

## Exact Solution of a One-Dimensional Multicomponent Lattice Gas with Hyperbolic Interaction

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We present the exact solution to a one-dimensional multicomponent quantum lattice model interacting by an exchange operator which falls off as the inverse sinh square of the distance. This interaction contains a variable range as a parameter and can thus interpolate between the known solutions for the nearest-neighbor chain and the inverse-square chain. The energy, susceptibility, charge stiffness, and the dispersion relations for low-lying excitations are explicitly calculated for the absolute ground state, as a function of both the range of the interaction and the number of species of fermions.

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In a recent Letter [1], to be called S2, Sutherland and Shastry solved a one-dimensional, continuum, multicomponent quantum many-body system with a hyperbolic exchange interaction, governed by the following Hamiltonian:

$$H_{\text{cont}} = -\frac{1}{2} \sum_j \frac{\partial^2}{\partial x_j^2} + \sum_{j>k} \frac{\lambda^2 - \lambda P_{jk}}{\sinh^2(x_j - x_k)}. \quad (1)$$

We have set  $\hbar = 1$ , the mass to be unity,  $P_{jk}$  is the exchange operator which permutes particles  $j$  and  $k$ , and we always have in mind the thermodynamic limit, when the number of particles  $N$  and the box size  $\Lambda$  both become infinite, while the density  $N/\Lambda = d$  remains finite. This model was shown to be integrable by Polychronakos [2]. In S2, scattering in this system was shown to be nondiffractive, and so the system could be solved exactly (in the thermodynamic limit, with finite size corrections being exponentially small in  $\Lambda$ ) by the asymptotic Bethe ansatz [3]. The system is characterized by the number of different types  $b$  ( $f$ ) of bosons (fermions) in the system; we write this as a  $B^b F^f$  system. Thus  $B$  ( $F$ ) represents a single component system of identical bosons (fermions) interacting with an interaction strength  $\lambda$  ( $\lambda \mp 1$ ).

This system has a classical limit, obtained by taking  $\lambda \rightarrow \infty$ , and, in the ground state, the particles freeze into a lattice with lattice spacing  $\Lambda/N = 1/d$ , so that  $x_j \rightarrow j/d$ . Expanding the Hamiltonian in  $1/\lambda$ , we find  $H_{\text{cont}} \rightarrow H_{\text{elas}} + 2\lambda H_{\text{latt}}$ , where  $H_{\text{elas}}$  represents the Hamiltonian for the elastic degrees of freedom—phonons and solitons—identical to the single component  $F$  system [4]. The additional term of order  $\lambda$  gives the Hamiltonian for the compositional degrees of freedom  $H_{\text{latt}}$ ,

$$H_{\text{latt}} = -\frac{1}{2} \sum_{j>k} \frac{1 + P_{jk}}{\sinh^2[(j - k)/d]}. \quad (2)$$

There is exactly one particle to a site. Thus, to order  $\lambda$ , the elastic and compositional degrees of freedom separate. Coupling between the two occurs through an

expansion of the interaction about the regular lattice, and so is of order  $\langle \delta x^2 \rangle \sim 1/\lambda$  and down by another factor of  $1/\lambda$ . (A similar procedure is used to derive the Heisenberg antiferromagnet from the half-filled Hubbard model.) This idea of freezing into a lattice has been used by Polychronakos [5] for the Haldane-Shastry chain ( $1/\tau^2$ ), and more generally in S2.

In this Letter, we will study the spectrum of this lattice Hamiltonian. The Hamiltonian has many realizations—as does the corresponding nearest-neighbor model—such as by spins, a (supersymmetric)  $t$ - $J$  model,  $SU(N)$  spins, etc. In fact, this system interpolates between the nearest-neighbor model [6,7] ( $d \rightarrow 0$ ), and the  $1/r^2$  lattice [1,8,9] ( $d \rightarrow \infty$ ) treated in S2. Our solution here will rely heavily on Sutherland's treatment [6]—to be called S1—of the nearest-neighbor system. Two results proven in S1 can easily be shown for the hyperbolic case: (i) the ground state of the  $B^b F^f$  system is identical to the ground state of the  $B F^f$  system, so for the ground state we need never consider more than one type of boson which we usually think of as a vacancy; (ii) the Hamiltonian with  $P_{jk} \rightarrow -P_{jk}$  is unitarily equivalent to  $H_{\text{latt}}$ , hence “ferro” and “antiferro” cases are both included in the ground state of  $H_{\text{latt}}$ . The pure (antiferro) ferro case is  $B^b (F^f)$ . Partial results concerning the integrability of this model have previously appeared in the literature [10].

Because the system is nondiffractive, the wave function must be given asymptotically as  $\Psi(x|Q) \sim \sum_P \Psi(P|Q) \exp[i \sum_{j=1}^N x_j k_{Pj}]$ , where  $x_1 < \dots < x_N$ ,  $P$  is one of the  $N!$  permutations of the  $N$  asymptotic momenta  $k_j$ ,  $Q$  is one of the  $N!$  rearrangements of the particles, and  $\Psi(P|Q)$  are  $(N!)^2$  amplitudes related by two-body scattering. The  $k$ 's are determined by a set of eigenvalue equations, which say that the phase shift of a particle going around the ring must be unity mod  $2\pi$ :

$$\exp[ik_j \Lambda] \prod_{m(\neq j)} - \exp[-i\theta_0(k_j - k_m)] \times X_{j,j-1} \cdots X_{j,1} X_{j,N} \cdots X_{j,j+1} \Psi = \Psi. \quad (3)$$

Here  $X_{j,m} = X(k_j - k_m)$  are two-body scattering operators identical to those for the  $\delta$ -function gas or the nearest-neighbor lattice model; the eigenvectors and eigenvalues are given in S1. The two-body phase shift  $\theta_0(k)$  is the phase shift for scattering from a  $1/\sinh^2(r)$  potential, and is given as  $\theta_0(k) = i \ln[\Gamma(1 + ik/2)\Gamma(\lambda - ik/2)]/\Gamma(1 - ik/2)\Gamma(\lambda + ik/2)$ .

Using the eigenvalues of the  $X$ 's from S1, one obtains coupled equations for the  $k$ 's, the first being

$$\Delta k = 2\pi I(k) + \sum_{k'} \theta_0(k - k') - \sum_{\alpha} \theta(2(k - \alpha)). \quad (4)$$

Here  $\theta(k)$  is the phase shift for a  $\delta$ -function gas:  $\theta(k) = 2 \arctan(k/2\lambda)$ . The  $I(k)$  are the quantum numbers, from  $\ln(\pm 1)/2\pi$ . The remaining equations are identical with those of the nearest-neighbor model; the  $k$ 's couple directly only to the second equation, which has the form  $0 = 2\pi J(\alpha) - \sum_k \theta(2(\alpha - k)) + \dots$ . The remaining terms and equations are the same as for the nearest-neighbor model, and their form and number depend upon what types of particles we have.

When we take the limit  $\lambda \rightarrow \infty$ , we must also rescale the asymptotic momenta  $k$ , defining a new variable  $x \equiv k/2\lambda$ . Likewise, the asymptotic momenta  $\alpha$  for the compositional degrees of freedom must be rescaled, but we use the same symbol, so  $\alpha \rightarrow \alpha/2\lambda$ . Then the phase shift for particle-particle scattering has an expansion in  $1/\lambda$  of the form  $\theta_0(k) \rightarrow 2\lambda\theta_0(x) + \theta_1(x) + \dots$ , where  $\theta_0'(x) = \frac{1}{2} \ln(1 + 1/x^2)$ . Finally, for one more redefinition, let  $\theta(x) \rightarrow 2 \arctan(x)$ .

We now expand the first equation in  $1/\lambda$ , obtaining order  $\lambda$ —which we call the zeroth order, since it starts the expansion—the equation  $\Delta x = \sum_x \theta_0(x - x')$ . Let  $N\rho(x)dx$  be the number of  $x$ 's in  $x \rightarrow x + dx$ . Then the equation for  $\rho$  becomes an integral equation, and upon differentiating  $1/2\pi d = (1/2\pi) \int_{-A}^A \theta_0'(x - x')\rho(x')dx' \equiv L\rho$ . Let  $L^{-1}$  be the inverse of  $L$ , so  $\rho = L^{-1}[1/2\pi d]$ . The normalization of  $\rho$  is  $1 = \int_{-A}^A \rho(x)dx \equiv \mathbb{1}^\dagger \rho$ . Before discussing the solution to this integral equation, we go on and examine the first order corrections to the equation for the  $x$ 's.

We write the  $x$ 's which satisfy the zeroth order equation as  $x_0$  and then look for corrections to the  $x$ 's as  $x = x_0 + \delta x/2\lambda + \dots$ . Let  $\gamma(x) = \delta x(x)\rho(x)$  and  $NR(\alpha)d\alpha$  is the number of  $\alpha$ 's in  $d\alpha$ . Then the first order equation is

$$L\gamma = \frac{I(x)}{N} + \frac{1}{2\pi} \int_{-A}^A \theta_0'(x - x')\rho(x')dx' - \frac{1}{2\pi} \int_{-B}^B \theta(2(x - \alpha))R(\alpha)d\alpha. \quad (5)$$

Now this equation is linear, so we can write  $\gamma = \gamma_0 + \gamma_1$ , where  $\gamma_0$  is the correction for the elastic modes, while  $\gamma_1$  is the correction from the compositional modes. We are

interested only in the compositional degrees of freedom in this Letter, so we write  $\gamma$  for  $\gamma_1$ , which obeys the integral equation  $L\gamma = -(1/2\pi) \int_{-B}^B \theta(2(x - \alpha))R(\alpha)d\alpha \equiv -g$ . Expanding the energy and momentum to the same order, we find

$$\begin{aligned} \Delta E/2\lambda N &= x^\dagger \gamma = -x^\dagger L^{-1}g = -g^\dagger L^{-1}x \\ &= -g^\dagger e = e^\dagger g = E_{\text{latt}}/N, \end{aligned} \quad (6a)$$

$$\begin{aligned} \Delta P/dN &= \mathbb{1}^\dagger \gamma/d = -\mathbb{1}^\dagger L^{-1}g/d \\ &= -g^\dagger L^{-1}/d = -2\pi g^\dagger \rho = P_{\text{latt}}/N. \end{aligned} \quad (6b)$$

Here,  $e$  and the derivative  $e'$  obey the equations  $Le = x$  and  $Le = x^2/2 - \mu$ ,  $e(\pm A) = 0$ . Remember, all integrals over  $x$  are integrals only over the interval  $-A$  to  $A$ . These expressions of Eq. (6) are exactly the energy and momentum for the lattice Hamiltonian  $H_{\text{latt}}$  of Eq. (2). The classical ground state energy can also be expressed as  $E/4\lambda^2 N = (x^2/2)^\dagger \rho = \mathbb{1}^\dagger e/2\pi d + \mu = \frac{1}{4} \sum_{j=1}^{\infty} \sinh^{-2}(j/d)$ . The only effect of the  $x$ 's—and thus the  $k$ 's—on the  $\alpha$ 's is through the expression

$$\begin{aligned} \sum_x \theta(2(\alpha - x)) \\ = N \int_{-A}^A \theta(2(\alpha - x))\rho(x)dx \equiv Np(\alpha). \end{aligned} \quad (7)$$

Returning to the expressions for the energy and momentum of the lattice Hamiltonian Eq. (6), and substituting the expression for  $g$ , we see that they can be written as  $P_{\text{latt}}/N = p^\dagger BR$ , and  $E_{\text{latt}}/N = \varepsilon^\dagger BR$ , with  $\varepsilon$  defined by  $\varepsilon(\alpha) \equiv \frac{1}{\pi} \int_{-A}^A \theta'(2(\alpha - x))e(x)dx$ . Eliminating  $\alpha$  between  $p(\alpha)$  and  $\varepsilon(\alpha)$  gives the dispersion relation  $\varepsilon(p)$  for a single excitation of the lattice Hamiltonian. This can also be expressed by the Fourier transform of the hopping matrix element, and so  $\varepsilon(p) = -\sum_{j=1}^{\infty} [1 - (-1)^j \cos(jp)]/\sinh^2(j/d)$ .

As will become clearer, the solution to the hyperbolic lattice problem depends crucially on the two classical ground state densities  $\rho(x)$  and  $e(x)$ , obeying integral equations with integral operator  $L$ . Unfortunately, these equations cannot be solved in closed form, even though we know the densities obey an infinite number of sum rules, arising from the classical limit of the Lax equations. The energy sum rule is the first of this hierarchy. However, we can derive what appears to be a convergent expansion for  $\rho(x)$  and  $e(x)$  in terms of the Chebyshev  $T$  and  $U$  polynomials, of the form

$$\rho(x) = \frac{1}{\pi\sqrt{A^2 - x^2}} \left[ 1 + \sum_{j=1}^{\infty} \rho_j T_{2j}(x/A) \right], \quad (8a)$$

$$e(x) = -2\sqrt{A^2 - x^2} \sum_{j=0}^{\infty} e_j U_{2j}(x/A). \quad (8b)$$

In the nearest-neighbor limit of  $A \rightarrow 0$ , when  $\theta'_0(x) \rightarrow -\ln|x|$ , one finds that the expansions stop at the first term, so that  $\rho(x) \rightarrow 1/\pi\sqrt{A^2 - x^2} \approx \delta(x)$ , and  $e(x) \rightarrow -2\sqrt{A^2 - x^2} \approx -\pi A^2 \delta(x)$ . This gives  $d^{-1} \rightarrow -\ln[A/2]$ . At the other limit of an inverse square interaction, when  $A \rightarrow \infty$ , then  $\theta'_0(x) \rightarrow \pi\delta(x)$ , so  $\rho(x) \rightarrow 1/2A$ ,  $d \rightarrow 2A/\pi$ , and  $e \rightarrow x^2 - 2\mu = x^2 - A^2$ .

We make extensive use of the solution from S1 for the nearest-neighbor chain, so we summarize the results here. We restrict ourselves in this Letter to the most interesting case of the  $F^f$  system, with  $N_j$  particles of type  $j$ , and  $N_1 \geq \dots \geq N_f$ . Let  $M_1 = N - N_1$ ,  $M_2 = N - N_1 - N_2$ ,  $\dots$ ,  $M_{f-1} = N_f$ . Then there are  $f - 1$  coupled equations for  $f - 1$  sets of roots, the  $j$ th set being  $M_j$  roots in number. For the ground state in the thermodynamic limit, these sets of roots distribute with densities  $r_j(\alpha)$  between the limits  $-b_j$  and  $b_j$ , normalized so that  $\int_{-b_j}^{b_j} r_j(\alpha) d\alpha = M_j/N \equiv m_j$  or  $\mathbb{1}^\dagger br = m$ . We have adopted a very useful notation where the  $f - 1$   $r$ 's and  $m$ 's are arranged as column vectors. Let  $K_n$  represent an integral operator with difference kernel  $K_n(\alpha) = \theta'(n\alpha)/2\pi$ ,  $\xi$  be a column vector with components  $x_j = \delta_{j1}K_2(\alpha)$ ,  $b$  a matrix projection operator which imposes the integration limits  $\pm b_j$  on  $\alpha_j$ , and  $K$  a symmetric matrix of integral operators with elements  $K_{j,m} = \delta_{j,m}K_2 - (\delta_{j,m-1} + \delta_{j,m+1})K_1$ . Then the ground state densities satisfy the  $f - 1$  coupled integral equations  $\xi = r + Kbr = (\mathbb{1} + K)b_r$ . With the Hamiltonian normalized as  $H_{nn} = -\sum_{j>k}[1 + P_{jk}]$ , the energy is  $E/N = -2\pi\xi^\dagger br$ . Finally, the momentum is  $P/N = \zeta^\dagger br$ , with  $\zeta^\dagger \equiv 2\pi\xi$ , which of course is zero for the ground state.

In the case of the absolute ground state, when there are equal numbers of each type of particle so  $n_j = N_j/N = 1/f$ , then all limits  $b_j = \infty$ , so  $b = \mathbb{1}$ , and the equation can be solved by Fourier transforms. The eigenvalues of the integral operators  $K_n$  are given by the Fourier transform of the kernel  $\tilde{K}_n(s) = e^{-|s|/n}$ . We see  $\tilde{K}_1 = (\tilde{K}_2)^2$ . We now define the resolvent operator  $J$  by  $(\mathbb{1} + J)(\mathbb{1} + K) = (\mathbb{1} + K)(\mathbb{1} + J) = \mathbb{1}$ . Then  $\mathbb{1} + J$  is also a symmetric matrix of integral operators with difference kernels, whose eigenvalues are given by Fourier transforms. The Fourier transform of the densities are then given as  $\tilde{r}_j(s) = \sinh[s(f -$

$j)/2]/\sinh(fs/2)$ . The ground state energy is calculated as  $E/N = -\int_{-\infty}^{\infty} e^{-|s|/2} \tilde{r}_1(s) ds = -2[\psi(1) - \psi(1/f)]/f$ , with  $\psi(x)$  as the digamma function.

For the excitations about the absolute ground state, there are  $f - 1$  branches. The energy and momentum for these branches are given as

$$\Delta E_j(\alpha) = \frac{2\pi}{f} \frac{\sin(\pi j/f)}{\cosh(2\pi\alpha/f) - \cos(\pi j/f)}, \quad (9a)$$

$$\Delta P_j(\alpha) = 2 \arctan[\cot(\pi j/2f) \tanh(\pi\alpha/f)] - \pi(1 - j/f). \quad (9b)$$

Eliminating  $\alpha$  between the two equations gives the dispersion relations. At  $k = 0$ , all branches have a common velocity  $v = 2\pi/f$ . This leads to our final result, that if we deviate slightly from equal filling, so the concentrations are  $n_j = 1/f + \delta n_j$ , then to leading order in the  $\delta n$ 's,  $\Delta E/N \approx \frac{\pi^2}{f} \sum_{j=1}^f \delta n_j^2$ , and thus the susceptibility exists and is isotropic.

Let us rewrite the coupled equations for the general problem as  $\hat{\xi} = (\mathbb{1} + K)b\hat{r}$ . The inhomogeneous term is given by Eq. (7) and can be rewritten as  $\hat{\xi} = \xi\rho$  by defining  $\xi$  as an a column vector of integral operators, with  $\xi_j = \delta_{1j}K_2$ . This is a natural extension of the previous notation. Since the equations are linear, the solution to the general problem is given by superposition as  $\hat{r} = r\rho$ ,  $r$  being the solution to the nearest-neighbor problem. For the lattice with equal filling, when  $b = \mathbb{1}$ ,  $r$  is just a vector of translationally invariant operators.

Examining the equations giving the energy and momentum, we see that they can be rewritten as  $P_{\text{latt}}/N = \rho^\dagger \zeta^\dagger b\hat{r} = \rho^\dagger \zeta^\dagger br\rho$  and  $E_{\text{latt}}/N = e^\dagger \xi^\dagger b\hat{r} = e^\dagger \xi^\dagger br\rho$ , so both can be expressed as quadratic forms in  $\rho$  and  $e$ . Here  $\zeta^\dagger \equiv 2\pi\xi$ . The concentrations  $n_j$  remain unchanged, due to the normalization of  $\rho$ . We see that singularities in the ground state only occur at surfaces of equal filling, the same as for the nearest-neighbor case. Again, taking the absolute ground state with equal filling, we can explicitly evaluate these expressions using the previous results, giving for the ground state energy

$$E_{\text{latt}}/N = -\int_{-A}^A e(x) dx \int_{-A}^A \rho(x') dx' \text{Re}[\psi(1/f + i(x - x')/f) - \psi(1 + i(x - x')/f)]/\pi f. \quad (10)$$

The ground state momentum is, of course, zero.

For the low-lying excitations, one calculates the shift in the asymptotic momenta, weighted by the ground state distributions, and this quantity does not depend on the momentum function  $p(\alpha)$ . Thus, the only change for the general case comes from the expressions

for the energy and momentum, so we can write for the excitations  $\Delta P_{\text{latt}} = \int_{-A}^A \Delta P_j(\alpha - x)\rho(x) dx$  and  $\Delta E_{\text{latt}} = -\int_{-A}^A \Delta E_j(\alpha - x)e(x) dx/2\pi$ , where  $\Delta P_j(\alpha)$  and  $\Delta E_j(\alpha)$  are the nearest-neighbor expressions of Eq. (9). The corresponding velocities for these gapless excitations are all identical and equal to  $v =$

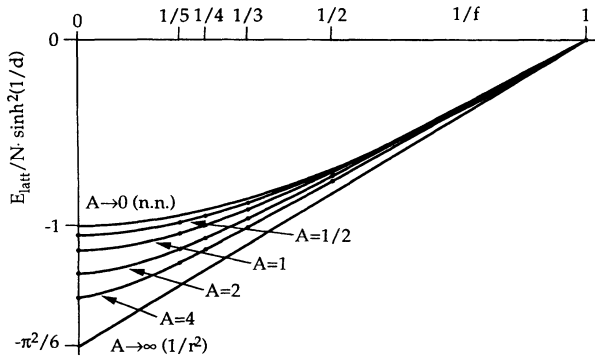


FIG. 1. The ground state energy  $E_{\text{latt}}/N$  is shown as a function of the number of species of fermions  $f$ , for representative values of the parameter  $A$ .

$\int_{-A}^A \exp(2\pi x/f) |e(x)| dx / f \int_{-A}^A \exp(2\pi x/f) \rho(x) dx$ . This expression is very much like that of a recently solved model for a magnetic fluid, which has the Heisenberg-Ising model imbedded in a continuum system [11].

In Figs. 1 and 2, we show the ground state energy and velocity, measured in terms of the nearest-neighbor

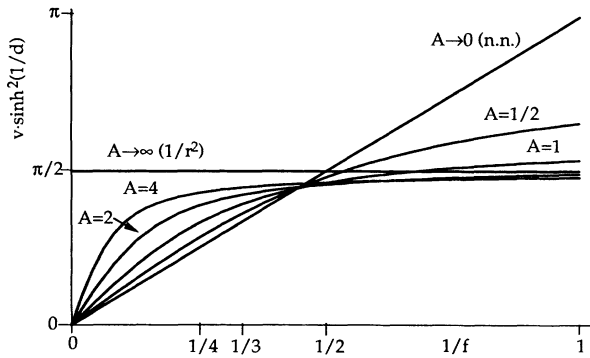


FIG. 2. The hydrodynamic velocity  $v$  is shown as a function of the number of species of fermions  $f$ , for representative values of the parameter  $A$ .

interaction energy  $\sinh^2(1/d)$ , as functions of the number of species of fermions  $f$ , for several typical densities  $d$ , and hence various ranges for the hyperbolic interaction. In both figures, we have taken as examples, limits of  $A = 1/2, 1, 2, 4$  corresponding to densities of  $d = 0.6735, 1.0388, 1.7293,$  and  $3.0596$ , respectively. In addition, we have the limit  $A \rightarrow 0, d \rightarrow 0$ , corresponding to the nearest-neighbor interaction, and the limit  $A \rightarrow \infty, d \rightarrow \infty$ , corresponding to the  $1/r^2$  interaction.

The susceptibility can be written using the above expression for  $v$  as  $\Delta E/N \approx \frac{\pi v}{2} \sum_{j=1}^f \delta n_j^2 \equiv \frac{\chi^{-1}}{2} \sum_{j=1}^f \delta n_j^2$ . Finally, by a thermodynamic argument [12], if we introduce fluxes  $\Phi_j = N\phi_j$  conjugate to the charge of type  $j$ , then  $\Delta E/N \approx \frac{v}{2\pi} \sum_{j=1}^f \delta \phi_j^2 \equiv \frac{D}{2} \sum_{j=1}^f \delta \phi_j^2$ , so the charge stiffness  $D$  obeys  $D\chi^{-1} = v^2$ .

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- [1] B. Sutherland and B. S. Shastry, Phys. Rev. Lett. **71**, 5 (1993).
- [2] A. P. Polychronakos, Phys. Rev. **69**, 703 (1992).
- [3] B. Sutherland, Rocky Mtn. J. Math. **8**, 431 (1978).
- [4] B. Sutherland, J. Math. Phys. **12**, 246 (1971); **12**, 251 (1971); Phys. Rev. A **4**, 2019 (1971); **5**, 1372 (1972).
- [5] A. P. Polychronakos, Phys. Rev. Lett. **70**, 2329 (1993).
- [6] B. Sutherland, Phys. Rev. B **12**, 3795 (1975).
- [7] C. K. Lai, J. Math. Phys. **15**, 1675 (1974).
- [8] B. S. Shastry, Phys. Rev. Lett. **60**, 639 (1988).
- [9] F. D. M. Haldane, Phys. Rev. Lett. **60**, 635 (1988).
- [10] V. I. Inozemtsev, J. Stat. Phys. **59**, 1143 (1990); M. Fowler and J. A. Minahan, Phys. Rev. Lett. **70**, 2325 (1993).
- [11] B. Sutherland and R. Römer, Phys. Rev. Lett. **71**, 2789 (1993).
- [12] B. Sutherland (to be published).