steps in the ECFL theory, such as the factorization of the Greens function into an auxiliary Greens function, the two self energies and the caparison function (see Eqs. (18), (19) and (21)) becomes very intuitive.

The analogy can be pushed further to establish a parallel between the λ parameter of the ECFL theory, and the small parameter of the Dyson–Maleev [16,17] theory, namely the inverse spin $\frac{1}{2s}$. Finally we are able to make contact with the illuminating work of Harris, Kumar, Halperin and Hohenberg [18]. In a detailed work these authors computed the Greens function of the spins for two sublattice antiferromagnet using the Dyson–Maleev scheme and extracted the lifetime of the magnons of the theory. We find that their calculation contains the precise Bosonic counterparts of the auxiliary Greens function and the second self energy Ψ defining the "caparison function" of the ECFL theory (see Eqs. (18), (19) and (21)). Unlike the spin problem with variable number of excitations, the t–J model has a fixed number of particles. Hence there are significant new elements in the ECFL theory involving the imposition of the Luttinger–Ward volume theorem, as discussed later.

A few comments on the canonical description of the equations of motion are appropriate. The general problem is to represent a time evolution of a state of the t–J model

$$[\psi]'_{final} = Q'_M \ldots Q'_2 . Q'_1 . [\psi]'_{initial}, \tag{2}$$

where the primed states and operators are in the t–J model Hilbert space defined with the three allowed states at each site as usual (see Section 2.1 for details). The operators Q'_j may be thought of as the exponential of the t–J Hamiltonian: $Q'_j \sim e^{-it_j H_{tJ}}$ written in terms of the projected operators. Since the algebra of the projected electrons is very inconvenient, one seeks a reframing of the problem into a canonical space. This involves mapping the states, the Hamiltonian and all other operators of the original theory, into the unconstrained Hilbert space of two Fermions at each site. This canonical space is of course described by the usual Fermi operators $C_{j\uparrow}, C_{j\downarrow}$ and their adjoints. This gives us an enlarged space with four states per site, with one redundant state corresponding to double occupancy, eliminated using Gutzwiller projection. There are various possibilities for doing this elimination leading to the different theories in literature. This includes the popular slave Boson or slave Fermion technique [19–21], where additional degrees of freedom, over and above the already enlarged 4 dimensional local state space, are introduced and finally eliminated as best as possible. This handling of the redundancy leads to gauge theories for the t–J model that are reviewed in Ref. [21].

In the enlarged state space let us block diagonalize the state space into physical and unphysical states and write the projection operator as

$$[\psi] = \begin{bmatrix} \psi^{ph} \\ \psi^{un} \end{bmatrix}; \qquad \hat{P}_G = \begin{bmatrix} \mathbb{1}^{ph} & 0 \\ 0 & 0 \end{bmatrix}, \tag{3}$$

where $\mathbb{1}^{ph}$ is the identity operator in the physical space. The relevant operators in the theory Q_j e.g. the Hamiltonian, the creation operators or the destruction operators, are now written in terms of the canonical Fermions:

$$Q_j = \begin{bmatrix} Q_j^{pp} & Q_j^{pu} \\ Q_j^{up} & Q_j^{uu} \end{bmatrix}. \tag{4}$$

The next goal of the construction is to ensure that a state resulting from the application of a sequence of operators on a projected state remains in the projected space, i.e.

$$[\psi]_{final} = Q_M \ldots Q_2 . Q_1 . \hat{P}_G . [\psi]_{initial}, \tag{5}$$

and $[\psi]_{final} = \hat{P}_G . [\psi]_{final}$. If this condition is not ensured, the projector has to be introduced at all intermediate time slices, thus making the calculations intractable. A sufficiency condition for this is the commutation $[Q_j, \hat{P}_G] = 0$ for all j. The slave Boson–Fermion technique uses the conservation of the Gutzwiller constraint by writing a suitable version of the Hamiltonian. This enables the use of a time independent Lagrange multiplier, as demonstrated in the work of Read and Newns [20]. In Section 3.1, we display a compact Hermitian representation that also achieves this, without however adding further states (beyond the four states) into the problem.

While the commutation condition $[Q_j, \hat{P}_G] = 0$ is sufficient, it is not necessary, and a much less restrictive condition can be found. We note that if the operators Q_j have a vanishing Q_j^{up} then the product in Eq. (5) remains in the physical sector with

$$[\psi]_{final} = \begin{bmatrix} Q_M^{pp} \ldots Q_2^{pp} . Q_1^{pp} . \psi_{initial}^{ph} \\ 0 \end{bmatrix}. \tag{6}$$

The property of a commuting projection operator $[Q_j, \hat{P}_G] = 0$, requires that $Q_j^{pu} = 0$ as well as $Q_j^{up} = 0$, whereas the vanishing property of the unphysical components noted in Eq. (6) requires only $Q_j^{up} = 0$. Then Q_j^{pu} as well as Q_j^{uu} are quite arbitrary. With this property, all the Q_j operators in Eq. (4) are block triangular

$$Q_j = \begin{bmatrix} Q_j^{pp} & Q_j^{pu} \\ 0 & Q_j^{uu} \end{bmatrix}. \tag{7}$$

In more formal terms the sufficiency condition with least constraints that leads to Eq. (6) (via the block triangularity Eq. (7)) is

$$(\mathbb{1} - \hat{P}_G) . Q_j . \hat{P}_G = 0. \tag{8}$$

This condition is also expressible as $[Q_j, \hat{P}_G] . \hat{P}_G = 0$; a conditional vanishing of the commutator, when right operating on projected states. This observation provides some intuition for why Eq. (8) is sufficient in the present context. In view of the block triangular operators in Eq. (7), the adjoint property, namely of representing conjugate operators by their matrix Hermitian conjugates, is lost in this representation. This is seen clearly in Eq. (1), where the first two operators are mutual adjoints in the defining representation, but not so in the canonical basis. In general this situation is expected to lead to non Hermitian Hamiltonians. The non Hermitian representation in Eq. (64) and Section 3.2 implements this idea and therefore leads to the most efficient canonical theory. We show that it exactly matches the minimal theory, found from the minimal description of the t–J model in terms of the Hubbard X operators and the Schwinger equations of motion. It is notable that the Gutzwiller

projection operator does not appear *explicitly* in the equations of motion, although it does play a crucial role in the canonical theory, and is at the root of its difficulty.

The plan of the paper is as follows. In Sections 2.1–2.3 we review the Schwinger equations of motion for the t–J model, and the ingredients of the recent method developed for a systematic expansion in a parameter λ. In Section 2.4 we summarize the general form of the Greens function at low frequencies near the Fermi surface, and obtain the prototypical spectral function of the theory. We summarize in Section 2.5 a kink in the electronic dispersion that arises from the theory, and seems to be closely related to that seen in many photoemission experiments. We also present simple but important ideas for analyzing photoemission data, with a view to isolating important feature of asymmetry predicted by the ECFL theory.

In Section 3 we formulate the "best possible" representation of the Hubbard operators in terms of canonical Fermions, as discussed above. Section 3.1 summarizes the well known representation and Section 3.2 implements the block triangular idea to obtain a non Hermitian method with least redundancy. Sections 3.3 and 3.4 give further details of the Hamiltonian in this representation and the proof of the antiperiodic temporal boundary conditions necessary for defining the new framework.

In Section 4, the above non Hermitian representation is used to analyze the nature of the Greens function of projected electrons. Quite remarkably this process also yields the Greens function as a convolution of an auxiliary Greens function and a caparison function, in complete parallel to that obtained from the Schwinger method employed in Sections 2.2 and 2.3. In Section 5 we generalize the above representation to define λ Fermions where the Gutzwiller projection is only partial, and becomes full at $\lambda = 1$. The equations of motion from these Fermions are shown to be those obtained in the λ expansion of Section 2.3.

In Section 6 we display a close analogy between the non Hermitian representation of the Gutzwiller projected electrons and the well known Dyson–Maleev representation of spin operators in terms of canonical Bosons. This connection also provides further meaning of the small parameter λ in the Fermion theory, as a parallel of the expansion parameter $\frac{1}{2s}$ of the Dyson–Maleev theory. A connection with the work of Harris, Kumar, Halperin and Hohenberg (HKHH) [18] is noted, who invented a method for computing the lifetime of spin waves in antiferromagnets, with considerable overlap with our representation of the Greens function with two self energies.

In Section 7, we cast the canonical theory in terms of Fermionic path integrals, and show how the exact Schwinger equations of motion can be obtained directly from this representation, thereby validating all the links in the argument. The subtle role of the Gutzwiller projection operator is explored, it does not appear explicitly in the equations of motion and yet plays an important role in the theory. In Section 8 we summarize the main points of the paper.

In Appendix A we summarize the derivation of the minimal equations of motion from the Schwinger viewpoint. In Appendices B–D we provide the details of the coherent state path integrals and the implementation of the Gutzwiller projection. In Appendix E we provide a more detailed interpretation of the caparison function in terms of a change of variable of the source fields.

2. Summary of the ECFL theory for the t–J model

2.1. The t–J model preliminaries

The well studied t–J model is a two component Fermi system on a lattice, defined on the restricted subspace of three local states, obtained by excluding all doubly occupied configurations. The allowed states are $|a\rangle$ with $a = 0, \uparrow, \downarrow$, and the double occupancy state $|\uparrow\downarrow\rangle$ is removed by the (Gutzwiller) projection operator. These Gutzwiller projected electron operators are denoted, in the convenient notation due to Hubbard, as $X_i^{a,b} = |a\rangle\langle b|$. Its Hamiltonian H_{tJ} is expressed in terms of the X operators so that the single occupancy constraint is explicit. Summing over repeated spin indices we write

$$H_{tJ} = H_t + H_J,$$
$$H_t = -\sum_{ij} t_{ij} X_i^{\sigma 0} X_j^{0\sigma} - \boldsymbol{\mu} \sum_i X_i^{\sigma\sigma},$$

$$H_J = \frac{1}{2}\sum_{ij} J_{ij}\left(\vec{S}_i.\vec{S}_j - \frac{1}{4}X_i^{\sigma\sigma}X_j^{\sigma'\sigma'}\right). \tag{9}$$

In computing the Green's functions we add two kinds of Schwinger sources to the Hamiltonian; the anticommuting Grassmann pair J, J^* coupling to electron creation and destruction operators, and the commuting potential $\mathcal{V}$, coupling to the charge as well as spin density. These sources serve to generate compact Schwinger equations of motion (EOM), and are set to zero at the end. Explicitly we write

$$\hat{\mathcal{A}}_S = \sum_i \int_0^\beta \hat{\mathcal{A}}_S(i,\tau)d\tau,$$

$$\hat{\mathcal{A}}_S(i,\tau) = \left[X_i^{\sigma 0}(\tau)J_{i\sigma}(\tau) + J_{i\sigma}^*(\tau)X_i^{0\sigma}(\tau)\right] + \mathcal{V}_i^{\sigma'\sigma}(\tau)X_i^{\sigma'\sigma}(\tau), \tag{10}$$

and all time dependences are as in $Q(\tau) = e^{\tau H_{tJ}}Qe^{-\tau H_{tJ}}$. The generating functional of Green's functions of the t–J model is

$$Z[J, J^*, \mathcal{V}] \equiv \mathrm{Tr}_{tJ}\, e^{-\beta H_{tJ}}T_\tau\left(e^{-\hat{\mathcal{A}}_S}\right). \tag{11}$$

It reduces to the standard partition function on turning off the indicated source terms. The Green's functions for positive times $0 \le \tau_j \le \beta$, are defined through the Martin–Schwinger prescription [22,23]:

$$\mathcal{G}_{\sigma\sigma'}(i\tau_i, f\tau_f) = -\frac{\left\langle T_\tau\left(e^{-\hat{\mathcal{A}}_S}X_i^{0\sigma}(\tau_i)X_f^{\sigma' 0}(\tau_f)\right)\right\rangle}{\langle T_\tau e^{-\hat{\mathcal{A}}_S}\rangle}. \tag{12}$$

The functional Z conveniently yields the Green's functions upon taking functional derivatives with respect to the sources, e.g.

$$\mathcal{G}_{\sigma\sigma'}(i\tau_i, f\tau_f) = \left(\frac{1}{Z}\frac{\delta^2 Z}{\delta J_{i\sigma}^*(\tau_i)\delta J_{f\sigma'}(\tau_f)}\right), \tag{13}$$

where the sources are turned off at then end. We note that n_σ, the number of particles per site, is determined from the number sum rule:

$$n_\sigma = \mathcal{G}_{\sigma\sigma}(i\tau^-, i\tau), \tag{14}$$

and $\boldsymbol{\mu}$ the chemical potential is fixed by this constraint.

2.2. The Schwinger equations of motion

The detailed theory of the t–J model developed so far [4,6] uses the Schwinger equations of motion. Since these equations play a fundamental role in the theory, we summarize next the equations of motion and their extension, obtained by introducing a parameter λ. We relegate to Appendix A the derivation of the "minimal theory" equations. In the minimal theory, the most compact set of Schwinger equations are established, and some redundant terms from Ref. [4] are omitted. This minimal version of the theory is important for the purposes of the present paper, since our goal in this paper is to recover these from a canonical formalism.

As the Schwinger school has [22,24,25] emphasized, a field theory is rigorously determined by its equations of motion plus the boundary conditions. We can also establish alternate descriptions such as path integrals formulations, from the requirement that they reproduce these equations of motion—we present an example of this approach in Section 7.2. In terms of the original description of the t–J model involving the Hubbard X operators, the Schwinger equation of motion is a partial differential equation in time and also a functional differential equation involving the derivatives with respect to a Bosonic source:

$$\left(\mathbf{g}^{-1}_{0,\sigma_i,\sigma_j}(i\tau_i, j\tau_j) - \hat{X}_{\sigma_i\sigma_j}(i\tau_i, j\tau_j) - Y_{1\sigma_i\sigma_j}(i\tau_i, j\tau_j)\right)$$
$$\times\, \mathcal{G}_{\sigma_j\sigma_f}(j\tau_j, f\tau_f) = \delta_{if}\delta(\tau_i - \tau_f)\left(\delta_{\sigma_i\sigma_f} - \gamma_{\sigma_i\sigma_f}(i\tau_i)\right), \tag{15}$$

where $\mathbf{g}_0$ is the noninteracting Green's function (Eq. (136)), $\hat{X}$ is a functional derivative operator (Eq. (130)), γ is the local Green's function obtained from $\mathcal{G}$ as $\gamma_{\sigma_a\sigma_b}(i\tau_i) = \sigma_a\sigma_b\mathcal{G}_{\bar{\sigma}_b\bar{\sigma}_a}(i\tau_i^-, i\tau_i)$ (see Eq. (135)) and Y_1 is the band hopping times γ (Eq. (131)); further details can be found in Appendix A. This equation has been written down in Refs. [4,6]: Antiperiodic boundary conditions with respect to both times (as in Eqs. (75) and (76)), and the number sum-rule (14) together with the equation of motion (15), specify the theory completely.

2.3. The λ expansion, the shift identities and second chemical potential u_0

The idea of introducing a parameter into the EOM (15) becomes quite natural when we observe the Schwinger EOM for the Hubbard model closely. These can be written schematically as $\left(\mathbf{g}_0^{-1} - U\delta/\delta\mathcal{V} - UG\right).G = \delta\,\mathbb{1}$. By scaling the interaction $U \to \lambda\, U$, with a parameter λ ($0 \le \lambda \le 1$), the interacting theory is connected continuously to the Fermi gas by tuning λ from 1 to 0. The standard perturbative expansion can be organized by counting the various powers of λ, setting $\lambda = 1$ at the end before evaluating the expressions [26]. Below in Section 5 we provide a more microscopic argument for introducing the λ parameter in the Hubbard X operators directly, this method leads back to the equations found here.

In the corresponding equation for the t–J model (15), we observe that the Green's function differs from that for the free Fermi gas through two terms on the left hand side, exactly as in the Hubbard model, but also through one term on the right hand side. Scaling these three terms by λ, we rewrite (15) schematically as:

$$\left(\mathbf{g}_0^{-1} - \lambda\hat{X} - \lambda Y_1\right).\mathcal{G} = \delta\,(\mathbb{1} - \lambda\gamma). \tag{16}$$

The strategy of the perturbative λ expansion method is to build up the solution of this equation at $\lambda = 1$ through a suitable expansion in λ, starting from the free Fermi limit $\lambda = 0$. Thus $\lambda < 1$ corresponds to the admixture of a finite fraction of double occupancy that vanishes at $\lambda = 1$. Insights from sum rules, the skeleton graph expansion and the physics of the Hubbard sub bands has played a major role in formulating a systematic λ expansion described in detail in Refs. [4,6].

Within this approach it is also necessary to add a term $\lambda u_0 \sum_i N_{i\uparrow}N_{i\downarrow}$ to the Hamiltonian, and a corresponding term to the EOM, so that the $\hat{X}$ and Y_1 in Eq. (16) are suitably redefined. Here u_0 is an extra Hubbard interaction type parameter that is *determined by a sum rule* as explained below. At $\lambda = 1$ such a term makes no difference since the double occupancy is excluded. This parameter u_0 also enables us to enforce a simple but crucial symmetry of the t–J model-*the shift invariance*, noted in Ref. [6]. This invariance arises from the twofold function of the hopping in the t–J model when expressed in terms of the canonical operators, of providing hopping as well as the four Fermion (interaction) terms. Therefore in an exact treatment, adding a constant times the identity matrix to the hopping matrix: $t_{ij} \to t_{ij} + \text{const} \times \delta_{ij}$, shifts the center of gravity band innocuously. In approximate implementations it has the unphysical effect of also adding to the interaction (i.e. four Fermion type) terms. Such a change must therefore be compensated by an adjustable parameter that can soak up this additive constant. Indeed u_0 provides precisely this type of a parameter. It also plays the role of a second chemical potential u_0 [6] to fix the number of Fermions in the auxiliary Green's function $\mathbf{g}$ through $n_\sigma = \mathbf{g}_{\sigma\sigma}(i\tau^-, i\tau)$, while the thermodynamical chemical potential $\boldsymbol{\mu}$ (residing in the non interacting $\mathbf{g}_0^{-1}$), is fixed by the number sum rule $n_\sigma = \mathcal{G}_{\sigma\sigma}(i\tau^-, i\tau)$ (Eq. (14)). Enforcing this shift invariance to each order in the λ expansion plays an important "watchdog" role on the λ expansion, in addition to other standard constraints such as the Ward identities.

To summarize some key points of the λ expansion, we first decompose the Greens function into the space time convolution of an auxiliary Greens function and a caparison function as:

$$\mathcal{G} = \mathbf{g}.\mu. \tag{17}$$

With this the operator in Eq. (16) acts on the two factors of Eq. (17), and breaks into two equations upon using the *ansatz* that $\mathbf{g}$ has a canonical structure $\left(\mathbf{g}_0^{-1} - \lambda\hat{X} - \lambda Y_1\right).\mathbf{g} = \delta\,\mathbb{1}$. The λ expansion [6] is then an iteration scheme that proceeds by an expansion of the caparison function $\mu(k)$ and Y_1

Table 1
A flowchart of the ECFL theory as developed in Refs. [4,6]. See Sections 2.2 and 2.3 for a detailed description.

Step(I)	Step(II)	Step(III)	Step(IV)	Step(V)	Step(VI)	Step(VII)
Green's function $\mathcal{G}$ in terms of Hubbard operators.	Exact Schwinger equations of motion for $\mathcal{G}$.	Product expression into canonical part $\mathbf{g}$ and adaptive spectral weight (caparison) part $\mu(k)$.	Exact equations for $\mathbf{g}(k)$ and $\mu(k)$.	Introduction of interpolating parameter λ connecting the Fermi gas to the extreme correlation limit.	Shift invariance requires second chemical potential u_0. Same sum rule for both Greens functions so that Fermi surface volume is conserved.	Successive orders in λ expansion satisfying shift invariance for practical calculations.
$\mathcal{G}$	$\partial_\tau \mathcal{G}$	$\mathcal{G}(k) = \mathbf{g}(k)\mu(k)$		$0 \le \lambda \le 1$	$\sum \mathcal{G} = \sum \mathbf{g} = \frac{n}{2}$	

($Y_1 = t\gamma$) in powers of λ. Dyson's skeleton graph idea is implemented by keeping the auxiliary $\mathbf{g}$ intact (i.e. unexpanded in λ), while all other variables are expanded in powers of λ *and* $\mathbf{g}$, thereby obtaining self consistent equations for $\mathbf{g}$ and the vertex functions. Successive levels of approximation are obtained by retaining increasing powers of λ. At each approximation level, we set $\lambda = 1$ before actually evaluating the expressions, and implement the antiperiodic boundary conditions (75), (76), and the number sum-rule $n_\sigma = \mathcal{G}_{\sigma\sigma}(i\tau^-, i\tau)$ (Eq. (14)).

Elaborating on the representation (17) of $\mathcal{G}$, we note that the γ term on the right hand side of (16) is due to the non canonical anticommutator of the projected Fermi operators. As noted in Ref. [4], this term contains the essential difficulty of the t–J problem, having no parallel in the (canonical) Hubbard type models. After turning off the sources, in the momentum–frequency space we can further introduce two self energies $\Psi(k, i\omega)$, and $\Phi(k, i\omega)$ with

$$\mu(\vec{k}, i\omega_n) = 1 - \frac{n}{2} + \Psi(\vec{k}, i\omega_n) \tag{18}$$

$$\mathbf{g}^{-1}(\vec{k}, i\omega_n) = \mathbf{g}_0^{(-1)}(\vec{k}, i\omega) - \Phi(\vec{k}, i\omega_n), \tag{19}$$

where the constant $\frac{n}{2}$ in Eq. (18) is fixed by the condition that Ψ vanishes at infinite frequency. The auxiliary Greens function satisfies a second sumrule analogous to Eq. (14), written in the Fourier domain:

$$(k_B T) \sum_{k,n} e^{i\omega_n 0^+} \mathbf{g}_{\sigma\sigma}(k, i\omega_n) = n_\sigma . \tag{20}$$

Clearly the same sumrule holds for $\mathcal{G}_{\sigma\sigma}(k, i\omega_n)$. Eq. (17) can now be written explicitly in the non-Dysonian form proposed in Refs. [4,5]

$$\mathcal{G}(\vec{k}, i\omega) = \frac{1 - \frac{n}{2} + \Psi(\vec{k}, i\omega)}{\mathbf{g}_0^{(-1)}(\vec{k}, i\omega) - \Phi(\vec{k}, i\omega)}. \tag{21}$$

As argued in [4,6,8,9], simple Fermi liquid type self energies Ψ and Φ can, in the combination above, lead to highly asymmetric (in frequency) Dyson self energies from the structure of Eq. (21), thus providing a considerable tactical advantage in describing extreme correlations. We further discuss the physical meaning of this decomposition and the twin self energies in Section 4. Table 1 provides an overview of the various steps in the construction of the theory.

2.4. $\mathcal{G}(\vec{k}, i\omega_n)$ and the low energy spectral function in ECFL theory

We summarize here the low temperature low energy theory near the Fermi surface that follows from the general structure of Eq. (21) in terms of a small number of parameters, upon assuming that the two self energies have a Fermi liquid behavior at low energies. In the limit of large dimensions, a similar exercise gives a very interesting spectral function that matches the exact solution of the

$U = \infty$ Hubbard model found from the dynamical mean field theory (DMFT) [9]. The presentation below generalizes that to include a momentum dependence that is absent in high dimensions, and is supplemented by a discussion of the behavior of the various coefficients as the density of electrons n approaches unity, or equivalently the hole density $\delta \to 0$.

The Dyson self energy can be inferred from a simple inversion, and has a strong set of corrections to the Fermi liquid theory that we delineate here. We assume here a Fermi liquid type state that survives the limit of small hold density $\delta \to 0$. In reality at very small δ several other broken symmetry states would compete and presumably win over the liquid state, so that this Fermi liquid state would be metastable. It characteristics are of interest and hence we proceed to describe these.

We study Eq. (21) by analytically continuing $i\omega \to \omega + i0^+$ and write

$$\mathbf{g}_0^{(-1)}(\vec{k}, i\omega) = \omega + \boldsymbol{\mu} - \left(1 - \frac{n}{2}\right)\varepsilon_k. \tag{22}$$

Let us define $\hat{k}$ as the *normal deviation* from the Fermi surface i.e. $\hat{k} = (\vec{k} - \vec{k}_F).\vec{k}_F/|\vec{k}_F|$, and the frequently occurring Fermi liquid function

$$\mathcal{R} = \pi\{\omega^2 + (\pi k_B T)^2\}. \tag{23}$$

We carry out a low frequency expansion as follows:

$$1 - \frac{n}{2} + \Psi(\vec{k}, \omega) = \alpha_0 + c_\Psi(\omega + v_\Psi\, \hat{k}\, v_f) + i\mathcal{R}/\gamma_\Psi + O(\omega^3), \tag{24}$$

where $\alpha_0 = 1 - \frac{n}{2} + \Psi_0$ is the constant term at the Fermi surface, and a similar expansion for $\Phi(\vec{k}, \omega)$ so that

$$\omega + \boldsymbol{\mu} - \left(1 - \frac{n}{2}\right)\varepsilon_k - \Phi(k, \omega) = (1 + c_\Phi)\left(\omega - v_\Phi\, \hat{k}\, v_f + i\mathcal{R}/\Omega_\Phi + O(\omega^3)\right), \tag{25}$$

where $v_f = (\partial_k \varepsilon_k)_{k_F}$ is the *bare* Fermi velocity. The expansion coefficients above are in principle functions of the location of $\vec{k}_F$ on the Fermi surface, and have suitable dimensions to ensure that Ψ is dimensionless and Φ is an energy. The dimensionless velocity renormalization constants v_Φ and v_Ψ capture the momentum dependence normal to the Fermi surface, arising from the two respective self energies. The Greens function near the Fermi surface can now be written as

$$\mathcal{G}(\vec{k}, \omega) \sim \frac{z_0}{\alpha_0}\left(\frac{\alpha_0 + c_\Psi(\omega + v_\Psi\, \hat{k}\, v_f) + i\mathcal{R}/\gamma_\Psi}{\omega - v_\Phi\, \hat{k}\, v_f + i\mathcal{R}/\Omega_\Phi}\right) \tag{26}$$

where $z_0 = \alpha_0/(1 + c_\Phi)$ is the net quasiparticle renormalization constant. The spectral function can be computed from $A(\vec{k}, \omega) = -\frac{1}{\pi}\Im m\, \mathcal{G}(\vec{k}, \omega + i0^+)$ in the ECFL form of a Fermi liquid function times a caparison function $\mu(k, \omega)$ as follows:

$$A(\vec{k}, \omega) = \frac{z_0}{\pi}\frac{\Gamma_0}{(\omega - v_\Phi\, \hat{k}\, v_f)^2 + \Gamma_0^2} \times \mu(k, \omega), \tag{27}$$

where the (Fermi liquid) width function

$$\Gamma_0(\hat{k}, \omega) = \eta + \frac{\pi(\omega^2 + (\pi k_B T)^2)}{\Omega_\Phi}, \tag{28}$$

with an extra phenomenological parameter η required to describe elastic scattering [14] in impure systems. The caparison function is

$$\mu(\hat{k}, \omega) = 1 - \frac{\omega}{\Delta_0} + \frac{v_0\, \hat{k}\, v_f}{\Delta_0}, \tag{29}$$

where we introduced an important (emergent) low energy scale combining the other parameters:

$$\Delta_0 = \alpha_0 \frac{\gamma_\Psi}{\Omega_\Phi - c_\Psi \gamma_\Psi} \tag{30}$$

and the dimensionless momentum dependence coefficient

$$\nu_0 = (\nu_\Psi \gamma_\Psi c_\Psi + \nu_\Phi \Omega_\Phi)/(\Omega_\Phi - c_\Psi \gamma_\Psi). \tag{31}$$

A cutoff $\theta\left(\mu(\hat{k},\omega)\right)$ is implicit in Eq. (29), so that the function $\mu(\hat{k},\omega)$ is assumed to be zero at large positive frequencies as discussed in Ref. [4]. The five final parameters defining the spectral function (27) are z_0, ν_0, ν_Φ, Ω_Φ, Δ_0. For fitting experimental data, it may be best to think of them as adjustable parameters that determine the line shapes, their asymmetries and also features in the spectral dispersions. In addition the η parameter is needed to describe impurities that are not contained in the microscopic theory. In the early fit [14] the total number of free parameters is even smaller—just two instead of five. The corrections to the Landau Fermi liquid theory are encapsulated in the caparison factor, which contains a correction term that is odd in frequency and seems to be ultimately responsible for the asymmetric appearance of the line shapes [14,8].

For reference we note that in the limit of high dimensions [9], the coefficient of the momentum dependent term ν_0 vanishes in Eq. (27), while the earlier fits to experiments in [14], it is non zero, and in modified fits [15] its magnitude is varied to get a good description of the constant energy cuts of the data.

It is useful to consider the approach to the Mott insulating limit, where the parameters behave in a specific fashion to satisfy the expected behavior. We consider the limit of density $\delta \to 0$, and a frequency scale $0 \leq |\omega| < \omega_c \sim \delta t$, where the above expression (27) may be expected to work. For reference, it is useful to note that in this limiting case, the widely used Gutzwiller–Brinkman–Rice theory [2,27] gives the quasiparticle propagator as:

$$G_{GBR}(\vec{k},\omega) \sim \frac{z}{\omega - z\,\hat{k}\,v_f}, \tag{32}$$

where z vanishes linearly with δ as $z = 2\delta/(1+\delta)$. This leads to a delta function spectral weight $A_{GBR} = z\,\delta(\omega - z\,\hat{k}\,v_f)$. In contrast Eq. (27) provides the spectral function at non zero T and ω.

As $n \to 1$ in Eq. (24) we expect that the constant $\Psi_0 \to -\frac{n}{2}$, in order to reach the Mott insulating limit continuously. This implies that $\alpha_0 \propto \delta$ in this regime, and this drives the various other coefficients as well. We summarize the expected behavior of the above five coefficients

$$\begin{aligned} &z_0 \to \overline{z}_0 \times \delta; \qquad \Delta_0 \to \overline{\Delta}_0 \times \delta; \qquad \Omega_\Phi \to \overline{\Omega}_\Phi \times \delta; \\ &\nu_0 \to \overline{\nu}_0 \times \delta; \qquad \nu_\Phi \to \overline{\nu}_\Phi \times \delta; \end{aligned} \tag{33}$$

by using an overline for denoting a non vanishing limit of the stated variable [9,28]. The scaling of the velocity constants ν is guided by the results in high dimensions, and ensure that the dispersing quasiparticles have a vanishing bandwidth as we approach the insulator—as emphasized by Brinkman and Rice [27]. From this we find that the ECFL spectral function (27) satisfies a simple homogeneity (i.e. scaling) relation valid in the low energy regime for a scale parameter s:

$$A(\hat{k}, s\,\omega | s\,T, s\,\delta) = A(\hat{k}, \omega | T, \delta), \tag{34}$$

where the dependence on the temperature and hole density are made explicit. The momentum variable does not scale with s due to the assumed behavior of the ν's. The scaling holds for $\eta = 0$, and generalizes to a non zero values if we scale $\eta \to s\,\eta$. This scaling relation describes a Fermi liquid including significant corrections to Fermi liquid theory through the caparison function. It rests upon the specific behavior for the coefficients as the density varies near the insulating state, unlike other generalized scaling relations that have been proposed in literature Ref. [29] for non Fermi liquid states. If set $s \times \delta = \delta_0$ with say $\delta_0 \lesssim .5$, then the ratio $\frac{\delta_0}{\delta} \gg 1$ and we infer

$$A(\hat{k}, \omega | T, \delta) \sim A\left(\hat{k},\ \omega\frac{\delta_0}{\delta}\middle|\ T\frac{\delta_0}{\delta}, \delta_0\right), \tag{35}$$

relating the low hole density system to an overdoped (i.e. high hole density) system at a high effective temperature. This relation provides basic intuition for why the t–J model, near the insulating limit behaves almost like a classical liquid, unless one fine tunes parameters very close to the $T = 0$, $\omega = 0$ limit.

2.5. Electronic origin of the low energy kink and further tests of dynamical asymmetry

In this section we summarize the origin of the important low energy *kink* feature of the dispersion relation obtained in the ECFL theory. Since a similar feature is seen in the experiments on angle resolved photoemission studies (ARPES) of various groups [30–32,14], it is worth clarifying the purely electronic origin of this feature within the ECFL theory. A higher (binding) energy kink is also seen and is well understood in terms of the behavior of the self energy over a greater range [5,9], and is not pursued here. Rather we focus on the low energy kink seen around −.05 eV in several compounds [30–32,14], and finds a natural interpretation within ECFL.

We also present a few experimentally testable features relating to *dynamical asymmetry*, i.e. the asymmetric in ω correction to the Fermi liquid theory contained in ECFL, arising from the caparison function in Eq. (27).

Let us assume that $|\omega| \ll \Gamma_0$ at low enough frequency relative to T so that we may treat Γ_0 as a constant. We may then bring Eq. (27) to an interesting form studied in Ref. [5] by defining variables

$$\epsilon = \frac{\omega - v_\Phi \, \hat{k} \, v_f}{\Gamma_0}$$
$$\sinh u_k = \frac{\Delta_0 + (v_0 - v_\Phi) \, \hat{k} \, v_f}{\Gamma_0}, \tag{36}$$

so that the spectral function reduces to the standard form occurring in the ECFL theory:

$$A(u_k, \epsilon) = A_0 \frac{\sinh u_k - \epsilon}{1 + \epsilon^2} \times \theta(\sinh u_k - \epsilon) \tag{37}$$

with $A_0 = \frac{z_0}{\Delta_0}$. This expression is valid for small enough ϵ [5,4], and can be viewed as the (weighted) sum of the real and imaginary parts of a simple damped oscillator with a scaled susceptibility $\chi(\epsilon) = 1/(\epsilon + i)$. It is interesting to note that the scaled spectral function (37) can be related to the (scaled) Fano line shape

$$A_{Fano}(q_f, \epsilon) \propto \frac{(q_f + \epsilon)^2}{(1 + \epsilon^2)}. \tag{38}$$

This spectrum is often considered with the Fano parameter $q_f > 0$, it is highlighted by a vanishing at negative energies $\epsilon = -q_f$, representing the destructive interference of a scattering amplitude with a background term arising from a continuum of states. However we can flip the sign of q_f and by choosing $q_f = -e^{u_k}$, we can relate these through

$$A(u_k, \epsilon) \propto \left(A_{Fano}(-e^{u_k}, \epsilon) - A_{Fano}(-e^{u_k}, \infty) \right). \tag{39}$$

For the purpose of representing ARPES spectral functions, the scaled spectral function (37) gains an advantage over the Fano line shape (38) by the absence of a background at large $|\epsilon|$. In relating them via Eq. (39), the background term in the Fano process is killed, while its interference with the peak is retained.

Unlike the simple Lorentzian obtained at $u_k \to \infty$, the energy variable enters the numerator as well as the denominator in both Eq. (37) and the Fano shape. This feature gives rise to the characteristic skew to the ECFL spectrum. The spectral function can be maximized with respect to the frequency at a fixed $\hat{k}$, yielding the energy distribution curve (EDC) dispersion E_k^*, or with respect to $\hat{k}$ at a fixed frequency ω, giving the momentum distribution curve (MDC) dispersion E_k. Let us introduce the convenient variables

$$r = \frac{v_0}{v_\Phi}, \tag{40}$$

giving the ratio of the two velocity factors. The ratio $r = 0$ in the limit of high dimensions [9]. In the simplified ECFL analysis in [4,14], we find $r > 1$ due to the suppression of v_Φ relative to v_0 by

a quasiparticle renormalization factor z_{FL}. We see below that the magnitude and sign of $(r-1)$ play a significant role in determining the location of the kink, and its observability in ARPES respectively. We also introduce a (linear in $\hat{k}\, v_f$) energy variable:

$$Q(\hat{k}) = \Delta_0 + (v_0 - v_\Phi)\, \hat{k}\, v_f. \tag{41}$$

In terms of these, the two dispersions are obtained as

$$E(k) \;= \frac{1}{2-r}\left(v_\Phi\, \hat{k}\, v_f + \Delta_0 - \sqrt{r(2-r)\,\Gamma_0^2 + Q^2}\right), \tag{42}$$

$$E^*(k) = \left(v_0\, \hat{k}\, v_f + \Delta_0 - \sqrt{\Gamma_0^2 + Q^2}\right). \tag{43}$$

Simplifying the notation, both energy dispersions are of the form $E \sim \gamma\, Q - \sqrt{Q^2 + M^2}$, i.e. the hybrid of a massless and a massive Dirac spectrum. As Q varies from $-\infty$ to ∞, the energy crosses over from $(\gamma+1)Q$ to $(\gamma-1)Q$, thus exhibiting a knee or a kink near $Q \sim 0$, with its sharpness determined by the "mass term". The mass term in the MDC spectrum depends on the ratio r, and this generally leads to a smaller magnitude. Upon turning off the decay rate Γ_0, both the EDC and MDC spectra reduce to the expected spectrum $\varepsilon_k = v_\Phi\, \hat{k}\, v_f$, arising from the pole of the auxiliary Greens function in Eq. (21). These expressions illustrate an unusual feature of this theory: the two dispersions are influenced by the emergent energy scale Δ_0, as well as the width Γ_0 (Eq. (28)).

The above dispersions exhibit an interesting *kink feature* at $Q = 0$ in both spectra. The condition $Q = 0$ locates the kink momentum as

$$(\hat{k}\, v_f)_{kink} = \frac{\Delta_0}{v_\Phi(1-r)}, \tag{44}$$

it corresponds to occupied momenta provided $r > 1$, we will confine to this case below. For the other case $r < 1$, a kink would arise in the unoccupied side, for this reason we do not pursue it here. For $|Q| \gg \Gamma_0$, the two dispersions asymptotically become $E^*(k) \sim (v_0 + (v_0 - v_\Phi)\, \mathrm{sign}(\hat{k}))\, \hat{k}\, v_f$ and $E(k) \sim \frac{1}{2-r}(v_\Phi + (v_0 - v_\Phi)\, \mathrm{sign}(\hat{k}))\, \hat{k}\, v_f$. Hence these spectra exhibit a change in velocity (i.e. slope) around $Q \sim 0$ of magnitude $2(v_0 - v_\Phi)v_F$ for the EDC and the usually larger $\frac{2}{2-r}(v_0 - v_\Phi)v_F$ for the MDC spectrum. The change in slope of the spectrum occurs over a range $\Delta Q \propto \Gamma_0$, thus becoming sharper as Γ_0 decreases.

The value of the EDC energy at the kink is found by substituting $Q = 0$ and gives

$$E^*(k_{kink}) = -\frac{r}{r-1}\Delta_0 - \Gamma_0. \tag{45}$$

The MDC spectrum shows a kink for $2 \geq r \geq 1$ at the same momentum (44), with energy

$$E(k_{kink}) = -\frac{1}{r-1}\Delta_0 - \Gamma_0\sqrt{\frac{r}{2-r}}, \tag{46}$$

this feature is sharper than in the EDC spectrum since the effective damping is smaller.

When $r > 2$, the MDC energy is real only for $|\hat{k}\, v_f| < (|\hat{k}\, v_f|)_{cutoff}$, where the (negative) momentum

$$(\hat{k}\, v_f)_{cutoff} = (\hat{k}\, v_f)_{kink} + \frac{\Gamma_0}{v_\Phi(r-1)}\sqrt{r(r-2)}.$$

For $\hat{k}\, v_f$ beyond the cut off, the root becomes complex implying the loss of a clear peak in the MDC spectrum. Thus the spectrum "fades" before reaching the kink momentum (44). Therefore in this case, the kink is less than ideal, unlike the EDC kink or the MDC kink for $1 \leq r \leq 2$, which should be visible on both sides of the kink momentum. From Eq. (33) we may extract the hole density dependence of all the kink parameters, while Γ_0, determining the kink width, is given in Eq. (28).

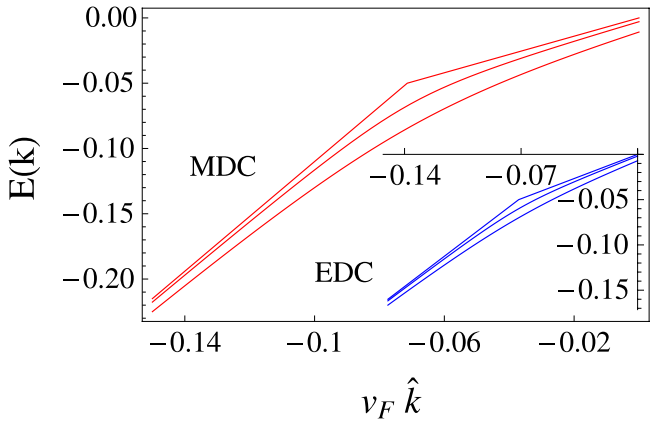

Fig. 1. A kink feature in the MDC dispersion relation $E(k)$ from Eq. (42) and in the inset from the EDC dispersion $E^*(k)$ (Eq. (43)) with parameters $\Delta_0 = .025$ eV, $\nu_0 = 1.05$, $\nu_\Phi = 0.7$ and three values of $\Gamma_0 = 0., .01, .02$ in eV from top to bottom. The kink is more pronounced in the MDC curve as discussed in text.

We observe in Fig. 1 that the kink becomes sharp when Γ_0 decreases. The MDC curves display a sharper kink than the EDC curves, this is easy to understand since the effective damping is smaller in this case, and also the net change in velocity across the kink is greater, as discussed above. From Eq. (28) we see various parameters that control Γ_0, in case of laser ARPES, it is argued [14] that η is small so we expect to see sharper kinks in this setup. Further, as T drops below T_c, the d-wave superconductor has gapless excitations along the nodal direction $\langle 11\rangle$, and the quasiparticles seen in this case are sharper. Theoretical considerations [33] show that in the superconducting state, a reduction in the available gapless states responsible for the linewidth implies a reduction of Γ_0 and hence to a sharper kink.

We next discuss the feature of *dynamical asymmetry* in the spectra. It is also important to note that the ECFL spectral function (27) has an unusual correction to the standard Fermi liquid part, embodied in the caparison function $\mu(k,\omega)$. This function is odd in frequency, thus disturbing the particle hole symmetry of the Fermi liquid part, and it grows in importance as we approach the insulating state since $\Delta_0 \to \delta\overline{\Delta}_0$ as indicated in Eq. (33). It is also interesting that the spectral line shape in the calculation of Anderson and Casey (AC) [34] as well as Doniach and Sunjic (DS) [35] also have such odd in ω corrections to the Fermi liquid part. In fact the AC result may be viewed as the vanishing of the scale $\Delta_0 \propto k_B T$ so that the ground state is non Fermi liquid like. At finite T and ω the AC and DS theories are parallel with the ECFL line shapes regarding the asymmetry as remarked in Ref. [8], and we wish to make a few comments about the experimental tests for such an asymmetry, going beyond standard measures such as the skewness factor.

DS [35] make the interesting point that the asymmetry is best isolated by looking at the inverse of the spectral function in a plot of

$$\frac{1}{A(k,\omega)} \quad vs \quad (\omega - E_k^*)^2, \tag{47}$$

where E_k^* is the peak location in the EDC. With this plot, a Fermi liquid yields two coincident straight lines above and below E_k^*, whereas an asymmetric contribution, as in Eq. (27) or the DS line shape [35], would split into two distinct non linear curves, from below and above E_k^*. The inversion of the spectral function is an interesting device, since it refocuses attention on the asymmetric parts. For very similar reasons Ref. [4] (Fig. 1 inset) also advocates plotting the inverse of the spectral function. On the other hand an untrained examination of the EDC curves invariably focuses on the close proximity of the peaks of $A(k,\omega)$, these are arguably the least interesting part of the asymmetry story!

In fact armed with the explicit knowledge of the spectral function of the ECFL theory in Eq. (27), we can aim to do better in establishing the asymmetry and in determining the various parameters. We first redefine the frequency by subtracting off the EDC peak value

$$\widetilde{\omega}_k = \omega - E_k^*, \tag{48}$$

so that the spectral peak occurs at $\widetilde{\omega}_k = 0$. The inverse spectral function can be computed as a function of $\widetilde{\omega}_k$ and reads:

$$\frac{A(k, E_k^*)}{A(k, E_k^* + \widetilde{\omega}_k)} = 1 + \frac{e^{u_k}}{2\Gamma_0} \times \frac{\widetilde{\omega}_k^2}{\Gamma_0 \cosh(u_k) - \widetilde{\omega}_k}, \tag{49}$$

where the peak value of the spectral function at $\widetilde{\omega}_k = 0$ is:

$$A(k, E_k^*) = \frac{A_0}{2} e^{u_k}. \tag{50}$$

We next construct the object $\mathcal{Q}(\widetilde{\omega}_k)$ from Eq. (49) by subtracting unity and cross multiplying:

$$\mathcal{Q}(\widetilde{\omega}_k) = \frac{\widetilde{\omega}_k^2}{A(k, E_k^*)/A(k, E_k^* + \widetilde{\omega}_k) - 1}. \tag{51}$$

This variable is designed to be a $\widetilde{\omega}_k$ independent constant in a simple Fermi liquid with a Lorentzian line shape (i.e. Eq. (27) without the caparison function μ). Here $\mathcal{Q}$ has dimensions of the square of energy, and when plotted against $\widetilde{\omega}_k$ in the small range surrounding zero i.e. $|\widetilde{\omega}_k| \leq \Gamma_0$ it exhibits a linearly decreasing behavior with $\widetilde{\omega}_k$ within the ECFL spectral function (27)

$$\mathcal{Q}(\widetilde{\omega}_k) = \Gamma_0^2 (1 + e^{-2u_k}) - \left(2\Gamma_0 e^{-u_k}\right) \widetilde{\omega}_k. \tag{52}$$

Note that this function is flat for the usual Fermi liquid state without asymmetric corrections, since in this case $u_k \to +\infty$. If found in data, this linear in $\widetilde{\omega}$ behavior is the distinctive aspect of the asymmetric line shapes. We can then read off various physical quantities once the curve of $\mathcal{Q}(\widetilde{\omega}_k)$ versus $\widetilde{\omega}_k$ is obtained. For this purpose we need the intercept $\mathcal{Q}(0)$ and the slope near the origin $(d\mathcal{Q}(\widetilde{\omega}_k)/d\widetilde{\omega}_k)_0$. Clearly the $\mathcal{Q}(\widetilde{\omega}_k)$ function will deviate from a straight line sufficiently far from $\widetilde{\omega}_k = 0$, and it will also be contaminated with background terms as well as noise. However, with high quality data this procedure could be useful in inverting the data to fit simple functional forms, and to make decisive tests of the predictions of the theories containing asymmetry, namely the DS and AC theories as well as ECFL.

3. Exact formulation in terms of a canonical Fermions

We will next rewrite this in canonical Fermi representation in *an enlarged Hilbert space* where double occupancy is permitted, and the singly occupied states form a subspace. We regard the physical subspace of states $|\Psi\rangle$ as those that satisfy the condition of single occupancy, i.e. $\hat{D}|\Psi\rangle = 0$ with the double occupancy operator $\hat{D}$ is given by:

$$\hat{D} = \sum_i \hat{D}_i, \qquad \hat{D}_i \equiv C_{i\uparrow}^\dagger C_{i\uparrow} C_{i\downarrow}^\dagger C_{i\downarrow}, \tag{53}$$

and $C_{i\sigma}$ and $C_{i\sigma}^\dagger$ denote the canonical Fermionic destruction and creation operators. The unphysical states contain one or more doubly occupied states. In terms of these, the Gutzwiller projector over all sites is written as:

$$\hat{P}_G = \prod_i \left(1 - \hat{D}_i\right). \tag{54}$$

This projection operator can be introduced into a partition function to deal with unphysical states, as we show below.

The next goal (see Table 1) is to write the most efficient representation in the enlarged space of the t–J model Green's functions, in terms of the canonical operators and the projection operator. As pointed out in the introduction, we note that pairs of operators that are mutual adjoints in the t–J model (e.g. $X_i^{0\sigma} = (X_i^{\sigma 0})^\dagger$), are allowed to be represented by operators that violate this adjoint property. The main result of this section is that this possibility leads to the most compact canonical theory; we term it the non-Hermitian theory. However we first warmup with a short summary of the more obvious Hermitian theory, which sets the stage for the main result.

3.1. A Hermitian canonical representation with redundancy

Projected Fermi operators distinguished by the hats can be written in a familiar construction [36]

$$\widetilde{C}_{i\sigma} = C_{i\sigma}(1 - N_{i\bar{\sigma}})$$
$$\widetilde{C}^{\dagger}_{i\sigma} = C^{\dagger}_{i\sigma}(1 - N_{i\bar{\sigma}}), \tag{55}$$

where $N_{i\sigma} = C^{\dagger}_{i\sigma} C_{i\sigma}$, and $N_i = \sum_{\sigma} N_{i\sigma}$, with the property that these conserve the number of doubly occupied sites locally:

$$[\widetilde{C}_{i\sigma}, \hat{D}_i] = 0, \qquad [\widetilde{C}^{\dagger}_{i\sigma}, \hat{D}_i] = 0 \tag{56}$$

and therefore also globally i.e. with $\hat{D}$ in place of $\hat{D}_i$. It implies that any Hamiltonian written in terms of these operators with hats commutes with the individual $\hat{D}_i$ as well as the global $\hat{D}$, and thus conserves the local symmetry of the model. Therefore acting within the physical subspace of states, (55) provide a faithful realization of the X^{ab}_i operators as $X^{0\sigma}_i \leftrightarrow \widetilde{C}_{i\sigma}$ and $X^{\sigma 0}_i \leftrightarrow \widetilde{C}^{\dagger}_{i\sigma}$, and clearly satisfies the mutual adjoint property. We are also interested in the product of two X's in order to represent the kinetic energy term of the effective Hamiltonian below. The optimal choice is seen to be

$$X^{\sigma 0}_i X^{0\sigma}_j \leftrightarrow C^{\dagger}_{i\sigma} C_{j\sigma} \left(1 - N_{i\bar{\sigma}} - N_{j\bar{\sigma}}\right). \tag{57}$$

While the choice

$$X^{\sigma 0}_i X^{0\sigma}_j \leftrightarrow \widetilde{C}^{\dagger}_{i\sigma} \widetilde{C}_{j\sigma} \tag{58}$$

is also a faithful representation, it contains an extra term $C^{\dagger}_{i\sigma} C_{j\sigma} N_{i\bar{\sigma}} N_{j\bar{\sigma}}$, over and above (57), which is redundant since (57) already commutes with (54).

Using (57) we write a canonical expression for the Hamiltonian

$$H_{tJ} \to \hat{H}_{eff} = \hat{H}_t + \hat{H}_J, \tag{59}$$

with

$$\hat{H}_t = \hat{T}_{eff} - \boldsymbol{\mu} \sum_i N_{i\sigma},$$
$$\hat{T}_{eff} = -\sum_{ij\sigma} t_{ij} C^{\dagger}_{i\sigma} C_{j\sigma} \left(1 - N_{i\bar{\sigma}} - N_{j\bar{\sigma}}\right), \tag{60}$$

we call this as the *symmetrized kinetic energy* in view of its obvious symmetry under the exchange $i \leftrightarrow j$, and write $\hat{H}_J \to \frac{1}{2}\sum_{ij} J_{ij} \left(\vec{S}_i.\vec{S}_j - \frac{1}{4} N_i N_j\right)$, with the spin and number operators written in terms of C's and $C^{\dagger}$'s without hats (since the occupancy of a site is unaffected by the exchange term). We easily verify that

$$[\hat{H}_{eff}, \hat{D}] = 0 = [\hat{H}_{eff}, \hat{P}_G], \tag{61}$$

therefore if we start with a state satisfying $\hat{D}|\Psi\rangle = 0$, i.e. in the singly occupied subspace, the resultant state $H_{eff}|\Psi\rangle$ remains in this subspace; we do not create doubly occupied states. We note that (61) implies that the operator (54) is invariant under time evolution through H_{eff}:

$$\hat{P}_G(\tau) = \hat{P}_G(0). \tag{62}$$

The partition functional as in (11), now defined with arbitrary τ_0:

$$Z = \mathrm{Tr}\, e^{-\beta \hat{H}_{eff}} T_{\tau} \left(e^{-\hat{\mathcal{A}}_S} \hat{P}_G(\tau_0)\right), \tag{63}$$

where the trace (unlike that in Eq. (11)), is over the entire canonical basis, i.e. includes doubly occupied states. For the observables as well as the source terms $\hat{\mathcal{A}}_S$, we use the replacement rules:

$$X^{0\sigma}_i \to \widetilde{C}_{i\sigma}, \qquad X^{\sigma 0}_i \to \widetilde{C}^{\dagger}_{i\sigma}, \qquad X^{\sigma\sigma'}_i \to C^{\dagger}_{i\sigma} C_{i\sigma'}, \tag{64}$$

to convert arbitrary expressions involving X_i^{ab} into those with the $\widetilde{C}$, $\widetilde{C}^\dagger$. Note that the density or spin density type variables are replaced by the canonical operators without a hat, since these commutes with the local $\hat{D}_i$.

We can compute the Green's functions in the enlarged (canonical) basis from

$$\mathcal{G}_{\sigma_i\sigma_f}(i\tau_i, f\tau_f) = -\frac{\left\langle T_\tau \left(e^{-\hat{\mathcal{A}}_S} \widetilde{C}_{i\sigma_i}(\tau_i) \widetilde{C}^\dagger_{f\sigma_f}(\tau_f)\, \hat{P}_G(\tau_0) \right) \right\rangle}{\left\langle T_\tau \left(e^{-\hat{\mathcal{A}}_S} \hat{P}_G(\tau_0) \right) \right\rangle}, \tag{65}$$

evaluated [23] at $\hat{\mathcal{A}}_S \to 0$. This relation can be replaced by differentiating the partition functional (63) with the Fermi sources J, J^*. Using the commutation of $\hat{P}_G$ or $\hat{D}$ with all operators and (62), we are free at this stage to locate place $\hat{P}_G$ at any specific time, without affecting the results. This formulation of the theory has parallels with the path integral representation of the electromagnetic field (QED) in the temporal gauge, where the scalar potential is chosen to be zero (i.e. $\phi(rt) = 0$) [37,38]. In this case the Gauss's law condition $\nabla.\vec{E}(r,t) = 0$ needs to be imposed at each time slice. However upon using $[H, \vec{E}] = \vec{\nabla} \times \vec{B}$, this object commutes with the Hamiltonian $[H, \vec{\nabla}.\vec{E}] = 0$, and therefore it suffices to impose this condition at the initial time. The situation has a clear analogy with Eq. (63), where it suffices to insert the projection operator at the initial time.

3.2. The hat removal rule and optimal non-Hermitian theory

The non-Hermitian theory arises when we inspect closely expressions of the type in (63), with the time τ_0 chosen as the earliest time 0^-. The general argument has been given in the introduction, we consider its specific application to the present problem next. Discretizing the time variables and expanding, we obtain a series containing expression of the type

$$\text{const} \times \sum \langle i|Q_1(\tau_1)\ldots Q_m(\tau_m)\hat{P}_G|i\rangle,$$

so that the first operator from the right $Q_m(\tau_m)$ acts upon a state which is Gutzwiller projected. Now the creation operators contained in the $Q(\tau)$'s are defined with the hats (see (55)) ensuring that they never create doubly occupied states. Next observe that destroying a particle cannot *create* a doubly occupied site. Therefore it cannot take a projected state out of this subspace! Therefore the operator $\widetilde{C}_{i\sigma}$ can as well be replaced by the destruction operator $C_{i\sigma}$ *without a hat*. We can iterate this argument for the next operator, which also acts on a Gutzwiller projected state, and so forth, leading to the hat removal rules. In this argument, we may replace the operator's $Q(\tau_m)$ by any expressions involving the destruction operators as well as creation operators with hats (as in (55)), and the same argument holds. More formally we may summarize by saying that the destruction operator *conditionally commutes* with the projection operator, when right-operating on projected states:

$$[C_{i\sigma}, \hat{P}_G]\hat{P}_G = 0, \tag{66}$$

although $[C_{i\sigma}, \hat{P}_G] \neq 0$, as one readily checks. Thus the commutator lives in an orthogonal subspace to that spanned by the Gutzwiller projected states. This property also extends to arbitrary functions $\hat{f}$ ($\hat{f} \equiv \hat{f}\{C_{i\sigma}\}, \{\widetilde{C}^\dagger_{j\sigma'}\}$) of the operators:

$$[\hat{f}, \hat{P}_G]\hat{P}_G = 0. \tag{67}$$

This property is just a rewriting of the important block triangularity condition of the operators noted in Eq. (8) leading to Eq. (7). We will make frequent use of this expression below.

We now turn to implementing this observation. Let us write the partition functional

$$Z = \text{Tr}\, e^{-\beta\hat{H}_{eff}}\, T_\tau \left(e^{-\hat{\mathcal{A}}_S} \hat{P}_G(0^-) \right), \tag{68}$$

and introduce the important abbreviation for averages:

$$\langle\langle A(\tau_1)B(\tau_2)\ldots\rangle\rangle \equiv \frac{1}{Z}\text{Tr}\, e^{-\beta\hat{H}_{eff}}\, T_\tau \left(e^{-\hat{\mathcal{A}}_S} A(\tau_1)B(\tau_2)\ldots \hat{P}_G(0^-) \right), \tag{69}$$

where notice that we located the projector at the *initial time*, by bringing it under the time ordering symbol.

We now state the crucial *hat-removal rule*: in all expressions of the type (68) and (69), the hats on *all destruction operators* can be removed

$$\widetilde{C}_{i\sigma}(\tau) \to C_{i\sigma}(\tau), \tag{70}$$

leaving the result unchanged. Notice that this rule can also be applied to H_{eff}, and the source terms $\hat{\mathcal{A}}_S$ containing the destruction operators $C_{i\sigma}$. Note that the *creation operators* cannot be 'un-hatted' in this fashion since these do create a doubly occupied site. Summarizing, we can use instead of (64), the more compact non-Hermitian rule

$$X_i^{0\sigma} \to C_{i\sigma}, \qquad X_i^{\sigma 0} \to \widetilde{C}_{i\sigma}^{\dagger} = C_{i\sigma}^{\dagger}(1 - N_{i\bar{\sigma}}), \qquad X_i^{\sigma\sigma'} \to C_{i\sigma}^{\dagger} C_{i\sigma'}. \tag{71}$$

We thus rewrite the sources (10) as:

$$\hat{\mathcal{A}}_S(i,\tau) = \left(\widetilde{C}_{i\sigma}^{\dagger}(\tau) J_{i\sigma}(\tau) + J_{i\sigma}^{*}(\tau) C_{i\sigma}(\tau)\right) + \mathcal{V}_i^{\sigma'\sigma}(\tau) C_{i\sigma'}^{\dagger}(\tau) C_{i\sigma}(\tau), \tag{72}$$

and the Green's function with imaginary time $0 \le \tau_i, \tau_j \le \beta$ is therefore written as:

$$\mathcal{G}_{\sigma_i\sigma_f}(i\tau_i, f\tau_f) = -\langle\langle C_{i\sigma_i}(\tau_i) \widetilde{C}_{f\sigma_f}^{\dagger}(\tau_f)\rangle\rangle, \tag{73}$$

analogous to (65) but with an unprojected destruction operator. We will show below that this is the most useful and compact expression for the Green's function. To complete the description of this theory, we turn to the task of specifying the Hamiltonian, and obtain the boundary conditions on the time variables. The last task is somewhat nontrivial since the projection operator does not commute with the other operators.

3.3. Hamiltonian in the symmetrized and minimal theories

In order to represent the Hamiltonian, the spin operators of the exchange part H_J are unambiguously expressed in terms of the $C_{i\sigma}$ and $C_{i\sigma}^{\dagger}$ operators without hats as in (71), since they preserve the occupation of a site. For the kinetic energy we could choose to work with (60), and thereby gain some advantage of dealing with a Hermitian Hamiltonian. This leads to the equations of motion termed the *the symmetrized theory* in Ref. [6]. Alternately we can implement the hat removal rule for the kinetic energy as well:

$$\hat{T}_{eff} = -\sum_{ij\sigma} t_{ij} \widetilde{C}_{i\sigma}^{\dagger} C_{j\sigma}. \tag{74}$$

This minimal version of the kinetic energy is clearly non-Hermitian. However, it has exactly the same action as the symmetrized version (59), when right-operating on the physical Gutzwiller projected states, as proved above. This leads to equations of motion of the *minimal theory* noted in Ref. [6] and elaborated upon in Refs. [10,9]. For completeness, we provide in Section 7.2 a brief derivation of these equations for the minimal case, using the above canonical representation, in place of the Schwinger equations.

3.4. Kubo–Martin–Schwinger antiperiodic boundary conditions

In working with the expressions (68), (71) and (73), we have assumed that all the times τ_j are positive and satisfy $0 \le \tau_j \le \beta$. The Green's function (12) satisfies the Kubo–Martin–Schwinger (KMS) anti-periodic boundary conditions [39]

$$\mathcal{G}(a\,\tau_i = 0, b\,\tau_f) = -\mathcal{G}(a\,\tau_i = \beta, b\,\tau_f), \tag{75}$$

$$\mathcal{G}(a\,\tau_i, b\,\tau_f = 0) = -\mathcal{G}(a\,\tau_i, b\,\tau_f = \beta), \tag{76}$$

Table 2
A summary of the representations of the Green's functions. The non-Hermitian minimal theory provides the most compact set of equations of motion, which are identical to those from the Hubbard–Gutzwiller theory in the second column. The absence of the adjoint property for the non-Hermitian theory arises from the asymmetric hat removal between the destruction and creation operators in the first two rows of the last column.

	Hubbard–Gutzwiller theory	(Canonical) Hermitian theory	(Canonical) Non-Hermitian theory
Operators:	$X^{\sigma 0}$ $X^{0\sigma}$ $X_i^{\sigma\sigma'}$	$\widetilde{C}_\sigma^\dagger = C_\sigma^\dagger\,(1-N_{\bar\sigma})$ $\widetilde{C}_\sigma = C_\sigma\,(1-N_{\bar\sigma})$ $C_{i\sigma}^\dagger C_{i\sigma'}$	$\widetilde{C}_\sigma^\dagger = C_\sigma^\dagger\,(1-N_{\bar\sigma})$ C_σ $C_{i\sigma}^\dagger C_{i\sigma'}$
Partition Functional: Z	$\mathrm{Tr}_{tJ} e^{-\beta H_{tJ}} T_\tau \left(e^{-\hat{\mathcal{A}}_S}\right)$	$\mathrm{Tr}\, e^{-\beta \hat{H}_{eff}} T_\tau \left(e^{-\hat{\mathcal{A}}_S} \hat{P}_G(\tau_0)\right)$; Arbitrary time τ_0 $(0 \le \tau_0 \le \beta)$.	$\mathrm{Tr}\, e^{-\beta \hat{H}_{eff}} T_\tau \left(e^{-\hat{\mathcal{A}}_S} \hat{P}_G(0^-)\right)$
Green's function: $-\mathcal{G}(1,1')$	$\langle T_\tau (e^{-\hat{\mathcal{A}}_S} X_1^{0\sigma} X_{1'}^{\sigma' 0})\rangle$	$\left\langle T_\tau \left(e^{-\hat{\mathcal{A}}_S} \widetilde{C}_\sigma(1) \widetilde{C}_{\sigma'}^\dagger(2) \hat{P}_G(\tau_0)\right)\right\rangle$ Arbitrary time τ_0 $(0 \le \tau_0 \le \beta)$.	$\left\langle T_\tau \left(e^{-\hat{\mathcal{A}}_S} C_\sigma(1) \widetilde{C}_{\sigma'}^\dagger(2) \hat{P}_G(0^-)\right)\right\rangle$
Remarks:	$H = H^\dagger$ in the defining representation.	Symmetrized theory $H = H^\dagger$	(i) Symmetrized theory: $\hat{H}_{eff} = H_{eff}^\dagger$ (ii) Minimal theory: $\hat{H}_{eff} \neq \hat{H}_{eff}^\dagger$

where the fixed time τ_f (τ_i) in the first (second) equations is assumed to satisfy $0 \le \tau \le \beta$. These conditions are usually proven by using the cyclic invariance of the trace [24], and translates easily to the canonical representation (65), with $\widetilde{C}$ and $\widetilde{C}^\dagger$ replacing the X operators (64).

In using the non-Hermitian representation (71) as in (73), we cannot use cyclicity of trace since the operator $\widetilde{C}$ does not commute with P_G. Remarkably enough, the conditional commutativity (66) and (67) suffices to guarantee the required antiperiodicity. In physical terms these proofs follow from the observation made above, the creation operators with hats, and destruction operators (without hats) preserve a Gutzwiller projected state within that subspace.

For simplicity we present the case with sources turned off i.e. $\mathcal{A} \to 0$, the more general case follows by a similar argument. From the definitions of the Green's functions, Eq. (76) is true since $\mathrm{Tr}\left(e^{-\beta H_{eff}} C_{a\sigma}(\tau_i)[\widetilde{C}_{b\sigma'}^\dagger(0), \hat{P}_G]\right)$ vanishes identically from Eq. (56).

In order to prove that Eq. (75) remains true, we need to show that the expression

$$\mathrm{Tr}\left(e^{-\beta H_{eff}} \widetilde{C}_{b\sigma'}^\dagger(\tau_f)[C_{a\sigma}(0), \hat{P}_G]\right) \tag{77}$$

vanishes, despite the non vanishing of the commutator in the expression. For this purpose, we utilize the conditional commutator (66) to write $[C_{a\sigma}(0), \hat{P}_G] = [C_{a\sigma}(0), \hat{P}_G](\mathbb{1} - \hat{P}_G)$. We next use cyclicity of trace and the simple identity (for any $\hat{Q}$): $\mathrm{Tr}\left((\mathbb{1} - \hat{P}_G)\hat{Q}\hat{P}_G\right) = 0$, to write the required expression (77) in the form

$$\mathrm{Tr}\left((\hat{P}_G - \mathbb{1}) e^{-\beta H_{eff}} \widetilde{C}_{b\sigma'}^\dagger(\tau_f) \hat{P}_G C_{a\sigma}(0)\right). \tag{78}$$

Using $(\hat{P}_G)^2 = \hat{P}_G$, we rewrite this as:

$$(\hat{P}_G - \mathbb{1}) e^{-\beta H_{eff}} \widetilde{C}_{b\sigma'}^\dagger(\tau_f) \hat{P}_G = [\hat{P}_G, e^{-\beta H_{eff}} \widetilde{C}_{b\sigma'}^\dagger(\tau_f)] \hat{P}_G.$$

This expression vanishes on using the conditional commutator (67), thereby proving the required result (75).

The two canonical theories providing an exact mapping of the original theory are summarized in Table 2.

4. The auxiliary Green's function and the caparison function using canonical Fermions

We next discuss the rationale for decomposing the Green's function into an auxiliary Greens function and a caparison function as in Ref. [4], using a simple argument from the exact formula

(73). This important part of the theory is also encountered in Section 6. In its simplest version this decomposition can be illustrated using the minimal theory, where the averages are defined as in Eq. (68), with the projection operator pinned at the initial time. We recall the Green's function from Eq. (73) $\mathcal{G}_{\sigma_i\sigma_f}(i\tau_i, f\tau_f) = -\langle\langle C_{i\sigma_i}(\tau_i)\widetilde{C}^\dagger_{f\sigma_f}(\tau_f)\rangle\rangle$, with the averages from Eq. (69). Expanding the $\widetilde{C}^\dagger$ operator this becomes

$$\mathcal{G}_{\sigma_i\sigma_f}(i\tau_i, f\tau_f) = -\langle\langle C_{i\sigma_i}(\tau_i)C^\dagger_{f\sigma_f}(\tau_f)\rangle\rangle + \langle\langle C_{i\sigma_i}(\tau_i)C^\dagger_{f\sigma_f}(\tau_f)N_{f\bar{\sigma}_f}(\tau_f)\rangle\rangle. \tag{79}$$

We next define the auxiliary Green's function as:

$$\mathbf{g}_{\sigma_i\sigma_j}(i\tau_i, j\tau_j) = -\langle\langle C_{i\sigma_i}(\tau_i)C^\dagger_{j\sigma}(\tau_j)\rangle\rangle, \tag{80}$$

and regarding the spin, space and time indices as matrix indices with a matrix inverse $\mathbf{g}^{-1}$. By separating the disconnected and connected parts ($\_c$) of the second term in (79) we write

$$\langle\langle C_{i\sigma_i}(\tau_i)C^\dagger_{f\sigma_f}(\tau_f)N_{f\bar{\sigma}_f}(\tau_f)\rangle\rangle = -\mathbf{g}_{\sigma_i\sigma_f}(i\tau_i, f\tau_f)\langle N_{f\bar{\sigma}_f}(\tau_f)\rangle + \langle\langle C_{i\sigma_i}(\tau_i)C^\dagger_{f\sigma_f}(\tau_f)N_{f\bar{\sigma}_f}(\tau_f)\rangle\rangle_c. \tag{81}$$

The connected part is written in terms of a second self energy Ψ defined as

$$\Psi_{\sigma_i\sigma_f}(i\tau_i, f\tau_f) = \mathbf{g}^{-1}_{\sigma_i\sigma_{\mathbf{k}}}(i\tau_i, \mathbf{k}\tau_{\mathbf{k}}) \times \langle\langle C_{\mathbf{k}\sigma_{\mathbf{k}}}(\tau_{\mathbf{k}})C^\dagger_{f\sigma_f}(\tau_f)N_{f\bar{\sigma}_f}(\tau_f)\rangle\rangle_c, \tag{82}$$

and assembling these we rewrite (79) as the product relation [4]

$$\begin{aligned}\mathcal{G}_{\sigma_i\sigma_f}(i\tau_i, f\tau_f) &= \mathbf{g}_{\sigma_i\sigma_{\mathbf{k}}}(i\tau_i, \mathbf{k}\tau_{\mathbf{k}})\mu_{\sigma_{\mathbf{k}}\sigma_f}(\mathbf{k}\tau_{\mathbf{k}}, f\tau_f),\\ \mu_{\sigma_i\sigma_f}(i\tau_i, f\tau_f) &= \delta(if)\left(1 - \langle N_{\bar{\sigma}_i}(\tau_i)\rangle\right) + \Psi_{\sigma_i\sigma_f}(i\tau_i, f\tau_f).\end{aligned} \tag{83}$$

There is a slight ambiguity in defining the two objects $\mathbf{g}$ and μ, since we have the freedom of adding a common function to the two parts of Eq. (79) that cancels out in the physical Greens function. Apart from this, we expect that the two objects in Eq. (83) are exactly equivalent to the auxiliary Greens function and the caparison factor in Eqs. (17)–(19) as found from the Schwinger method.

We observe from the expression (82) that if the averages are (temporarily) computed in a standard Feynman Dyson theory, then Ψ is essentially the self energy of a Hubbard type model, made dimensionless by dropping an explicit interaction constant U. Indeed this is the key observation made in Ref. [4], on the basis of the λ expansion, where the two self energies are argued to be generically Fermi liquid-like and similar to each other. An energy scale (Δ) emerges from a ratio of their imaginary parts, and controls the significant asymmetry seen in the spectral functions.

5. The λ-Fermions

A natural question is whether Eq. (16), explicitly containing the parameter λ, can arise in a microscopic theory where λ enters in a fundamental way, as opposed to the "engineering approach" in Section 2.3. A set of λ-Fermi operators are defined below, as generalized version of the non-Hermitian representation (71) with a parameter $\lambda \in [0, 1]$ providing a continuous interpolation between the free Fermi and extremely correlated limits:

$$\begin{aligned}X^{\sigma 0}_i(\lambda) &\to C^\dagger_{i\sigma}(1 - \lambda C^\dagger_{i\bar{\sigma}}C_{i\bar{\sigma}})\\ X^{0\sigma}_i(\lambda) &\to C_{i\sigma}\\ X^{\sigma\sigma'}_i(\lambda) &\to C^\dagger_{i\sigma}C_{i\sigma'}.\end{aligned} \tag{84}$$

Clearly $\lambda = 0$ gives us back the canonical Fermion operators, whereas $\lambda = 1$ gives the Gutzwiller projected Hubbard X operators [3] as in (71), provided the states are Gutzwiller projected. A feature of this representation is the loss of the adjoint property, i.e. $\left(X^{\sigma 0}_i(\lambda)\right)^\dagger \neq X^{0\sigma}_i(\lambda)$, unless $\lambda = 0$.

These operators satisfy a λ dependent (graded) Lie algebra with fundamental brackets that are partly Fermionic and partly Bosonic. Using the canonical anticommutation relations of the $C, C^\dagger$ operators, we work out the fundamental Fermionic bracket:

$$\{X^{0\sigma_i}_i(\lambda), X^{\sigma_j 0}_j(\lambda)\} = \delta_{ij}\{\delta_{\sigma_i\sigma_j} - \lambda\,\sigma_i\sigma_j X^{\bar{\sigma}_i\bar{\sigma}_j}_i(\lambda)\}. \tag{85}$$

We next evaluate the fundamental Bosonic bracket

$$[X_i^{0\sigma_i}(\lambda), X_j^{\sigma_j\sigma_k}(\lambda)] = \delta_{ij}\delta_{\sigma_i\sigma_j}X_i^{0\sigma_k}(\lambda) \tag{86}$$

$$[X_i^{\sigma_i 0}(\lambda), X_j^{\sigma_j\sigma_k}(\lambda)] = -\delta_{ij}\delta_{\sigma_i\sigma_k}X_i^{\sigma_j 0}(\lambda). \tag{87}$$

Here (87) requires a brief calculation [40] invoking the Pauli principle vanishing of $C_\sigma^\dagger C_\sigma^\dagger \to 0$. On the other hand (86) is elementary, due to the absence of λ in both sides of the equation. At $\lambda = 1$ these reduce to the relevant subset of the Hubbard algebra [3] found from the fundamental definition $X_i^{ab} = |a\rangle\langle b|$.

The representation (84) does not at general λ reproduce the "half bracket", or product relations expected for *projection operators*. We find that

$$X_i^{\sigma 0}(\lambda)X_i^{0\sigma'}(\lambda) \neq X_i^{\sigma\sigma'}(\lambda), \tag{88}$$

$$X_i^{0\sigma'}(\lambda)X_i^{\sigma 0}(\lambda) \neq X_i^{00}\delta_{\sigma\sigma'}. \tag{89}$$

The exceptions are at $\lambda = 0$, where it is trivially true, and non trivially at $\lambda = 1$, where Gutzwiller projection of the allowed states does restore this property when right-operating on the projected states. In the Green's functions below, we will equate the *averages* of both sides of Eq. (88). This equality of the averages acts as the number constraint and fixes the chemical potential $\boldsymbol{\mu}$. In doing so, the average of Eq. (89) is not constrained and takes on a suitable value determined by the anticommutation relation (85).

This representation can be used to define a many-body problem where the λ dependent EOMs for the Green's functions constructed from (84) can be written down. Observe that the EOMs for the Green's functions only require the use of (85) and the Heisenberg equations of motion, and in turn these arise from the basic Lie commutators (anticommutators) of the type given in (86) and (87). The calculation does not ever require the use of product relations of the type (88). It then follows that we can replace the t–J Hamiltonian and the operators in the original theory by their λ-versions, i.e. replacing $X_i^{ab} \to X_i^{ab}(\lambda)$, and thereby obtain equations that yield (16). This procedure then provides a (continuous) interpolation between the free Fermi and extremely correlated regimes by varying λ from 0 to 1. Let us first demonstrate this by a brief calculation.

5.1. The λ-Fermion theory equations of motion

Using the λ Fermions, we define the Green's function as

$$\mathcal{G}^{(\lambda)}_{\sigma_i\sigma_f}(i\tau_i, f\tau_f) = -\langle T_\tau X_i^{0\sigma_i}(\tau_i, \lambda)X_f^{\sigma_f 0}(\tau_f, \lambda)\rangle_{(\lambda)} \tag{90}$$

where with arbitrary $\hat{A}$

$$\langle\hat{A}\rangle_\lambda \equiv -\frac{\mathrm{Tr}\, e^{-\beta H_{eff}(\lambda)}T_\tau\left(e^{-\hat{\mathcal{A}}_S(\lambda)}\hat{A}\right)}{Z(\lambda)},$$
$$Z(\lambda) = \mathrm{Tr}\, e^{-\beta H_{eff}(\lambda)}T_\tau\left(e^{-\hat{\mathcal{A}}_S(\lambda)}\right). \tag{91}$$

In this expression $H_{eff}(\lambda)$ is given by Eq. (92) and $\hat{\mathcal{A}}_S(\lambda)$ is obtained from (10), with the replacement $X_i^{ab} \to X_i^{ab}(\lambda)$:

$$H_{eff}(\lambda) = -\sum_{ij} t_{ij}X_i^{\sigma 0}(\lambda)X_j^{0\sigma}(\lambda) - \boldsymbol{\mu}\sum_i N_{i\sigma} + \lambda\frac{1}{2}\sum_{ij}J_{ij}\left(\vec{S}_i.\vec{S}_j - \frac{1}{4}N_{i\sigma}N_{j\sigma'}\right)$$
$$+ u_0\,\lambda\sum_i N_{i\uparrow}N_{i\downarrow}, \tag{92}$$

where u_0 is now the "second chemical potential". The scaling of the J term with λ is optional, and done here so that we obtain the Fermi gas at $\lambda = 0$. Using Eq. (84), we see that this Hamiltonian is linear in λ and interpolates between the free Fermi gas and the fully interacting model, when acting on suitably projected states. The equation of motion of $\mathcal{G}^{(\lambda)}$ can be obtained using the commutation relations (85)–(87), the calculation is parallel to that in Appendix A. In brief, Eqs. (132) and (133) are unchanged by working with $X(\lambda)$'s, and in place of Eq. (137) we obtain

$$\begin{aligned}\mathbf{g}^{-1}_{0,\sigma_i,\sigma_j}(i\tau_i, j\tau_j)\mathcal{G}^{(\lambda)}_{\sigma_j\sigma_f}(j\tau_j, f\tau_f) \;=\; & \delta(\tau_i-\tau_f)\delta_{ij}(1-\lambda\,\gamma_{\sigma_i\sigma_f}(i\tau_i)) \\ & -\lambda\sum_{j\sigma_j} t_{ij}(\sigma_i\sigma_j)\,\left\langle T_\tau\left(X_i^{\bar\sigma_i\bar\sigma_j}(\tau_i)X_j^{0\sigma_j}(\tau_i)\,X_f^{\sigma_f 0}(\tau_f)\right)\right\rangle_{(\lambda)} \\ & +\frac{1}{2}\sum_{j\sigma_j} J_{ij}(\sigma_i\sigma_j)\left\langle T_\tau\left(X_j^{\bar\sigma_i\bar\sigma_j}(\tau_i)X_i^{0\sigma_j}(\tau_i)X_f^{\sigma_f 0}(\tau_f)\right)\right\rangle_{(\lambda)} \\ & -\frac{1}{2}\lambda u_0\sum_{\sigma_j}(\sigma_i\sigma_j)\left\langle T_\tau\left(X_i^{\bar\sigma_i\bar\sigma_j}(\tau_i)X_i^{0\sigma_j}(\tau_i)X_f^{\sigma_f 0}(\tau_f)\right)\right\rangle_{(\lambda)}, \end{aligned} \tag{93}$$

where the λ dependence of the X operators is implicit. The higher order Green's functions may be expressed as functional derivatives with respect to the Bosonic source $\mathcal{V}$, in the same fashion as in Appendix A. The exchange term J_{ij} does not pick up a factor of λ through the EOM since it conserves double occupancy. We can choose to additionally scale it with λ as $J_{ij} \to \lambda J_{ij}$, so that at $\lambda = 0$ we obtain the Fermi gas. This choice seems reasonable in the liquid phase of the electrons, in other phases it is easy enough to recover from this scaling if needed. To save writing the u_0 term is absorbed as $J_{ij} \to J_{ij} - u_0\delta_{ij}$, with this the resulting equation is

$$\begin{aligned}&\left(\mathbf{g}^{-1}_{0,\sigma_i,\sigma_j}(i\tau_i, j\tau_j) - \lambda\,\hat{X}_{\sigma_i\sigma_j}(i\tau_i, j\tau_j) - \lambda\,Y_{1\sigma_i\sigma_j}(i\tau_i, j\tau_j)\right)\times\mathcal{G}^{(\lambda)}_{\sigma_j\sigma_f}(j\tau_j, f\tau_f) \\ &\quad = \delta_{if}\delta(\tau_i-\tau_f)\left(\delta_{\sigma_i\sigma_f} - \lambda\,\gamma_{\sigma_i\sigma_f}(i\tau_i)\right).\end{aligned} \tag{94}$$

The constitutive relation determining the chemical potential is taken as

$$\begin{aligned} n_{i\sigma} \;&=\; \langle X_i^{\sigma 0}(\tau,\lambda)X_i^{0\sigma}(\tau^-,\lambda)\rangle_{(\lambda)}, \\ &=\; \mathcal{G}^{(\lambda)}_{\sigma\sigma}(i,\;\tau^-,\tau), \end{aligned} \tag{95}$$

rather than $n_{i\sigma} = \langle X_i^{\sigma 0}(\tau,\lambda)X_i^{0\sigma}(\tau,\lambda)\rangle_{(\lambda)}$ [41]. This limiting process corresponds to enforcing the half bracket relation Eq. (88) *as an average*. (95) is exact for the fully projected operators where $\lambda = 1$, while for other values of λ it is guided by the requirement of continuity in λ. In the same spirit, we express the function γ in Eq. (16) as

$$\gamma_{\sigma\sigma'}(i\tau) = \sigma\sigma'\mathcal{G}_{\bar\sigma'\bar\sigma}(i\tau^-, i\tau), \tag{96}$$

while the direct computation using Eq. (85) would yield identical times, rather than the split times in Eq. (96). An iteration scheme for solving these equations using ideas of the skeleton expansion is detailed in Refs. [6,7], and hence we skip the details.

A very simple example can be given to illustrate the role of λ and u_0, where the skeleton expansion is avoided. Let us consider the atomic limit of the λ-Fermions theory. We consider the Hamiltonian $H_0 = -\boldsymbol{\mu}\sum_\sigma N_\sigma + \lambda u_0 N_\uparrow N_\downarrow$ with $u_0 \geq 0$. The Green's function in Eq. (90) can be calculated easily using the EOM technique as:

$$\mathcal{G}(i\omega_n) = \frac{1-n_{\bar\sigma}}{i\omega_n + \boldsymbol{\mu}} + \frac{(1-\lambda)n_{\bar\sigma}}{i\omega_n + \boldsymbol{\mu} - \lambda u_0}. \tag{97}$$

At $\lambda = 0$ or 1, this yields the exact atomic limit result, and provides a smooth interpolation between these limits. The positive energy pole at $\lambda u_0 - \boldsymbol{\mu}$ does not contribute to the occupancy for a sufficiently large u_0 and low T. In the more realistic case with non zero hopping discussed in Refs. [6,7], the energy u_0 is non-trivially fixed by a second sum rule (20), and the iteration procedure is more

complex, involving the skeleton expansion. While the atomic limit example is quite explicit, it does not generalize in any simple way to the case of finite hopping, and therefore is somewhat trivial.

We next remark on some consequences of the λ expansion in the intermediate region $\lambda < 1$, that follow from general principles. Let us first summarize the high frequency limit of the Green's functions. When $i\omega_n \to \infty$, the local Green's function falls off as $\mathcal{G}(i\omega_n) \to a_G/i\omega_n$. Here the constant $a_G = \langle\{\hat{C}, \hat{C}^\dagger\}\rangle$, with $\hat{C}$, $\hat{C}^\dagger$ the two appropriate operators involved in $\mathcal{G}$, it is a measure of the total fraction of states. In the Hubbard model $a_G = 1$, since we have canonical operators, and implicitly $|\omega_n| \gg U$ as well. However for the t–J model we obtain $a_G = (1-n/2)$, with a net deficit of $n/2$ states from the Hubbard model. This deficit is accounted for by the upper Hubbard band that is ignored in the t–J model. The lower Hubbard band thus contains a fraction $1 - n/2$ of all the states, of which we account for $n/2$ as the occupied states (with two spin projections available), and $1 - n$ as the unoccupied part of the lower Hubbard band. These $1 - n$ states are available for charge excitations in the t–J model, and freeze out towards the insulating limit. Summarizing, in this picture we have $n/2$ occupied and $1 - n$ unoccupied states in the lower Hubbard band, and $n/2$ states at high energy of $O(U)$.

In the λ expansion, from Eq. (16) we have $a_G = 1 - \lambda\gamma$, where γ is further expanded in λ. On enforcing the number sum rule (95) we find that the effective number of states described by this theory can be decomposed into $n/2$ occupied states and $(1 - n) + (n/2 - \lambda\gamma)$ unoccupied states. These are to be taken as the low energy sector of a fiduciary Hamiltonian. The fraction $(n/2 - \lambda\gamma)$ vanishes only when $\lambda = 1$ and is otherwise an unspecified surplus of states in the low energy sector. An unbalanced state count of this type is to be expected when we have non-unitary evolution. Indeed in the second order λ expansion carried out numerically, a similar excess of states is found [7, Section (2), last paragraph]. Another related consequence is that the spectral function positivity, requiring unitary evolution, can no longer be guaranteed in finite orders of the λ expansion. This feature is well recognized in Ref. [4], where it is noted that the occupied states with $\omega < 0$ are essentially unaffected by this problem.

6. Analogy with the Dyson–Maleev representation of spin operators

The non-Hermitian representation in Eq. (71) of the Gutzwiller projected electron operators, when used with the averaging in Eq. (69), was shown in Section 3.2 to provide an exact mapping of the t–J model. Reflecting on this result, the author realized recently that the mapping Eq. (71) is the Fermionic analog of the Dyson–Maleev representation for spin operators [16,17], used to understand spin wave interactions in magnets (see Table 3).

With the advantage of hindsight, this connection seems natural. The Gutzwiller projected electronic X^{ab} operators defined by Hubbard [3], generate a non canonical algebra of Fermions that is (partly) given in Eqs. (85)–(87) with $\lambda = 1$. On the other hand the spin operators provide the best studied non canonical Bosonic algebras. The spins are not quite Bosons, they are equivalent to "hard core" Bosons—with infinite on site repulsion, in parallel to the infinite U in the extremely correlated electron problem. In order to avoid dealing with the infinite energy of the hard core, several other representations of spins were invented, such as the Holstein Primakoff method [44]. Dyson's use of a non-Hermitian representation provides the most compact canonical description of the spin operators. In fact it is analogous to the non-Hermitian mapping of the Fermionic Gutzwiller problem in Eq. (71).

Dyson's representation, later streamlined by Maleev [17], may be written with $n_i = b_i^\dagger b_i$ as

$$\begin{aligned} S_i^+ &= (2s)\, b_i^\dagger \left(1 - \frac{n_i}{2s}\right) \\ S_i^- &= b_i \\ S_i^z + s &= n_i, \end{aligned} \tag{98}$$

where $\vec{S}_i.\vec{S}_i = s(s+1)$ and b_i, $b_i^\dagger$ are canonical Bose operators. The Boson vacuum state $b_i|vac\rangle = 0$ is mapped as $|vac\rangle \leftrightarrow |\downarrow, \downarrow, \downarrow \ldots \downarrow\rangle$, so that the action of $b_i^\dagger$ creates spin reversals. Their number is cut off such that $n_i \le (2s)$, thereby defining the physical states. Under these conditions Eq. (98) is

Table 3
A comparison of the Dyson–Maleev representation for spins and the non-Hermitian representation (71) for two component Fermions $\sigma = \pm 1$ with $\bar{\sigma} = -\sigma$. At $\lambda = 1$ the Fermion mappings provide a faithful representation of Gutzwiller projected Fermi operators X_i^{ab}, *acting to the right* on states with single occupancy, since their action produces states that remain in this space. The representation is non self adjoint, i.e. its left operation on Dirac bra states is not faithful. The situation has an exact parallel in the Dyson–Maleev representation. The Dyson projection operator $\hat{P}_D$ for integer $2s$ and the Gutzwiller projection operator $\hat{P}_G$ at $\lambda = 1$, play a similar role in filtering out unphysical states. The role of the parameter λ away from 0, 1 is similar to that of $\frac{1}{2s}$, extending the Dyson–Maleev representation to spin values that are neither integer or half integer. The last three rows show the auxiliary Green's function, the caparison function and the second self energy in terms of the Bosons from Eqs. (100) and (101). These follow from the work of Harris, Kumar, Halperin and Hohenberg [18] adapted to the ferromagnet. The corresponding Fermionic objects are discussed in Section 4 and detailed in Eqs. (82) and (83).

	Spins: The Dyson–Maleev mapping		Fermions: The non-Hermitian mapping	
Destruction operator	S_i^-	b_i	$X_i^{0\sigma}$	$C_{i\sigma}$
Creation operator	S_i^+	$(2s)\, b_i^\dagger (1 - \frac{n_i}{2s})$	$X_i^{\sigma 0}$	$C_{i\sigma}^\dagger (1 - \lambda N_{i\bar{\sigma}})$
Density operator(s)	$S_i^z + s$	$n_i = b_i^\dagger b_i$	$X_i^{\sigma\sigma'}$	$C_{i\sigma}^\dagger C_{i\sigma'}$
Projection operator	$\hat{P}_D$	$\prod_i \{\sum_{m=0}^{2s} \delta_{n_i,m}\}$	$\hat{P}_G$	$\prod_i (1 - N_{i\uparrow} N_{i\downarrow})$, for $\lambda = 1$
Vacuum	$\vert \downarrow\downarrow \ldots \downarrow\rangle$	$\vert 00 \ldots 0\rangle$	$\vert Vac\rangle$	$\vert 00 \ldots 0\rangle$
Small parameter & its range	$\frac{1}{2s}$	$\frac{1}{2s} \in [0, 1]$	λ	$\lambda \in [0, 1]$
Auxiliary Green's function		$\mathbf{g}(i, j) = -\langle\langle b_i b_j^\dagger \rangle\rangle$		$\mathbf{g}(i, j) = -\langle\langle C_{i\sigma} C_{j\sigma}^\dagger \rangle\rangle$
Caparison function		$\mu(i, j) = \delta_{ij}(1 - \frac{1}{2s}\langle n_j\rangle) + \frac{1}{2s}\Psi(i, j)$		$\mu(i, j) = \delta_{ij}(1-\lambda\gamma)+\lambda\Psi(i, j)$
Second Self energy Ψ		$\Psi(i, j) = \mathbf{g}^{-1}(i, \mathbf{a})\langle\langle b_{\mathbf{a}} b_j^\dagger n_j\rangle\rangle_c$		$\Psi(i, j) = \mathbf{g}^{-1}(i, \mathbf{a})\langle\langle C_{\mathbf{a}\sigma} C_{j\sigma}^\dagger N_{j\bar{\sigma}}\rangle\rangle_c$

shown to provide a faithful representation of the angular momentum operators, when right-operating on physical states. Under the action of the operators in (98), the physical states form an invariant subspace of the extended Bose Hilbert space, and are selected by projection. The Dyson projection operator $\hat{P}_D$ acts on the Bose state space and leaves the physical states unchanged while annihilating states with $n_i > (2s)$.

It is now evident that the Dyson–Maleev representation has a strong formal similarity to the minimal representation (71). The Dyson projector $\hat{P}_D$ plays a role parallel to that of the Gutzwiller projector $\hat{P}_G$ in (71) in our theory. The parallel further deepens in the path integral representation of the Fermions that we discuss below. The interesting work of Douglass [42], following Langer's [43] path integral program for Bosons, employs the projection operator $\hat{P}_D$ in the same spirit to our usage below.

The work of Harris, Kumar, Halperin and Hohenberg (HKHH) [18] extended Dyson's method to two sublattice antiferromagnets, and provided a non trivial generalization to study the lifetime of the excitations. Details of the ECFL formalism turn out to have points of overlap with those in HKHH that are worth noting. In particular HKHH decompose the physical Green's function into a space time convolution of two parts. These parts are precisely the Bosonic analogs of the ECFL breakup of the physical Green's function, into an auxiliary Green's function $\mathbf{g}(k)$ and a caparison function $\mu(k)$, as detailed in Ref. [4] and in Section 4.

The computation of the Green's function by HKHH [18] was carried out for the two sublattice antiferromagnet. In order to avoid dealing with the added complexity of the two sublattice problem, we translate their method to the Dyson problem of the dynamical Green's function of the ferromagnet. We use a notation that brings out the close parallel with the product *ansatz* used in ECFL [4].

The calculation, paraphrasing that of HKHH, proceeds as follows. In order to compute the imaginary time Green's function $\mathcal{G}(i, j) = -\langle\langle S_i^- S_j^+ \rangle\rangle$ with the shorthand spacetime notation $i \equiv (r_i, \tau_i)$, the repeated index summation (integration) convention and denoting the averages as $\langle\langle Q \rangle\rangle = \mathrm{Tr}(e^{-\beta H}$

$T_\tau Q P_D)/\mathrm{Tr} e^{-\beta H} P_D$, we write from (98)

$$\frac{1}{2s}\mathcal{G}(i,j) = -\left\langle\!\!\left\langle b_i b_j^\dagger \left(1 - \frac{1}{2s} n_j\right)\right\rangle\!\!\right\rangle. \tag{99}$$

Separating out the disconnected part we write $\langle\langle b_i b_j^\dagger n_j\rangle\rangle = \langle\langle b_i b_j^\dagger\rangle\rangle\langle n_j\rangle + \langle\langle b_i b_j^\dagger n_j\rangle\rangle_c$, and defining the auxiliary Green's function $\mathbf{g}(i,j) = -\langle\langle b_i b_j^\dagger\rangle\rangle$ as well as its inverse through $\mathbf{g}(i,\mathbf{k})\mathbf{g}^{-1}(\mathbf{k},j) = \delta(i,j)$, we arrive at

$$\frac{1}{2s}\mathcal{G}(i,j) = \mathbf{g}(i,j)\left(1 - \frac{1}{2s}\langle n_j\rangle\right) + \frac{1}{2s}\mathbf{g}(i,\mathbf{k})\,\Psi(\mathbf{k},j), \tag{100}$$

$$\Psi(i,j) = \mathbf{g}^{-1}(i,\mathbf{a})\langle\langle b_{\mathbf{a}} b_j^\dagger n_j\rangle\rangle_c. \tag{101}$$

We use a notation with sums over repeated bold indices everywhere. We can rewrite (100) as a convolution of the auxiliary Green's function $\mathbf{g}$ and a caparison function μ, in the form $\frac{1}{2s}\mathcal{G}(i,j) = \mathbf{g}(i,\mathbf{k})\mu(\mathbf{k},j)$, where $\mu(i,j) = \delta_{ij}(1 - \frac{1}{2s}\langle n_j\rangle) + \frac{1}{2s}\Psi(i,j)$. The auxiliary Green's function is defined in terms of its own self energy Φ through the usual Dyson equation $\mathbf{g}^{-1}(i,j) = \mathbf{g}_0^{-1}(i,j) - \Phi(i,j)$. Thus the physical Green's function $\mathcal{G}$ is determined in terms of the two self energies $\Phi(k,\omega)$ and $\Psi(k,\omega)$. Written in $(k,\ i\omega)$ space, this is identical to the functional form in ECFL theory Eq. (21)!

The corresponding Fermionic objects are discussed in Section 4 and detailed in Eqs. (82) and (83). On comparing the two we recognize that the structure of Eqs. (100) and (101) is the exact parallel of the ECFL theory for the Green's function written in the notation of Ref. [4]. In the HKHH paper, the objects evaluated amount to these two ECFL self energies, by the correspondence $\Psi(k,\omega) \leftrightarrow \Lambda(k,\omega)$ (see [18, Eq. (C10)]), and $\Phi(k,\omega) \leftrightarrow \Sigma(k,\omega)$ (see [18, Eq. (2.22)]). It is worth noting further that the role of the parameter λ in the ECFL theory is in close parallel to that of $\frac{1}{2s}$ in the magnon problem. Expansions in these two "small parameters" serve to organize the calculations.

The product ansatz in ECFL [4,6] was originally arrived at in Ref. [4] by analyzing the Schwinger equations and insisting on a canonical Green's function to be factored out from the physical $\mathcal{G}$. The calculation of HKHH, on the other hand, was through a different route using insights from the Feynman diagrams applied to the four Boson operators in (99). It is satisfying that the two independent calculations, one for Gutzwiller projected Fermions and the other for hard-core Bosons, lead to such a close parallel, expressed most naturally in the twin self energy representation (100) and (101).

A few additional comments on the role of the projection operator in the two problems are relevant here. Dyson demonstrated in his non-Hermitian representation that magnon interactions at low temperatures lead to T^4 type corrections to the magnetization of the ideal spin wave theory. He argued that the projection operator $\hat{P}_D$ is largely irrelevant in the ferromagnet, and provided an estimate of corrections to the low T behavior arising from this neglect. For the antiferromagnet, HKHH similarly argued that the projector is unimportant at low T, and gave an estimate of the expected corrections. The corrections are larger than in the ferromagnet, and yet smaller than most quantities of interest at low T. The density of excitations is small at low T in the magnetic problem, and thus provides a basis for ignoring the projection operator. However in the Fermion problem studied here, the particle density is never too small in the interesting regime, and hence the projection operator must be respected. Interestingly enough, the projector does not explicitly appear in the Schwinger EOM (16), but it does determine the choice of the correct constitutive relation (14). Thus the projection operator plays a significant role in enforcing the Luttinger–Ward theorem [45] for the volume of the Fermi surface.

Another major difference between the Fermionic and the spin problems is the role of the second Lagrange multiplier u_0, when the parameter $\lambda < 1$. In the Fermi problem, it is essential to change the Hamiltonian by adding the term $\lambda u_o \sum_i N_{i\uparrow} N_{i\downarrow}$, in addition to replacing the projected X_i^{ab} by $X_i^{ab}(\lambda)$. This is required in order to satisfy the shift identities, and as explained in Ref. [6], the parameter u_0 is fixed by a number sum rule on the auxiliary Green's function. The problem of magnetic excitations does not have a counterpart to this term. However, we can imagine extending the Dyson–Maleev and HKHH formalism to an extremely correlated Bose liquid with a fixed number of Bosons, e.g. ^{4}He on a suitable substrate giving rise to a lattice model with hard core repulsion. In such a case, a corresponding theory parallel to ECFL can be developed, requiring both the shift identities and a second Lagrange multiplier u_0 disfavoring multiple occupancy to satisfy these.

7. Path integrals

7.1. Canonical path integral representation

We now introduce path integrals to represent the partition functional (68), wherein the operators are replaced by anticommuting c-numbers, i.e. the Grassmann variables. We will keep the discussions to a minimum since excellent references can be consulted for details [46–49]. We map the operators as $C_{i\sigma} \to c_{i\sigma}$, $\widetilde{C}_{i\sigma} \to \widetilde{c}_{i\sigma} \equiv c_{i\sigma}(1 - c^*_{i\bar{\sigma}} c_{i\bar{\sigma}})$, $C^\dagger_{i\sigma} \to c^*_{i\sigma}$, $\widetilde{C}^\dagger_{i\sigma} \to \widetilde{c}^*_{i\sigma} \equiv c^*_{i\sigma}(1 - c^*_{i\bar{\sigma}} c_{i\bar{\sigma}})$. The time dependence is dealt with using a standard Trotter decomposition of the non commuting pieces [49]. Handling the Gutzwiller projector is discussed below and in Appendix C. It is understood that when the Trotter index M is finite, we have a discretized time representation, so that when $M \to \infty$, we obtain the continuous time path integrals. We work initially with the discrete time version since somewhat subtle identities such as the Pauli principle and the Gutzwiller projection identities can be verified explicitly. We now write the partition functional Z (68), in terms of Grassmann variables at discrete times $c_{i\sigma}(\tau_j)$ and $c^*_{i\sigma}(\tau_j)$, and a global integration over all Grassmann variables with the conventional definition [49]:

$$Z^{(M)}[J^*, J, \mathcal{V}] = \int_c P_G(\tau_1, \tau_0)\, e^{-\mathcal{A}^{(M)}_{Tot}},$$
$$\mathcal{A}_{Tot} = \mathcal{A}^{(M)}_0 + \mathcal{A}^{(M)}_S + \mathcal{A}^{(M)}_t + \mathcal{A}^{(M)}_J. \tag{102}$$

We detail the various contributions next; the free Fermi term is given by

$$\frac{1}{\Delta\tau}\mathcal{A}^{(M)}_0 = \sum_{j=0}^{M-1} \left[c^*_{i\sigma}(\tau_{j+1})\delta_{\tau_j} c_{i\sigma}(\tau_j) - \boldsymbol{\mu}\; n_{j\sigma}(\tau_j)\right], \tag{103}$$

with the finite difference operator δ_{τ_j} defined through

$$\delta_{\tau_j} F(\tau_j) \equiv \frac{1}{\Delta\tau}\left\{F(\tau_{j+1}) - F(\tau_j)\right\}. \tag{104}$$

As $M \to \infty$, we note that δ_{τ_j} reduces to the derivative operator ∂_τ, and we obtain the integral $\mathcal{A}_0 = \int_0^\beta d\tau\; c^*_{i\sigma}(\tau)(\partial_\tau - \boldsymbol{\mu}) c_{i\sigma}(\tau)$, and in that limit $Z^{(M)} \to Z[J^*, J, \mathcal{V}]$. The source term $\mathcal{A}^{(M)}_S(\tau_{j+1}, \tau_j)$ obtained from (72) is given by

$$\mathcal{A}^{(M)}_S = \sum_i \left[\widetilde{c}^*_{i\sigma}(\tau_{j+1}) J_{i\sigma}(\tau_{j+1}) + J^*_{i\sigma}(\tau_{j+1}) c_{i\sigma}(\tau_j)\right] + \left[\mathcal{V}_i^{\sigma'\sigma}(\tau_{j+1}) c^*_{i\sigma'}(\tau_{j+1}) c_{i\sigma}(\tau_j)\right]. \tag{105}$$

As in (72), the projected variable with a hat appears in the creation operator and nowhere else in this expression. The Hamiltonian (59) gives rise to two parts of the action. The hopping term is given by

$$\mathcal{A}^{(M)}_t = \Delta\tau \sum_j T_{eff}(\tau_j) \to \int_0^\beta d\tau\; T_{eff}(\tau), \tag{106}$$

with T_{eff} from Eq. (60) or Eq. (74):

$$T_{eff}^{Sym}(\tau_j) = -\sum_{lm\sigma} t_{lm}\, c^*_{l\sigma}(\tau_{j+1}) c_{m\sigma}(\tau_j) \times \left(1 - n_{l\bar{\sigma}}(\tau_j) - n_{m\bar{\sigma}}(\tau_j)\right),$$

$$T_{eff}^{Min}(\tau_j) = -\sum_{lm\sigma} t_{lm}\, c^*_{l\sigma}(\tau_{j+1}) c_{m\sigma}(\tau_j) \times \left(1 - n_{l\bar{\sigma}}(\tau_j)\right), \tag{107}$$

where (107) corresponds to the symmetrized theory of (64) and (107) to the minimal version of (71). The exchange part of the action is given by

$$\mathcal{A}_J^{(M)} = \Delta\tau \sum_j H_J(\tau_j) \to \int_0^\beta d\tau\, H_J(\tau),$$

$$H_J(\tau_j) \equiv -\frac{1}{4}\sum_{lm} J_{lm}\, \sigma_1\sigma_2 \times c^*_{l\sigma_1}(\tau_{j+1}) c^*_{m\bar{\sigma}_1}(\tau_{j+1}) c_{m\bar{\sigma}_2}(\tau_j) c_{l\sigma_2}(\tau_j). \tag{108}$$

Where possible we simplify the notation by dropping the superscript M; most expressions provide sufficient context for this and there should be no confusion. Thus we will write $\mathcal{G}^{(M)}_{\sigma\sigma'}(a\tau_i, b\tau_f) \to \mathcal{G}_{\sigma\sigma'}(a\tau_i, b\tau_f)$ and $Z^{(M)} \to Z$ etc. below. When no confusion is likely we will refer to $Z[J^*, J, \mathcal{V}]$ as simply Z, and also abbreviate terms such as $\mathcal{H}_{eff}(\tau_{j+1}, \tau_j)$ to $\mathcal{H}_{eff}(\tau_j)$ or even more simply to $\mathcal{H}_{eff}$. Eq. (102) is almost in the form of a canonical partition function for unprojected electrons, but with an important difference. The extra term in the integration measure is the Gutzwiller projector written in Grassmann variables. These variables arise *at the initial and next time instant only* and the rest of the time variables have only the standard measure of unity. Explicitly we find

$$P_G(\tau_1, \tau_0) \equiv \prod_{i=1}^{N_s} \left(1 - c^*_{i\uparrow}(\tau_1) c_{i\uparrow}(\tau_0) c^*_{i\downarrow}(\tau_1) c_{i\downarrow}(\tau_0)\right), \tag{109}$$

it has all creation (destruction) operators at $j = 1$ ($j = 0$), and N_s is the number of sites. In Appendix D, we summarize the Pauli principle and Gutzwiller identities obeyed by the present coherent state representation, these represent an important aspect of the strong correlation problem. We will also recycle the notation of (63) for the average in this distribution of any function Q of the Grassmann variables:

$$\langle\langle Q \rangle\rangle = \frac{\|Q\|}{Z}, \quad \text{with } \|Q\| = \int_c P_G(\tau_1, \tau_0) e^{-\mathcal{A}}\, Q, \tag{110}$$

a useful abbreviation (110), and drop the superscript (M). This representation of the path integral with a constraining projection factor at only the initial time has a resemblance to the that in the canonical quantization of the electromagnetic field in the temporal gauge [37,38], as already noted in the introduction. The Green's functions follow from Eq. (73) using $\delta/\delta J(\tau_j) \to \frac{1}{(\Delta\tau)} d/dJ(\tau_j)$ [50]:

$$\mathcal{G}_{\sigma_i\sigma_f}(i\tau_i, f\tau_f) = \frac{1}{Z} \|\widetilde{c}^*_{f\sigma_f}(\tau_f)\, c_{i\sigma}(\tau_i)\|. \tag{111}$$

7.2. Equations of motion from path integral representation

In this section we obtain the Schwinger equations of motion of ECFL (see Ref. [6] and especially Appendix A Eq. (138)), directly from the path integral representation given above thus providing a non trivial check on the representation. To obtain Eq. (138), we initially set the Fermionic sources to zero, the Bosonic sources are turned off at the very end. The equations of motion are most easily found using a Grassmann integration identity:

$$\int_c P_G(\tau_1, \tau_0) \frac{\delta}{\delta c^*_{i\sigma_i}(\tau_{i+1})} \left[\widetilde{c}^*_{f\sigma_f}(\tau_{f+1})\, e^{-\mathcal{A}_{Tot}}\right] = 0. \tag{112}$$

This identity is a straight forward generalization of the theorem on vanishing of a total derivative [46], including a non trivial measure P_G (109) where the time arguments are greater than all time

arguments in (109), i.e. $i, f \geq 1$. It is proved by the usual logic for Grassmann variables; the derivative $\frac{\delta}{\delta c^*_{i\sigma_i}(\tau_{i+1})}$ is in addition to an integration over $c^*_{i\sigma_i}(\tau_{i+1})$ contained in the overall integration. We next recall that the highest possible degree of a polynomial in any Grassmann variable is unity. The above expression vanishes upon further noting that Grassmann integration and Grassmann differentiation are identical. The same identity is valid if we replace $\widetilde{c}^*_{f\sigma_f}(\tau_{f+1})$ by any other allowed Grassmann variable U, subject to the double occupancy restriction, and similarly with V (see Ref. [51]). In summary, an abstract equation of motion, following from $\int P_G \frac{\delta}{\delta V}(Ue^{-A_{Tot}}) = 0$ and Fermionic U, V reads

$$\left\| \frac{\delta U}{\delta V} \right\| + \left\| U \frac{\delta \mathcal{A}_S}{\delta V} \right\| + \left\| U \frac{\delta \mathcal{A}_0}{\delta V} \right\| + \left\| U \frac{\delta \mathcal{A}_t}{\delta V} \right\| + \left\| U \frac{\delta \mathcal{A}_J}{\delta V} \right\| = 0. \tag{113}$$

7.3. Equation for $\mathcal{G}_{\sigma_i\sigma_f}(i,f)$

Our first task is to find an equation for the Green's function [50]—we use (113) with $U = \widetilde{c}^*_{f\sigma_f}(\tau_f)$ and $V = c^*_{i\sigma_i}(\tau_i)$. We compute the various pieces of (113) next.

Denoting

$$\widehat{\gamma}_{\sigma_i\sigma_f}(i) \equiv \sigma_i\sigma_f c^*_{i\bar{\sigma}_i}(\tau_{i+1}) c_{\,i\bar{\sigma}_f}(\tau_i), \tag{114}$$

and using the convention that repeated spin indices are summed over, we obtain the first result:

$$\frac{\delta}{\delta c^*_{i\sigma_i}(\tau_{i+1})} \widetilde{c}^*_{f\sigma_f}(\tau_{f+1}) = \delta_{\tau_i\tau_f}\delta_{if} \left\{ \delta_{\sigma_i\sigma_f} - \widehat{\gamma}_{\sigma_i\sigma_f}(i) \right\}. \tag{115}$$

We obtain

$$\frac{1}{\Delta\tau} \frac{\delta \mathcal{A}_0}{\delta c^*_{i\sigma_i}(\tau_{i+1})} = \delta_{\tau_i} c_{i\sigma_i}(\tau_i) - \boldsymbol{\mu}\, c_{i\sigma_i}(\tau_i), \tag{116}$$

$$\frac{1}{\Delta\tau} \frac{\delta \mathcal{A}_S}{\delta c^*_{i\sigma_i}(\tau_{i+1})} = \mathcal{V}_i^{\sigma_i\sigma_j}(\tau_{i+1})\, c_{i\sigma_j}(\tau_i) + \left\{ \delta_{\sigma_i\sigma_j} - \widehat{\gamma}_{\sigma_i\sigma_j}(i) \right\} J_{i\sigma_j}(\tau_{i+1}), \tag{117}$$

$$\frac{1}{\Delta\tau} \frac{\delta \mathcal{A}_t^{Sym}}{\delta c^*_{i\sigma_i}(\tau_{i+1})} = -t_{ij} c_{j\sigma_i}(\tau_i) + t_{ij} \left[\hat{\gamma}_{\sigma_i\sigma_j}(i\tau_i)\, c_{j\sigma_j}(\tau_i) + c^*_{j\bar{\sigma}_i} c_{j\bar{\sigma}_i} c_{j\sigma_i} + c^*_{j\bar{\sigma}_i} c_{i\bar{\sigma}_i} c_{i\sigma_i} \right],$$

$$\frac{1}{\Delta\tau} \frac{\delta \mathcal{A}_t^{Min}}{\delta c^*_{i\sigma_i}(\tau_{i+1})} = -t_{ij} c_{j\sigma_i}(\tau_i) + t_{ij} \hat{\gamma}_{\sigma_i\sigma_j}(i\tau_i)\, c_{j\sigma_j}(\tau_i)$$

$$\frac{1}{\Delta\tau} \frac{\delta \mathcal{A}_J}{\delta c^*_{i\sigma_i}(\tau_{i+1})} = -\frac{1}{2} J_{ij}\sigma_i\sigma_j\, c^*_{j\bar{\sigma}_i}(\tau_{i+1}) c_{j\bar{\sigma}_j}(\tau_i) c_{i\sigma_j}(\tau_i), \tag{118}$$

We combine the two terms as:

$$\frac{1}{\Delta\tau} \frac{\delta(\mathcal{A}_t + \mathcal{A}_J)}{\delta c^*_{i\sigma_i}(\tau_{i+1})} = -\sum_j t_{ij} c_{j\sigma_i}(\tau_i) + A_{i\sigma_i}(\tau_{i+1}, \tau_i), \tag{119}$$

the first (linear) term in Fermions is separated out in this expression, and $A_{i\sigma_i}$, detailed below in Eq. (120), is obtained by combining all the *three Fermion* contributions in Eqs. (117) and (118). In the minimal case we get from Eqs. (107), (108), (114) and (115)

$$A^{Min}_{i\sigma_i} = t_{ij} \widehat{\gamma}_{\sigma_i\sigma_j}(i\tau_i)\, c_{j\sigma_j}(\tau_i) - \frac{1}{2} J_{ij} \widehat{\gamma}_{\sigma_i\sigma_j}(j\tau_i) c_{i\sigma_j}(\tau_i), \tag{120}$$

in agreement with Eq. (22) of Ref. [6], and the symmetrized case is obtained in a similar way. Combining these (with $J \to 0$) we get the EOM in discrete time space:

$$\left[\left\{\boldsymbol{\mu} - \delta_{\tau_i} - \mathcal{V}_i^{\sigma_i\sigma_j}(\tau_{i+1})\right\}\delta_{i,j} + t_{ij}\right] \|\widetilde{c}^*_{f\sigma_f}(\tau_{f+1})c_{j\sigma_i}(\tau_i)\| - \|\widetilde{c}^*_{f\sigma_f}(\tau_{f+1})A_{i\sigma_i}(\tau_{i+1},\tau_i)\|$$
$$= \delta_{if}\|(\delta_{\sigma_i\sigma_f} - \widehat{\gamma}_{\sigma_i\sigma_f}(i))\| \frac{\delta_{\tau_i,\tau_f}}{\Delta\tau}. \tag{121}$$

We next take the continuum limit in time τ_i; with $\Delta\tau \to 0$, and using $\frac{\delta_{\tau_i,\tau_f}}{\Delta\tau} \to \delta(\tau_i - \tau_f)$, and using the non interacting Fermi Green's function from (136), and implementing the basic Schwinger identity for representing higher order correlation functions as source derivatives:

$$\|\widetilde{c}^*_{f\sigma_f}(\tau_f)A_{i\sigma_i}(\tau_i)\| = \hat{X}_{\sigma_i\sigma_j}(i\tau_i, j\tau_j)\|\widetilde{c}^*_{f\sigma_f}(\tau_f)c_{j\sigma_j}(\tau_j)\| \tag{122}$$

where $\hat{X}$ is a functional derivative operator defined more completely below in (130). With this preparation we can rewrite Eq. (121) as

$$\left(\mathbf{g}^{-1}_{0,\sigma_i,\sigma_j}(i\tau_i, j\tau_j) - \hat{X}_{\sigma_i\sigma_j}(i\tau_i, j\tau_j)\right) \|\widetilde{c}^*_{f\sigma_f}(\tau_f)c_{j\sigma_j}(\tau_j)\| = \delta_{if}\|(\delta_{\sigma_i\sigma_f} - \widehat{\gamma}_{\sigma_i\sigma_f}(i))\|\delta(\tau_i - \tau_f). \tag{123}$$

Here and elsewhere since τ_j repeats in product, it is assumed to be integrated between $0 \le \tau_j \le \beta$, this rule is analogous to the spin index summation rule. We next divide by Z, use (110) to define the Green's function, and also define

$$Y_{1\sigma_i\sigma_j}(i\tau_i, j\tau_j) = \frac{1}{Z}(\hat{X}_{\sigma_i\sigma_j}(i\tau_i, j\tau_j)Z), \tag{124}$$

to rewrite (123) in the same form as Eq. (138)

$$\left(\mathbf{g}^{-1}_{0,\sigma_i,\sigma_j}(i\tau_i, j\tau_j) - \hat{X}_{\sigma_i\sigma_j}(i\tau_i, j\tau_j) - Y_{1\sigma_i\sigma_j}(i\tau_i, j\tau_j)\right)$$
$$\times \mathcal{G}_{\sigma_i\sigma_j}(i\tau_i, j\tau_j) = \delta_{if}\delta(\tau_i - \tau_f)\left[\delta_{\sigma_i\sigma_f} - \gamma_{\sigma_i\sigma_f}(i)\right], \tag{125}$$

where

$$\gamma_{\sigma_i\sigma_f}(i) \equiv \langle\langle\widehat{\gamma}_{\sigma_i\sigma_f}(i)\rangle\rangle. \tag{126}$$

This is readily seen to be identical to the direct definition given before in (135). We next use $\tilde{i} \equiv (i, \tau_i, \sigma_i)$ as an abbreviation for the (space, time, spin) indices, and use the repeated index summation convention. Here summation stands for spin and spatial sums, and temporal integrals in the standard intervals. With this we can write $\hat{X}_{\sigma_i\sigma_j}(i\tau_i, j\tau_j) \leftrightarrow \hat{X}_{\tilde{i}\tilde{j}}$, and similarly for $\mathbf{g}_0^{-1}$, $\mathcal{G}$ and Y_1. The variable (126) is local and needs the extra definition $\gamma_{\sigma_i\sigma_f}(i\tau_i)\delta_{if}\delta(\tau_i - \tau_f) \leftrightarrow \gamma_{\tilde{i}\tilde{f}}$ and also denote $\delta_{if}\delta_{\sigma_i\sigma_f}\delta(\tau_i - \tau_f) \leftrightarrow \delta_{\tilde{i}\tilde{f}}$. With these, the matrix product form of Eq. (123) reads:

$$\left(\mathbf{g}^{-1}_{0,\tilde{i}\tilde{j}} - \hat{X}_{\tilde{i},\tilde{j}} - Y_{1\tilde{i},\tilde{j}}\right) \mathcal{G}_{\tilde{j},\tilde{f}} = (\delta_{\tilde{i}\tilde{f}} - \gamma_{\tilde{i}\tilde{f}}). \tag{127}$$

This is exactly the form of the Schwinger equation for the Green's function obtained from the continuous time Heisenberg equations of motion (138) in [4,6], using the above abbreviation convention.

In order to obtain an expression for $\hat{X}$, we note a useful relationship involving the action on the partition functional (102) of the operator $D_{\sigma_i\sigma_j}(i) \equiv \sigma_i\sigma_j\delta/\delta\mathcal{V}_i^{\bar{\sigma}_i\bar{\sigma}_j}$ (from Eq. (39) of Ref. [6])

$$D_{\sigma_i\sigma_j}(i)Z[\mathcal{V}] = -\|\hat{\gamma}_{\sigma_i\sigma_j}(i)\|, \tag{128}$$

so that:

$$\|\widetilde{c}^*_{f\sigma_f}(\tau_{f+1})A^{Min}_{i\sigma_i}\| = -t_{ij}D_{\sigma_i\sigma_j}(i)\|\widetilde{c}^*_{f\sigma_f}(\tau_{f+1})c_{j\sigma_j}(\tau_i)\| + \frac{1}{2}J_{ij}D_{\sigma_i\sigma_j}(j)\|\widetilde{c}^*_{f\sigma_f}(\tau_{f+1})c_{i\sigma_j}(\tau_i)\|, \tag{129}$$

and comparing with Eq. (122) we conclude

$$\hat{X}_{\sigma_i\sigma_j}(i\tau_i, j\tau_j) = \delta(\tau_i - \tau_j)(-t_{ij}D_{\sigma_i\sigma_j}(i) + \delta_{ij}\sum_k \frac{1}{2}J_{ik}D_{\sigma_i\sigma_j}(k\tau_i)), \tag{130}$$

where the derivative $D_{\sigma_i\sigma_j}(k\tau_i)$ is at spatial site k and time τ_i. The corresponding Y_1 (with a similar convention as above) in Eq. (124) is

$$Y_{1\sigma_i\sigma_j}(i\tau_i, j\tau_j) = -\delta(\tau_i - \tau_j)\left(-t_{ij}\gamma_{\sigma_i\sigma_j}(i) + \delta_{ij}\sum_k \frac{1}{2}J_{ik}\gamma_{\sigma_i\sigma_j}(k\tau_i)\right). \tag{131}$$

Analogous expressions for the symmetrized case, for A_i, $\hat{X}$ and Y_1 parallel to Eqs. (130), (131) and (120), can be obtained by using the symmetrized version (top line) of Eq. (117). This expression agrees with Eq. (43) of Ref. [6], and their minimal version obtained after dropping the second and fourth term. We have thus verified that the exact equations of motion are obtained from the path integral representation outlined here, constituting a non trivial check on the formalism.

8. Conclusions

In this work we have presented a simpler method to obtain the ECFL theory that complements the Schwinger method used earlier. This new method brings an important analogy to the Dyson–Maleev theory to attention, and this connection helps us to get a different perspective on the main results of ECFL, in particular the novel non-Dysonian representation of the Greens function is placed on a firm foundation. The path integral method is used to set up an alternate quantum field theory with a non Hermitian Hamiltonian, and it is proven to be valid by reproducing the Schwinger equations of motion.

We draw particular attention to the scaling result for the spectral function (34) and (35) in Section 2.4. Here the low energy spectral function is shown to satisfy a simple relation involving the hole density that throws light on the ever shrinking regime of validity of the Landau Fermi liquid, as we approach the insulating state. Finally the discussion of the alternate ways to analyze the ARPES line shapes discussed in Section 2.5 should be of interest to the ARPES community, as also the discussion of the electronic origin of a kink in the EDC energy dispersion.

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Appendix A. Summary of the minimal theory and its Schwinger equations of motion

In order to make the discussions reasonably self contained, we provide a brief discussion of the minimal equations of motion for the Green's function. These are obtained through the usual Schwinger method used in Refs. [4,6]. These equations are a subset of the ones given in Ref. [6], and can be obtained by omitting certain extra terms therein, which were added to satisfy a symmetry property. We term these equations are *the minimal theory*, since no terms are added or dropped, and the expressions are not reducible by any other argument. We also indicate the generalization to include the parameter λ in these equations, to facilitate comparing with the equations in this work.

Using the Hamiltonian (9) we note the important commutator (given in Ref. [6]):

$$[H_{tJ}, X_i^{0\sigma_i}] = \sum_j t_{ij}X_j^{0\sigma_i} + \boldsymbol{\mu}X_i^{0\sigma_i} - \sum_{j\sigma_j} t_{ij}(\sigma_i\sigma_j)X_i^{\bar{\sigma}_i\bar{\sigma}_j}X_j^{0\sigma_j} + \frac{1}{2}\sum_{j\neq i} J_{ij}\,(\sigma_i\sigma_j)X_j^{\bar{\sigma}_i\bar{\sigma}_j}X_i^{0\sigma_j}. \tag{132}$$

Temporarily ignoring the Fermionic sources:

$$[\hat{\mathcal{A}}_S(i\tau_i), X_i^{0\sigma_i}] = -\mathcal{V}_i^{\sigma_i\sigma_j}X_i^{0\sigma_j}, \tag{133}$$

and combining with the Heisenberg equation of motion, we see that the Green's function satisfies the EOM

$$\partial_{\tau_i}\mathcal{G}_{\sigma_i\sigma_f}(i,f) = -\delta(\tau_i-\tau_f)\delta_{ij}(1-\gamma_{\sigma_i\sigma_f}(i\tau_i)) - \left\langle T_\tau\left(e^{-\hat{\mathcal{A}}_S}[H_{tJ}+\hat{\mathcal{A}}_S(i,\tau_i), X_i^{0\sigma_i}(\tau_i)]\, X_f^{\sigma_f 0}(\tau_f)\right)\right\rangle \tag{134}$$

where the local Green's function

$$\gamma_{\sigma_a\sigma_b}(i\tau_i) = \sigma_a\sigma_b\mathcal{G}_{\bar{\sigma}_b\bar{\sigma}_a}(i\tau_i^-, i\tau_i). \tag{135}$$

Substituting and using the Fermi gas (i.e. free) Green's function:

$$\mathbf{g}^{-1}_{0,\sigma_i,\sigma_j}(i\tau_i, j\tau_j) = \left\{\delta_{\sigma_i\sigma_j}\left[\delta_{ij}(\boldsymbol{\mu}-\partial_{\tau_i})+t_{ij}\right]-\delta_{ij}\mathcal{V}_i^{\sigma_i\sigma_j}(\tau_i)\right\}\delta(\tau_i-\tau_j), \tag{136}$$

we obtain (using the repeated index summation and integration convention of Ref. [6])

$$\begin{aligned}\mathbf{g}^{-1}_{0,\sigma_i,\sigma_j}(i\tau_i, j\tau_j)\mathcal{G}_{\sigma_j\sigma_f}(j\tau_j, f\tau_f) &= \delta(\tau_i-\tau_f)\delta_{ij}(1-\gamma_{\sigma_i\sigma_f}(i\tau_i)) \\ &\quad - \sum_{j\sigma_j} t_{ij}(\sigma_i\sigma_j)\left\langle T_\tau\left(X_i^{\bar{\sigma}_i\bar{\sigma}_j}(\tau_i)X_j^{0\sigma_j}(\tau_i)X_f^{\sigma_f 0}(\tau_f)\right)\right\rangle \\ &\quad + \frac{1}{2}\sum_{j\sigma_j} J_{ij}(\sigma_i\sigma_j)\left\langle T_\tau\left(X_j^{\bar{\sigma}_i\bar{\sigma}_j}(\tau_i)X_i^{0\sigma_j}(\tau_i)X_f^{\sigma_f 0}(\tau_f)\right)\right\rangle.\end{aligned} \tag{137}$$

We next express the higher order Green's function in terms of the derivatives of the lower one to obtain the Schwinger EOM:

$$\begin{aligned}&\left(\mathbf{g}^{-1}_{0,\sigma_i,\sigma_j}(i\tau_i, j\tau_j) - \hat{X}_{\sigma_i\sigma_j}(i\tau_i, j\tau_j) - Y_{1\sigma_i\sigma_j}(i\tau_i, j\tau_j)\right)\mathcal{G}_{\sigma_j\sigma_f}(j\tau_j, f\tau_f) \\ &= \delta_{if}\delta(\tau_i-\tau_f)\left(\delta_{\sigma_i\sigma_f}-\gamma_{\sigma_i\sigma_f}(i\tau_i)\right),\end{aligned} \tag{138}$$

where we used the functional derivative operator

$$D_{\sigma_i\sigma_j}(i\tau_i) = \sigma_i\sigma_j\frac{\delta}{\delta\mathcal{V}_i^{\bar{\sigma}_i\bar{\sigma}_j}(\tau_i)} \tag{139}$$

and the composite derivative operator

$$\hat{X}_{\sigma_i\sigma_j}(i\tau_i, j\tau_j) = \delta(\tau_i-\tau_j)\left(-t_{ij}D_{\sigma_i\sigma_j}(i\tau_i)+\delta_{ij}\sum_k\frac{1}{2}J_{ik}D_{\sigma_i\sigma_j}(k\tau_i)\right), \tag{140}$$

where the derivative $D_{\sigma_i\sigma_j}(k\tau_i)$ is at spatial site k and time τ_i. The corresponding Y_1 (with a similar convention as above) in Eq. (124) is

$$Y_{1\sigma_i\sigma_j}(i\tau_i, j\tau_j) = -\delta(\tau_i-\tau_j)\left(-t_{ij}\gamma_{\sigma_i\sigma_j}(i\tau_i)+\delta_{ij}\sum_k\frac{1}{2}J_{ik}\gamma_{\sigma_i\sigma_j}(k\tau_i)\right). \tag{141}$$

Eqs. (138), (140) and (141) define the minimal theory. For reference we note that Ref. [6] gives these equations, and goes on to add terms that account for the symmetrized theory with a Hermitian H_{eff}. We also note that Eq. (138) can be generalized to include the λ parameter by scaling $\hat{X}_{\sigma_i\sigma_j}, Y_{i\sigma_i\sigma_j}, \gamma_{\sigma_i\sigma_j} \to \lambda\hat{X}_{\sigma_i\sigma_j}, \lambda Y_{i\sigma_i\sigma_j}, \lambda\gamma_{\sigma_i\sigma_j}$.

Appendix B. Coherent state definitions

We use standard anticommuting Grassmann variables [46] to represent the canonical Fermions C and $C^\dagger$ for each spin and site. In brief we note the anticommuting property $\left\{c_i, c_j^*\right\} = 0 = \left\{c_i, c_j\right\} = \left\{c_i^*, c_j^*\right\} = \left\{c_i, C_j^\dagger\right\}$. Suppressing j and spin index the Fermi coherent states are given as usual by:

$$\begin{aligned}
|c\rangle &= e^{-c\,C^\dagger}|vac\rangle = \left(1 - c\,C^\dagger\right)\,|vac\rangle \\
\langle c| &= \langle vac|e^{-C\,c^*} = \langle vac|\left(1 - \,C\,c^*\right) \\
\langle c|c'\rangle &= 1 + c^*c' = e^{c^*c'},
\end{aligned} \tag{142}$$

where $|vac\rangle$ is the vacuum state. We use the abbreviation to denote coherent state integrals:

$$\int_c = \int dc^*\,dc. \tag{143}$$

The basic integrals are

$$\begin{aligned}
\int_c \left(1, c^*, c, cc^*\right) &= (0, 0, 0, 1) \\
\int_c e^{-c^*c} &= 1.
\end{aligned} \tag{144}$$

The completeness relation reads:

$$\int_c e^{-c^*c}\,|c\rangle\langle c| = |vac\rangle\langle vac| + C^\dagger|vac\rangle\langle vac|C \equiv \mathbb{1}, \tag{145}$$

and the trace over Fermionic variables is given by:

$$\mathrm{Tr}A = \int_c e^{-c^*c}\,\langle -c|A|c\rangle. \tag{146}$$

Appendix C. Path integral representation

We now introduce path integrals to represent Eq. (68) leading to (102). Towards this end let us write

$$\begin{aligned}
\beta\hat{H}_{eff} + \hat{\mathcal{A}}_S &= \int_0^\beta \hat{\mathscr{H}}(\tau)d\tau \\
\hat{\mathscr{H}}(\tau) &\equiv \hat{H}_{eff} + \sum_i \hat{\mathcal{A}}_S(i,\tau).
\end{aligned} \tag{147}$$

The integral is represented by a finite sum over M intervals, and the limit $M \to \infty$ taken at the end, thus

$$\int_0^\beta \hat{\mathscr{H}}(\tau)d\tau \to \lim_{M\to\infty} \Delta\tau \sum_{j=1,M} \hat{\mathscr{H}}(\tau_j), \tag{148}$$

where we defined

$$\begin{aligned}
\tau_j &= \Delta\tau \times j = \frac{j\,\beta}{M}, \\
\Delta\tau &= \frac{\beta}{M}, \quad j = 1, M.
\end{aligned} \tag{149}$$

Thus with $\hat{\mathcal{H}}(j) \equiv \hat{\mathcal{H}}(\tau_j)$ arranged to be in normal ordered form (creation operators to the left of the destruction operators) we write Trotters formula for the exponential

$$\begin{aligned} Z^{(M)} &= \int_{c(0)} e^{-c^*_{i\sigma}(0)c_{i\sigma}(0)} \langle -c(0)|e^{-\Delta\tau\hat{\mathcal{H}}(\tau_M)}\, e^{-\Delta\tau\hat{\mathcal{H}}(\tau_{M-1})} \dots\, e^{-\Delta\tau\hat{\mathcal{H}}(\tau_2)} e^{-\Delta\tau\hat{\mathcal{H}}(\tau_1)}\, \hat{P}_G|c(0)\rangle \\ &= \int_c e^{-\sum_{j=1}^{M} c^*_{i\sigma}(j)c_{i\sigma}(j)}\, \langle c(M)|e^{-\Delta\tau\hat{\mathcal{H}}(\tau_M)}|c(M-1)\rangle \dots\, |c(2)\rangle\langle c(2)| \\ &\quad \times e^{-\Delta\tau\hat{\mathcal{H}}(\tau_2)}|c(1)\rangle\langle c(1)|e^{-\Delta\tau\hat{\mathcal{H}}(\tau_1)}\, \hat{P}_G|c(0)\rangle. \end{aligned} \tag{150}$$

Anti periodic boundary conditions are used: $c(\tau_M) = -c(\tau_0)$ and we set at each time slice τ_j the coherent state $|c(j)\rangle = \prod_{i\sigma} |c_{i\sigma}(\tau_j)\rangle$ as a global product over all sites and both spins, and the symbol $\int_c$ represents integration over all the sites spins and time slices. The site index i and spin σ are implicitly summed over. Recall that $\hat{P}_G$ is brought to the extreme right of the product. We calculate as usual:

$$\begin{aligned} &\langle c(\tau_{j+1})|c(\tau_j)\rangle = e^{c^*(\tau_{j+1})c(\tau_j)} \\ &\langle c(\tau_{j+1})|e^{-\Delta\tau\hat{\mathcal{H}}(\tau_{j+1})}|c(\tau_j)\rangle \equiv e^{c^*(\tau_{j+1})c(\tau_j) - \Delta\tau\mathcal{H}(\tau_{j+1},\tau_j)} + O(\Delta\tau^2) \\ &\mathcal{H}(\tau_{j+1},\tau_j) \equiv \frac{\langle c(\tau_{j+1})|\hat{\mathcal{H}}(\tau_{\tau_{j+1}})|c(\tau_j)\rangle}{\langle c(\tau_{j+1})|c(\tau_j)\rangle}. \end{aligned} \tag{151}$$

The last term needs careful attention, we note

$$\begin{aligned} \langle c(\tau_1)|e^{-\Delta\tau\hat{\mathcal{H}}(\tau_1)}\, \hat{P}_G|c(\tau_0)\rangle &= \langle c(\tau_1)|(1-\Delta\tau\hat{\mathcal{H}}(\tau_1))\, \hat{P}_G|c(\tau_0)\rangle + O(\Delta\tau^2) \\ &= \langle c(\tau_1)|c(\tau_0)\rangle(1-\Delta\tau\mathcal{H}(\tau_1,\tau_0))\, P_G(\tau_1,\tau_0) + O(\Delta\tau^2) \\ &= \langle c(\tau_1)|c(\tau_0)\rangle e^{-\Delta\tau\mathcal{H}(\tau_1,\tau_0)}\, P_G(\tau_1,\tau_0) + O(\Delta\tau^2), \end{aligned} \tag{152}$$

where Eq. (109) details the expression for $P_G(\tau_1, \tau_0)$, it contains variables at the initial and next time instant only. Combining all terms, we get the expression (102). We have thrown out terms of $O(\Delta\tau)^2$ in obtaining Eq. (102), and hence it is important to keep track of the Pauli principle identities, discussed above in Eqs. (156) and (158). Note that for arbitrary τ

$$\langle \tau|C^\dagger_{i\sigma}|\tau_j\rangle = c^*_{i\sigma}(\tau)e^{c^*_{i\sigma}(\tau)c_{i\sigma}(\tau_j)} = -\frac{\delta}{\delta c_{i\sigma}(\tau_j)}\langle \tau|\tau_j\rangle. \tag{153}$$

In view of this relation we note the following mappings:

$$\begin{aligned} \langle \psi|C^\dagger_{i\sigma}|\tau_j\rangle &\to -\frac{\delta}{\delta c_{i\sigma}(\tau_j)}\langle \psi|\tau_j\rangle, \\ \langle \psi|C_{i\sigma}|\tau_j\rangle &\to c_{i\sigma}(\tau_j)\langle \psi|\tau_j\rangle, \\ \langle \tau_j|C^\dagger_{i\sigma}|\psi\rangle &\to c^*_{i\sigma}(\tau_j)\langle \tau_j|\psi\rangle, \\ \langle \tau_j|C_{i\sigma}|\psi\rangle &\to \frac{\delta}{\delta c^*_{i\sigma}(\tau_j)}\langle \tau_j|\psi\rangle, \end{aligned} \tag{154}$$

Let us show how the commutation works here:

$$(CC^\dagger + C^\dagger C)|c\rangle = \left(-C\frac{\delta}{\delta c} + C^\dagger c\right)|c\rangle = \left(\frac{\delta}{\delta c}C - cC^\dagger\right)|c\rangle = \left(\frac{\delta}{\delta c}c + c\frac{\delta}{\delta c}\right)|c\rangle = |c\rangle. \tag{155}$$

Appendix D. Pauli and Gutzwiller exclusion identities

It is worth highlighting a few conventions about (102) and related expressions. These are designed to retain some of the most important features of strongly interacting electrons *on a lattice*. In contrast, in a theory of electrons in the continuum, these constraints are of no special consequence since coincident spatial points have a measure of zero. We first discuss the Pauli principle related rules

referring to the same spin spices, and then the Gutzwiller projection related rules relating to opposite spin species, these are operative when two electronic operators have coincident space and time coordinates.

- (I) When two coincident times in a product of operators *on the same lattice site* and same spin arise, we follow the convention of immediate evaluation of the product. By evaluation, we understand that the product of two similar Grassmann variables is set to zero, and for dissimilar Grassmann variables (e.g. c and c^*) at a common time, both of them are integrated out immediately. This leads to the basic set of *Pauli exclusion identities* at equal times as one easily verifies:

$$\begin{aligned} &c_{i\sigma}(\tau_j)c^*_{i\sigma}(\tau_j) \to 1\\ &c^*_{i\sigma}(\tau_j)c_{i\sigma}(\tau_j) \to -1\\ &c_{i\sigma}(\tau_j)c_{i\sigma}(\tau_j) \to 0\\ &c^*_{i\sigma}(\tau_j)c^*_{i\sigma}(\tau_j) \to 0. \end{aligned} \tag{156}$$

- (II) We denote the number operator as

$$n_{i\sigma}(\tau_j) \equiv c^*_{i\sigma}(\tau_{j+1})\, c_{i\sigma}(\tau_j), \tag{157}$$

where we observe that the c^* has the immediately later time argument than that of c, this comes about from representing $\langle j+1|C^\dagger C|j\rangle = c^*(\tau_{j+1})c(\tau_j) \times \langle j+1|j\rangle$. Using this we will verify the *second set* of Pauli exclusion identities

$$\begin{aligned} &n_{i\sigma}(\tau_{j+1})n_{i\sigma}(\tau_j) = n_{i\sigma}(\tau_j)\\ &c^*_{i\sigma}(\tau_{j+1})n_{i\sigma}(\tau_j) = 0\\ &n_{i\sigma}(\tau_j)c^*_{i\sigma}(\tau_j) = c^*_{i\sigma}(\tau_{j+1})\\ &n_{i\sigma}(\tau_j)c_{i\sigma}(\tau_j) = 0\\ &c_{i\sigma}(\tau_j)n_{i\sigma}(\tau_{j-1}) = c_{i\sigma}(\tau_{j-1}). \end{aligned} \tag{158}$$

(III) We next obtain the important *Gutzwiller exclusion identity*. Calling the ith term in the product (109) as $P^{(i)}_G(\tau_1, \tau_0)$; we see that

$$\begin{aligned} &n_{i\uparrow}(\tau_1)n_{i\downarrow}(\tau_1)P^{(i)}_G(\tau_1, \tau_0) = n_{i\uparrow}(\tau_1)n_{i\downarrow}(\tau_1)\\ &-c^*_{i\uparrow}(\tau_2)c_{i\uparrow}(\tau_0)c^*_{i\downarrow}(\tau_2)c_{i\downarrow}(\tau_0) \sim 0. \end{aligned} \tag{159}$$

The last line follows upon expanding $\tau_2 (\equiv\ \tau_1 + \Delta\tau)$ about τ_1. The assumption that terms of $O(\Delta\tau)$ are negligible is implicit in the entire path integral formulation. This shows that the double occupancy type terms $n_{i\uparrow}(\tau_1)n_{i\downarrow}(\tau_1)$ that occur at any site lead to vanishing contribution, thus enforcing Gutzwiller projection. We can extend this argument to other times $\tau_j \geq \tau_1$:

$$c_{i\uparrow}(\tau_j)c_{i\downarrow}(\tau_j)\ldots P^{(i)}_G(\tau_1, \tau_0) = 0, \tag{160}$$

where the dots indicate contributions from intermediate times. These contributions, after Grassmann integration over the terms at intermediate times, must necessarily end up with $\ldots c_{i\uparrow}(\tau_1)c_{i\downarrow}(\tau_1)P^{(i)}_G(\tau_1, \tau_0)$. Expanding this factor leads to $c_{i\uparrow}(\tau_1)c_{i\downarrow}(\tau_1) - c_{i\uparrow}(\tau_0)c_{i\downarrow}(\tau_0)$, and therefore vanishes to $O(\Delta\tau)$, as in the argument in (159).

Appendix E. Interpreting the caparison factor in the Schwinger method

Within the Schwinger method, or the related path integral formulation given above, the decomposition of $\mathcal{G}$ is best done by rescaling the source terms by a factor determined through a self consistent argument given next. A convenient method is to work in the presence of the Fermionic sources, which allows us to start with the non vanishing average of a Fermi operator:

$$\xi_{i\sigma_i}(\tau_i) \equiv \frac{1}{Z}\|X^{0\sigma_i}_i(\tau_i)\| = \frac{1}{Z}\|c_{i\sigma_i}(\tau_i)\|, \tag{161}$$

and further abbreviate $\xi_{i\sigma_i}(\tau_i) \leftrightarrow \xi_{\tilde{i}}$. A creation variable average $\xi^*_{\tilde{i}} \equiv \frac{1}{Z}\|X_i^{\sigma_i 0}(\tau_i)\|$ is also useful. The variable $\xi_{\tilde{i}}$ satisfies the functional differential equation that we study next:

$$(\mathbf{g}^{-1}_{0,\tilde{i},\tilde{j}} - \hat{X}_{\tilde{i}\tilde{j}})(Z\xi_{\tilde{j}}) = Z(\delta_{\tilde{i},\tilde{k}} - \gamma_{\tilde{i},\tilde{k}})J_{\tilde{k}}, \tag{162}$$

or using Eq. (124)

$$(\mathbf{g}^{-1}_{0,\tilde{i},\tilde{j}} - \hat{X}_{\tilde{i}\tilde{j}} - Y_{1\,\tilde{i}\tilde{j}})\,\xi_{\tilde{j}} = (\delta_{\tilde{i},\tilde{k}} - \gamma_{\tilde{i},\tilde{k}})J_{\tilde{k}}. \tag{163}$$

This equation can be arrived at within the path integral representation (73), by using a variant of (112) after omitting the Fermionic creation type variable in the square bracket; and of course with a non vanishing Fermi source term. Alternately we can take the Heisenberg equations of motion in terms of the original expressions in terms of the X operators (11) and (13). The agreement between the two methods can be checked easily, and provides a strong check on the path integral formulation.

The Green's function is found from a variant of Eq. (13):

$$\mathcal{G}_{\tilde{i}\tilde{f}} - \xi^*_{\tilde{f}}\,\xi_{\tilde{i}} = \frac{\delta\xi_{\tilde{i}}}{\delta J_{\tilde{f}}}, \tag{164}$$

and on taking the limit $J \to 0, J^* \to 0$, all the single Fermi expectations ξ, ξ^* vanish. Taking the J derivative of (164), we see that (127) follows, so we will work with this equation from here.

The main objective from this point onwards, is to cast Eq. (163) or Eq. (127) into a form where the expressions on right are in the canonical form, i.e. where the time dependent γ term is gotten rid of in favor of a suitable constraint [4,52,53]. The occurrence of the factor $\mathbb{1} - \gamma$ multiplying the source J in Eq. (163) suggests that one should scale the source J by a suitable time dependent factor to obtain new sources $\mathcal{J}$. The factor can be adjusted self consistently, so as to extract a canonical Green's function. Thus we scale

$$J_{\tilde{i}} = (\mu^{-1})_{\tilde{i}\tilde{j}}\,\mathcal{J}_{\tilde{j}}, \tag{165}$$

so that

$$\frac{\delta}{\delta J_{\tilde{f}}} = \frac{\delta}{\delta \mathcal{J}_{\tilde{k}}}\mu_{\tilde{k}\tilde{f}}, \tag{166}$$

where (μ^{-1}) is the *matrix inverse* of μ. These equations have inverses that are easily obtained. The matrix μ is dependent on the Fermi sources only indirectly, and this dependence may be neglected since the Fermi sources are turned off in the sequel. However it is allowed to be a functional of the Bosonic sources $\mathcal{V}$, thereby giving us considerable flexibility in defining it, we must also then be careful in locating it relative to the operator $\hat{X}$, since it contains derivatives with respect to $\mathcal{V}$.

In view of Eq. (166) we obtain the product relation

$$\begin{aligned}\mathcal{G}_{\tilde{i}\tilde{f}} &= \mathbf{g}_{\tilde{i}\tilde{k}}\,\mu_{\tilde{k}\tilde{f}} + \xi^*_{\tilde{f}}\,\xi_{\tilde{i}},\\ \mathbf{g}_{\tilde{i}\tilde{k}} &= \frac{\delta\xi_{\tilde{i}}}{\delta\mathcal{J}_{\tilde{k}}}.\end{aligned} \tag{167}$$

The goal is to choose μ such that the so defined $\mathbf{g}$ satisfies a canonical equation, i.e. the analog of Eq. (127), but without the γ term on the right. For this purpose we can differentiate Eq. (163) with the scaled source field $\mathcal{J}_{\tilde{k}}$, taking care to observe the non commutation of $\mathcal{J}_{\tilde{k}}$ with the derivative term $\hat{X}$. This process yields the equations:

$$(\mathbf{g}^{-1}_{0,\tilde{i},\tilde{j}} - \hat{X}_{\tilde{i}\tilde{j}} - Y_{1\,\tilde{i}\tilde{j}})\,\mathbf{g}_{\tilde{j}\tilde{k}} = (\delta_{\tilde{i},\tilde{j}} - \gamma_{\tilde{i},\tilde{j}})(\mu^{-1})_{\tilde{j}\tilde{k}} - \left[\hat{X}_{\tilde{i}\tilde{j}}(\mu^{-1})_{\tilde{l}\tilde{k}}\right]\mathbf{g}_{\tilde{j}\tilde{m}}\mu_{\tilde{m}\tilde{l}}, \tag{168}$$

where the square brackets demarcate the terms acted upon, by the derivative operators in $\hat{X}$. Eq. (168) exhibits a separation of variables, all the dependence on μ is confined to the right hand side, and hence we set both sides to the identity matrix:

$$(\mathbf{g}^{-1}_{0,\tilde{i},\tilde{j}} - \hat{X}_{\tilde{i}\tilde{j}} - Y_{1\,\tilde{i}\tilde{j}})\,\mathbf{g}_{\tilde{j}\tilde{k}} = \delta_{\tilde{i}\tilde{k}}, \tag{169}$$

and

$$(\delta_{\tilde{i},\tilde{j}} - \gamma_{\tilde{i},\tilde{j}})(\mu^{-1})_{\tilde{j}\tilde{k}} - \left[\hat{X}_{\tilde{i}\tilde{j}}(\mu^{-1})_{\tilde{l}\tilde{k}}\right]\mathbf{g}_{\tilde{j}\tilde{m}}\mu_{\tilde{m}\tilde{l}} = \delta_{\tilde{i}\tilde{k}}. \tag{170}$$

Multiplying through with μ and using

$$\left[\hat{X}_{\tilde{i}\tilde{j}}(\mu^{-1})_{\tilde{l}\tilde{k}}\right]\mu_{\tilde{m}\tilde{l}} = -\left[\hat{X}_{\tilde{i}\tilde{j}}\mu_{\tilde{m}\tilde{l}}\right](\mu^{-1})_{\tilde{l}\tilde{k}}, \tag{171}$$

we rewrite Eq. (170) as

$$\mu_{\tilde{i}\tilde{f}} = (\delta_{\tilde{i}\tilde{f}} - \gamma_{\tilde{i}\tilde{f}}) + \mathbf{g}_{\tilde{j}\tilde{k}}\left[\hat{X}_{\tilde{i}\tilde{j}}\mu_{\tilde{k}\tilde{f}}\right]. \tag{172}$$

We next show that Eqs. (169) and (170) can be rewritten in terms of the two self energies Φ and Ψ used in Refs. [4,6]. We need the relation analogous to Eq. (171) to simplify Eq. (169):

$$\left[\hat{X}_{\tilde{i}\tilde{j}}\mathbf{g}_{\tilde{j}\tilde{k}}\right] = -\mathbf{g}_{\tilde{j}\tilde{k}}\left[\hat{X}_{\tilde{i}\tilde{j}}\mathbf{g}^{-1}_{\tilde{k}\tilde{l}}\right]\mathbf{g}_{\tilde{l}\tilde{k}}. \tag{173}$$

Therefore we write the two equations as

$$\begin{aligned}(\mathbf{g}^{-1}_{0,\tilde{i},\tilde{j}} - \Phi_{\tilde{i}\tilde{j}} - Y_{1\,\tilde{i}\tilde{j}})\,\mathbf{g}_{\tilde{j}\tilde{k}} &= \delta_{\tilde{i}\tilde{k}},\\ \mu_{\tilde{i}\tilde{f}} &= (\delta_{\tilde{i}\tilde{f}} - \gamma_{\tilde{i}\tilde{f}}) + \Psi_{\tilde{i}\tilde{f}},\end{aligned} \tag{174}$$

where the two self energies Φ and Ψ are functions obtained by iteration, and have a finite limit on turning off the Bosonic source $\mathcal{V}$. These are obtained from the above as

$$\begin{aligned}\Phi_{\tilde{i}\tilde{j}} &= -\mathbf{g}_{\tilde{m}\tilde{k}}\left[\hat{X}_{\tilde{i}\tilde{m}}\mathbf{g}^{-1}_{\tilde{k}\tilde{j}}\right]\\ \Psi_{\tilde{i}\tilde{f}} &= \mathbf{g}_{\tilde{j}\tilde{k}}\left[\hat{X}_{\tilde{i}\tilde{j}}\mu_{\tilde{k}\tilde{f}}\right].\end{aligned} \tag{175}$$

Using the definition of $\hat{X}$ and of various vertex functions, we can verify that these are precisely the pair of equations that we obtained in Refs. [4,6] for the two self energies of $\mathcal{G}$.

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[41] This alternative choice, upon using Eq. (85) leads to

$$n_{i\sigma} = \frac{1}{\lambda}\{1 - \mathcal{G}^{(\lambda)}_{\bar\sigma\bar\sigma}(i\tau^- i\tau) + \mathcal{G}^{(\lambda)}_{\bar\sigma\bar\sigma}(i\tau^+ i\tau)\}.$$

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Erratum

Erratum to: "Theory of extreme correlations using canonical Fermions and path integrals" [Ann. Phys. 343 (2014) 164–199]

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ABSTRACT

Erratum for paper "Theory of extreme correlations using canonical Fermions and path integrals", B.S. Shastry, Ann. Phys. 343, 164–199 (2014) http://dx.doi.org/10.1016/j.aop.2014.02.005 AOP69618.

The following errors should be corrected in the paper.

(1) Page 172 below Eq. (22).
Replace
$\hat{k} = (\vec{k} - \vec{k}_F).\vec{k}_F/|\vec{k}_F|$
by
$\hat{k} = (\vec{k} - \vec{k}_F).\nabla\varepsilon_k/|\nabla\varepsilon_k|$.

(2) Page 175 Second paragraph.
Replace
"Upon turning off …"
by
"For $Q > 0$, upon turning off …".

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(3) Page 175 Third line after Eq. (44).
Replace
$sign(\hat{k})$
by
$sign(-Q)$.
(4) Page 175 Fourth line after Eq. (44).
Replace
$sign(\hat{k})$
by
$sign(-Q)$.
(5) Eq. (45) should read

$$E^*(k_{kink}) = -\frac{1}{r-1}\Delta_0 - \Gamma_0.$$

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