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Conservation laws in the continuum $1/r^2$ systems

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Abstract. We study the conservation laws of both the classical and the quantum mechanical continuum $1/r^2$ type systems. For the classical case, we introduce new integrals of motion along the recent ideas of Shastry and Sutherland (SS), supplementing the usual integrals of motion constructed much earlier by Moser. We show by explicit construction that one set of integrals can be related algebraically to the other. The difference of these two sets of integrals then gives rise to yet another complete set of integrals of motion. For the quantum case, we first need to resum the integrals proposed by Calogero, Marchioro and Ragnisco. We give a diagrammatic construction scheme for these new integrals, which are the quantum analogues of the classical traces. Again we show that there is a relationship between these new integrals and the quantum integrals of SS by explicit construction. Finally, we go to the asymptotic or low-density limit and derive recursion relations of the two sets of asymptotic integrals.

1. Introduction

The integrability of both the classical and the quantum one-dimensional problem of N particles interacting via the two-body potentials $V_0(x) = g^2/x^2$, $V_t(x) = g^2\Phi^2 \sin^{-2}[\Phi x]$ and $V_h(x) = g^2\Phi^2 \sinh^{-2}[\Phi x]$ has been shown more than two decades ago by Moser [1] (for the classical problem) and Calogero, Marchioro and Ragnisco (CMR) [2] (for the quantum problem), both groups exploiting a technique due to Lax [3]. These early results have been reviewed, extended and collected nicely both for the classical and the quantum cases by Olshanetsky and Perelomov in [4, 5].

For the classical systems, integrability restricts the motion in terms of action-angle variables onto a torus in phase space. However, for the quantum case, integrability leads to solvability only for those special cases which support scattering, i.e. systems which fly apart when the walls of the box are removed. In these cases, integrability implies conservation of individual momenta and thus the wavefunction is given asymptotically by Bethe's ansatz. For the above interaction potentials, Sutherland [6] has exploited this fact to determine the properties of the quantum systems in the thermodynamic limit.

Recently, Shastry and Sutherland (SS) have given an independent proof of integrability of the quantum many-body problem and constructed new integrals of motion [7]. However, for any finite number of particles N, we know that in principle we have exactly N conserved quantities. Therefore we expect the new integrals of motion to be related to the integrals constructed by CMR. It is the aim of the present work to elucidate some of the features

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of the new integrals of motion and to show their relation to the integrals of CMR. We emphasize that this new proof of integrability has also made possible the application of the ideas of the asymptotic Bethe ansatz to the $1/r^2$ models with quantum exchange [8].

In section 2 we show that the new construction of SS gives integrals of motion also for the classical problem. We next explicitly calculate these new integrals up to $n = 4$ and compare them to the integrals of CMR. This then gives rise to yet another set of integrals K_n. Section 3 is devoted to a comparison of the two series of integrals of motion for the quantum case. The integrals given by CMR are not extensive quantities and we need to resum them via an application of the linked cluster theorem. In section 4, we take the asymptotic or low-density limit of the problem and section 5 summarizes and discusses our results.

2. The classical case

The Hamiltonian of primary interest for our present work is given as

$$H = \sum_i p_i^2 + \lambda(\lambda-1)\Phi^2 {\sum_{ij}}' \sinh^{-2}[\Phi(x_i - x_j)]. \tag{1}$$

The interaction term reduces to V_0 in the limit of high densities (or $\Phi \to 0$) and the trigonometric interaction V_{t} is just the analytic continuation of $\Phi \to \mathrm{i}\Phi$. Here and in the following, we will use the primed sum $\sum'$ to indicate that the summation runs over unequal indices only.

2.1. Moser's invariants

Let us briefly recall the method of [1, 3]. We introduce the Lax pair L, M,

$$L_{jk} = p_j\delta_{jk} + \mathrm{i}(1-\delta_{jk})\sqrt{\lambda(\lambda-1)}\Phi \coth[\Phi(x_j - x_k)] \tag{2}$$

$$M_{jk} = 2\sqrt{\lambda(\lambda-1)}\Phi^2\left[\delta_{jk}{\sum_l}' \sinh^{-2}[\Phi(x_j - x_l)] + (1-\delta_{jk})\sinh^{-2}[\Phi(x_j - x_k)]\right]. \tag{3}$$

The classical equations of motion then imply the matrix equation

$$\frac{\mathrm{d}L}{\mathrm{d}t} = \{L, H\} = \mathrm{i}[ML - LM] \tag{4}$$

where we define the Poisson brackets as

$$\{F, G\} = \sum_{j=1}^{N} \frac{\partial F}{\partial x_j}\frac{\partial G}{\partial p_j} - \frac{\partial F}{\partial p_j}\frac{\partial G}{\partial x_j}.$$

The time evolution of L consequently is an isospectral deformation,

$$L(t) = \exp\left[\mathrm{i}\int_0^t M(\tau)\,\mathrm{d}\tau\right]L(0)\exp\left[-\mathrm{i}\int_0^t M(\tau)\,\mathrm{d}\tau\right]$$

and the integrals of motion are simply given as the traces

$$T_n = \mathrm{Tr}\, L^n(t) = \mathrm{Tr}\, L^n. \tag{5}$$

We also need to show that the T_n's are in involution, for example, $\{T_n, T_m\} = 0$. Using the Jacobi relation for Poisson brackets, we see that

$$\{H, \{T_n, T_m\}\} = \{T_n, \{H, T_m\}\} - \{T_m, \{H, T_n\}\} \tag{6}$$

and thus $\{T_n, T_m\}$ is also an integral of motion. But, allowing the system to evolve in time, all particles scatter, the Lax matrix itself evolves into

$$L \xrightarrow{t\to\infty} L^\infty = \begin{cases} L_{jj} = k_j & \\ L_{jk} = +\mathrm{i}\sqrt{\lambda(\lambda-1)} & j > k \\ L_{jk} = -\mathrm{i}\sqrt{\lambda(\lambda-1)} & j < k \end{cases} \tag{7}$$

and so the coordinate dependence vanishes. Thus the Poisson bracket $\{T_n, T_m\}$ evaluates to zero. We remark that this is the procedure that we use to prove involution for all the integrals constructed in the following chapters.

Let us define $\alpha_{jk} = \sqrt{\lambda(\lambda-1)}\Phi \coth[\Phi(x_j - x_k)]$ and $\alpha_{jj} = 0$. Then a direct calculation of the integrals of motion up to $n = 4$ gives

$$T_1 = \sum_i p_i = P \tag{8a}$$

$$T_2 = \sum_i p_i^2 + \sum_{ij}{}' \alpha_{ij}^2 \tag{8b}$$

$$T_3 = \sum_i p_i^3 + 3\sum_{ij}{}' \alpha_{ij}^2 p_i \tag{8c}$$

$$T_4 = \sum_i p_i^4 + 2\sum_{ij}{}' \alpha_{ij}^2 (p_j^2 + p_i p_j + p_i^2) + \mathrm{Tr}\,\alpha^4. \tag{8d}$$

Using

$$\alpha_{ij}^2 = \Phi\sqrt{\lambda(\lambda-1)}[\sinh^{-2}[\Phi(x_j - x_k)] + 1]$$

we see that

$$T_2 = H + \Phi^2\lambda(\lambda-1)N(N-1).$$

Note that due to the antisymmetry $\alpha_{ij} = -\alpha_{ji}$, only even powers of α—and thus integer powers of λ—will appear in all these expressions.

Let us now define the classical down-boost [4]

$$X = \sum_{j=1}^{N} x_j. \tag{9}$$

We then find easily that

$$\{X, T_n\} = nT_{n-1}. \tag{10}$$

Further, Jacobi's identity gives

$$\{X, \{T_n, T_m\}\} = (n-1)\{T_{n-1}, T_m\} + (m-1)\{T_{m-1}, T_n\}. \tag{11}$$

As a particular case, suppose $n = 2$, so $n - 1 = 1$ and $T_{n-1} = P$. Then by translation invariance $\{P, T_n\} = 0$, so we conclude that if $\{H, T_n\} = 0$, then $\{H, T_{n-1}\} = 0$. In particular, $\{H, T_N\} = 0$ implies that all T_n are integrals. Finally, we may construct all integrals of motion from T_N by repeatedly using the boost X in the representation

$$X = \sum_{j=1}^{N} \frac{\partial}{\partial p_j}. \tag{12}$$

2.2. Shastry's invariants

In [7], Shastry and Sutherland provide a set of integrals of motion for the quantum problem. However, mimicking their arguments, we can straightforwardly construct integrals of motion for the classical case, too. Let us introduce the singular matrix $\Delta_{jk} = 1$ for all i, j and the vector $\eta_j = 1$ for all j. Then we define integrals of motion such that

$$J_n = \mathrm{Tr}[L^n(t)\Delta] = \eta^\dagger L^n(t)\eta = \sum_{i_1,i_2,\ldots,i_{n+1}} L_{i_1i_2}L_{i_2i_3}\cdots L_{i_{n-1}i_n}L_{i_ni_{n+1}} \tag{13}$$

with the Lax matrix L given as before. We then have

$$\frac{\mathrm{d}J_n}{\mathrm{d}t} = \frac{\mathrm{d}}{\mathrm{d}t}\{\mathrm{Tr}[\exp[\mathrm{i}Mt]L^n(t)\exp[-\mathrm{i}Mt]\Delta]\} \tag{14}$$

$$= \mathrm{i}\,\mathrm{Tr}[ML^n(t)\Delta - L^n(t)M\Delta] \tag{15}$$

$$= \mathrm{i}\left[\mathrm{Tr}[L^n(t)\Delta M] - \mathrm{Tr}[L^n(t)M\Delta]\right] \tag{16}$$

$$= 0 \tag{17}$$

since $M\Delta = \Delta M = 0$ as shown in SS. Involution for these integrals of motion is proven by the same asymptotic argument as before. A direct calculation of the conserved quantities of SS up to $n = 4$ gives

$$J_1 = \sum_i p_i \tag{18a}$$

$$J_2 = \sum_i p_i^2 + {\sum_{ij}}' \alpha_{ij}^2 - {\sum_{ijk}}' \alpha_{ij}\alpha_{jk} \tag{18b}$$

$$J_3 = \sum_i p_i^3 + 3{\sum_{ij}}' \alpha_{ij}^2 p_i - {\sum_{ijk}}' \alpha_{ij}\alpha_{jk}(p_i + p_j + p_k) \tag{18c}$$

$$J_4 = \sum_i p_i^4 + 2{\sum_{ij}}' \alpha_{ij}^2[p_i^2 + p_ip_j + p_j^2] + \mathrm{Tr}\,\alpha^4 + \sum_{i\neq j\neq k\neq l\neq m\neq i} \alpha_{ij}\alpha_{jk}\alpha_{kl}\alpha_{lm} - {\sum_{ijk}}' \alpha_{ij}\alpha_{jk}[p_i^2 + p_j^2 + p_k^2 + p_ip_j + p_jp_k + p_kp_i]. \tag{18d}$$

Again the Hamiltonian can be found in the $n = 2$ term, $J_2 = H + \Phi^2\lambda(\lambda-1)N(N^2-1)/3$ and again only even powers of α appear in the expressions of the J_n's.

The action of the down-boost on these new integrals of motion is as in equation (10), for example, $\{X, J_n\} = nJ_{n-1}$. Much more useful is the up-boost Y which we define as

$$Y = \sum_i x_i p_i^2 + {\sum_{ij}}' (x_i + x_j)\alpha_{ij}^2/2 \tag{19}$$

in analogy with the up-boost operator $\sum_n n\boldsymbol{S}_n\boldsymbol{S}_{n+1}$ in the Heisenberg model. Unfortunately, this up-boost only works, if we restrict ourselves to the potential V_0 such that $\alpha_{ij}^2 = \lambda(\lambda-1)/(x_i-x_j)^2$. In this case, we find by explicit construction that $\{Y, J_n\} = (n+1)J_{n+1}$. The Jacobi relation $\{J_m, \{Y, J_n\}\} = \{Y, \{J_m, J_n\}\} - \{J_n, \{J_m, Y\}\}$ now gives

$$(n+1)\{J_m, J_{n+1}\} = \{Y, \{J_m, J_n\}\} - (m+1)\{J_n, J_{m+1}\}. \tag{20}$$

Thus, if $\{J_m, J_n\} = 0$ and $\{J_{m+1}, J_n\} = 0$, we also have $\{J_m, J_{n+1}\} = 0$. We emphasize that the up-boost (19) seems to work only for the special potential V_0.

2.3. Relation between invariants

We can again use the Jacobi relation to show that the Poisson bracket $\{T_n, J_m\}$ is an integral of motion which, in the asymptotic limits, evaluates to zero. The difference between the integrals of motion of Moser and SS then gives rise to yet another set of constants,

$$K_n = J_n - T_n = \sum_{i_1 \neq i_{n+1}} L_{i_1 i_2} L_{i_2 i_3} \cdots L_{i_{n-1} i_n} L_{i_n i_{n+1}}. \tag{21}$$

Various terms in the J_n's can be simplified with the help of the coth rule,

$$\alpha_{ij}\alpha_{jk} + \alpha_{ij}\alpha_{ki} + \alpha_{jk}\alpha_{ki} = -\Phi^2\lambda(\lambda - 1) \tag{22}$$

and hence we find

$$K_1 = 0 \tag{23a}$$

$$K_2 = \Phi^2\lambda(\lambda - 1)N(N-1)(N-2)/3 \tag{23b}$$

$$K_3 = \Phi^2\lambda(\lambda - 1)(N-1)(N-2)P \tag{23c}$$

$$K_4 = \Phi^2\lambda(\lambda - 1)(N-2)[(N-2)T_2 + P^2] + [\Phi^2\lambda(\lambda - 1)(N-1)(N-2)]^2/9. \tag{23d}$$

Note that K_3 is the first term that is not a simple constant, and in order to make the K_n's a complete set of integrals of motion, we may simply use K_{N+1} and K_{N+2}. Thus we conclude that by construction, we can express Shastry's integrals of motion in terms of Moser's and *vice versa*. We emphasize that this relationship is not linear, but only algebraic as seen from the existence of the P^2 term in K_4.

Taking the limit $\Phi \to 0$, we see that the K_n's are zero. Thus only for the simplest case of the Calogero potential $V_0(x) = g^2/x^2$ do we find that the Moser set of integrals of motion is identical to the set of SS.

3. The quantum case

In the quantum case, the elements of the Lax L and M matrices become operators themselves, i.e. the momentum operator is $p_j = -\mathrm{i}\partial/\partial x_j$ and we have the commutation relation $[x_j, p_k] = \mathrm{i}\delta_{jk}$. Since operator elements do not necessarily commute, we always mean an ordered product of elements when we multiply matrices in the following.

3.1. Calogero's invariants

The early work of Calogero *et al* [2] quantized the classical Lax equation, by antisymmetrizing the right-hand side of equation (4). The proof of invariance of the traces then does no longer hold. However, CMR also showed that after replacing the classical variables with the corresponding quantum mechanical operators, we can define new integrals of motion I_n such that

$$\Delta(\beta) \equiv \det[1 - \beta L] \equiv 1 + \sum_{n=1}^{N} (-\beta)^n I_n. \tag{24}$$

CMR then go on to argue that these I_n are conserved, $[I_n, H] = 0$, and in involution, $[I_n, I_m] = 0$. The later result is again proved† by use of the asymptotic limit as in the last

† We would still need to prove that, for $t \to \pm\infty$, L takes the form of L^∞, at least in the sense of weak convergence. However, we have explicitly checked the commutation relations of the I_n's for up to $N = 5$ particles and $n = 5$.

section. A direct calculation of the conserved quantities of CMR up to $n = 5$ for H yields

$$I_1 = \sum_i p_i \tag{25a}$$

$$I_2 = \tfrac{1}{2}I_1^2 - \tfrac{1}{2}\left[\sum_i p_i^2 + \sum_{ij}{}' \alpha_{ij}^2\right] \tag{25b}$$

$$I_3 = \tfrac{1}{6}\sum_{ijk}{}' p_i p_j p_k - \tfrac{1}{2}\sum_{ijk}{}' \alpha_{jk}^2 p_i \tag{25c}$$

$$I_4 = \tfrac{1}{4!}\sum_{ijkl}{}' p_i p_j p_k p_l - \tfrac{1}{4}\sum_{ijkl}{}' \alpha_{ij}^2 p_k p_l - \tfrac{1}{4}\sum_{ijkl}{}' \alpha_{ij}\alpha_{jk}\alpha_{kl}\alpha_{li} + \tfrac{1}{8}\sum_{ijkl}{}' \alpha_{ij}^2\alpha_{kl}^2 \tag{25d}$$

$$I_5 = \tfrac{1}{5!}\sum_{ijklm}{}' p_i p_j p_k p_l p_m - \tfrac{1}{12}\sum_{ijklm}{}' \alpha_{ij}^2 p_k p_l p_m - \tfrac{1}{4}\sum_{ijklm}{}' \alpha_{ij}\alpha_{jk}\alpha_{kl}\alpha_{li} p_m + \tfrac{1}{8}\sum_{ijklm}{}' \alpha_{ij}^2\alpha_{kl}^2 p_m. \tag{25e}$$

Note that the Hamiltonian can be found in the term in parenthesis in I_2.

Let us define a quantum down-boost operator analogous to the classical boost [5]. With $X = \sum_{j=1}^N x_j$ as before, we then find

$$[X, I_m] = \mathrm{i}(N - m + 1)I_{m-1}. \tag{26}$$

Using Jacobi's identity for commutators, we can easily show that as previously, $[H, I_n] = 0$ implies $[H, I_{n-1}] = 0$ and thus $[H, I_N] = 0$ implies all I_n are integrals. A particularly nice result is to write $I_N = \det L$, treat the momenta p_j as classical c-numbers since there are no ordering ambiguities, and use the representation

$$X = \sum_{j=1}^N i\frac{\partial}{\partial p_j} \tag{27}$$

to generate all I_n in the quantum case.

Of special importance in the following will be that, as in the classical invariants by Moser, α will only appear in even powers in the I_n's. Therefore, λ will occur with integer powers only and terms such as $[\lambda(\lambda - 1)]^{3/2}$ do not exist.

3.2. Shastry's invariants

In [7], Shastry and Sutherland provide a proof of integrability in the quantum case via an entirely different method. The Hamiltonian H is given as before but the Lax matrices now read

$$L_{jk}^{\mathrm{SS}} = p_j\delta_{jk} + i(1 - \delta_{jk})\lambda\Phi\coth[\Phi(x_j - x_k)] \tag{28}$$

$$\equiv p_j\delta_{jk} + \mathrm{i}(1 - \delta_{jk})\chi_{jk} \tag{29}$$

$$M_{jk}^{\mathrm{SS}} = 2\lambda\Phi^2\left[\delta_{jk}\sum_l{}' \sinh^{-2}[\Phi(x_j - x_l)] + (1 - \delta_{jk})\sinh^{-2}[\Phi(x_j - x_k)]\right] \tag{30}$$

with $\chi_{ii} = 0$. SS define their conserved quantum integrals of motion as in equation (18), for example, $J_n = \eta^\dagger(L^{\mathrm{SS}})^n\eta$. The new Lax matrices obey the ordered Lax equation

$$[L^{\mathrm{SS}}, H] = M^{\mathrm{SS}}L^{\mathrm{SS}} - L^{\mathrm{SS}}M^{\mathrm{SS}} \tag{31}$$

and we may easily prove invariance via

$$[J_n, H] = \eta^\dagger[(L^{\mathrm{SS}})^n, H]\eta = \eta^\dagger[M^{\mathrm{SS}}(L^{\mathrm{SS}})^n - (L^{\mathrm{SS}})^n M^{\mathrm{SS}}]\eta = 0 \tag{32}$$

since as before $\eta^\dagger M^{\mathrm{SS}} = M^{\mathrm{SS}}\eta = 0$. A direct calculation of the conserved quantities of SS up to $n = 4$ yields

$$J_1 = \sum_i p_i \tag{33a}$$

$$J_2 = \sum_i p_i^2 + {\sum_{ij}}'(\chi_{ij}^2 + \chi'_{ij}) - {\sum_{ijk}}'\chi_{ij}\chi_{jk} \tag{33b}$$

$$J_3 = \sum_i p_i^3 + 3{\sum_{ij}}'(\chi_{ij}^2 + \chi'_{ij})p_i - {\sum_{ijk}}'\chi_{ij}\chi_{jk}(p_i + p_j + p_k) \tag{33c}$$

$$\begin{aligned} J_4 = \sum_i p_i^4 &+ 2{\sum_{ij}}'(\chi_{ij}^2 + \chi'_{ij})[p_i^2 + p_i p_j + p_j^2] + \sum_{i\neq j\neq k\neq l\neq m\neq i} \chi_{ij}\chi_{jk}\chi_{kl}\chi_{lm} \\ &+ \mathrm{Tr}\,\chi^4 - {\sum_{ijk}}'\chi_{ij}\chi_{jk}[p_i^2 + p_j^2 + p_k^2 + p_i p_j + p_j p_k + p_k p_i] \\ &+ 2\mathrm{i}{\sum_{ij}}'\chi''_{ij}p_j + 4\mathrm{i}{\sum_{ij}}'\chi_{ij}\chi'_{ij}p_j + \mathrm{i}{\sum_{ijk}}'\chi_{ij}\chi'_{jk}(p_j - p_k) \\ &- {\sum_{ij}}'\chi'''_{ij} - 2\sum_{ij}\chi_{ij}\chi''_{ij} + 2{\sum_{ijk}}'\chi_{ij}\chi''_{jk} - {\sum_{ij}}'(\chi'_{ij})^2 + {\sum_{ijk}}'\chi'_{ij}\chi'_{jk} \\ &+ 3{\sum_{ijk}}'\chi_{ij}^2\chi'_{jk} + 2{\sum_{ij}}'\chi_{ij}^2\chi'_{ij} - {\sum_{ijkl}}'[\chi_{ij}\chi'_{jk}\chi_{kl} + 2\chi_{ij}\chi_{jk}\chi'_{kl}] + {\sum_{ijk}}'\chi_{ij}\chi_{jk}\chi'_{ki}. \end{aligned} \tag{33d}$$

The derivative χ'_{jk} is defined by the commutator $[p_j, \chi_{jk}^{(n)}] \equiv -\mathrm{i}\chi_{jk}^{(n+1)}$. See the appendix for an explicit list of derivatives.

Using $\chi'_{ij} = -\Phi^2\lambda\sinh^{-2}[\Phi(x_i - x_j)]$, we see that just as in the classical case J_2 contains the Hamiltonian, i.e. $J_2 = H + \Phi^2\lambda^2 N(N^2 - 1)/3$. However, the interaction strength $\lambda(\lambda - 1)$ in the Hamiltonian could only be obtained with the modified form of the Lax matrix L^{SS}. Also, the λ dependence of the constant term in the above equation is different from its classical counterpart. We remark that the last terms in equation (33*b*) and (33*c*) can again be written as a constant and a constant $\times \sum_i p_i$ by the coth rule of equation (22).

The down-boost operator acts as before, for example, $[X, J_n] = \mathrm{i}nJ_{n-1}$. In case of the potential V_0, we may also use the up-boost of equation (19) in operator form as

$$Y = \sum_i (x_i p_i^2 + p_i^2 x_i)/2 + {\sum_{ij}}'(x_i + x_j)\alpha_{ij}^2/2. \tag{34}$$

Then $[Y, J_n] = \mathrm{i}(n+1)J_{n+1}$ and we again have from the Jacobi identity

$$\mathrm{i}(n+1)[J_m, J_{n+1}] = [Y, [J_m, J_n]] - \mathrm{i}(m+1)[J_n, J_{m+1}] \tag{35}$$

so if $[J_m, J_n] = 0$ and $[J_{m+1}, J_n] = 0$, this then implies $[J_m, J_{n+1}] = 0$. We remark that an operator similar to our up-boost operator Y, which we constructed in analogy to the boost in the Heisenberg model, has been found previously by Wadati *et al* in the context of an investigation of the systems with algebraic potential V_0 [9].

Finally, we note another interesting property of these integrals of motion. Let Ψ_0 denote the ground state of the model, then it has been shown in [10] that $\sum_j L_{ij}^{\mathrm{SS}}\Psi_0 = 0$ for all $i = 1, \ldots, N$. Therefore, we see that

$$\Psi_0^\dagger J_n \Psi_0 = 0 \tag{36}$$

for all n. Thus all the J_n's somehow know about the ground state and subtract the appropriate expectation values, for example, the ground-state expectation value of the Hamiltonian is just the above constant, $\Phi^2\lambda^2 N(N^2-1)/3$.

3.3. Perturbation theory in the Lax matrices

Looking at equation (25), we see that each I_n, $n > 1$ in fact contains various powers of I_1. Furthermore, in the thermodynamic limit, the I_n's are not extensive quantities. Thus the situation seems to be similar to the usual problem of *connected* and *disconnected* pieces of diagrams encountered in perturbation theory. In brief, CMR's I_n seems to contain disconnected pieces and we hope that by a linked cluster expansion, we can write new integrals of motion with connected graphs only.

Let us be specific. With the help of the fermionic coherent path integral [11], we may rewrite the determinant

$$\Delta(\beta) = \det[1-\beta L] \tag{37}$$

$$= \int \prod_a \mathrm{d}c_a^* \,\mathrm{d}c_a \exp\left[-\sum_{jk} c_j^*[\delta_{jk}-\beta L_{jk}]c_k\right] \tag{38}$$

$$= \int \prod_a \mathrm{d}c_a^* \,\mathrm{d}c_a \exp\left[-\beta\sum_{jk} c_j^*(\delta_{jk}/\beta)c_k - c_j^* L_{jk}c_k\right] \tag{39}$$

where $c_a^*, c_a, a = 1,\ldots,N$ are Grassmann variables. Note first that we may write this expression both for a classical L *and* a quantum L. The fact that the elements of a quantum matrix will not necessarily commute with each other is taken care of by the Grassmann nature of the integration: each momentum p_i will only encounter indices $j \neq i$, otherwise the integration measure will have expressions such as $c_i c_i$ or $c_i^* c_i^*$ which are zero.

When we now include a dummy time dependence for the Grassmann variables, i.e. $c_a^{(*)} = c_a^{(*)}(t)$, we can write

$$\Delta(\beta) = \int \prod_a \mathrm{d}c_a^*(\tau)\,\mathrm{d}c_a(\tau)$$
$$\times \exp\left[-\int_0^\beta \mathrm{d}t\left(\sum_{jk} c_j^*(t)(\delta_{jk}/\beta)c_k(t) - c_j^*(t)L_{jk}c_k(t)\right)\right] \tag{40}$$

$$= \Delta_0\left\langle \exp\left[-\int_0^\beta \mathrm{d}t\left(\sum_{jk} -c_j^*(t)L_{jk}c_k(t)\right)\right]\right\rangle_0 \tag{41}$$

where the average is defined as

$$\langle F(c_a^*(t_i)c_b^*(t_j)\ldots c_g(t_k)c_h(t_l)\ldots)\rangle_0 = \frac{1}{\Delta_0}\int\prod_{a'} \mathrm{d}c_{a'}^*(\tau)\,\mathrm{d}c_{a'}(\tau)$$
$$\times \exp\left[-\int_0^\beta \mathrm{d}t\sum_{j'} c_{j'}^*(t)(1/\beta)c_{j'}(t)\right]F(c_a^*(t_i)c_b^*(t_j)\ldots c_g(t_k)c_h(t_l)\ldots). \tag{42}$$

This is very much like a path integral description of a many-body partition function Z. We further note that the interaction part $V = \sum_{jk} c_j^*(t)L_{jk}c_k(t)$ is just the super Lax operator $\mathcal{L}$ of SS.

The perturbation expansion is obtained by expanding equation (40) in a power series

$$\Delta(\beta)/\Delta_0 = \sum_{n=0}^{\infty} \frac{(-1)^n}{n!} \int_0^{\beta} \mathrm{d}t_1\, \mathrm{d}t_2 \ldots \mathrm{d}t_n$$
$$\times \left\langle \sum_{i_1 j_1} c^*_{i_1}(t_1) L_{i_1 j_1} c_{j_1}(t_1) \cdots \sum_{i_n j_n} c^*_{i_n}(t_n) L_{i_n j_n} c_{j_n}(t_n) \right\rangle_0 \tag{43}$$
$$\equiv \sum_{n=0}^{\infty} \frac{(-1)^n}{n!} \Delta_n \tag{44}$$

and $\Delta_n \sim \beta^n I_n$. The last equation is obtained by comparison with equation (24) and $\Delta_0 = 1$. Note that $I_n = 0$ in equation (44) for all $n > N$. For example, for $N = 2$, we have

$$I_3 \sim \sum_{i_1 j_1 i_2 j_2}^{N=2} \sum_{i_3 j_3}^{N=2} L_{i_1 j_1} L_{i_2 j_2} L_{i_3 j_3} \langle c^*_{i_1} c_{j_1} c^*_{i_2} c_{j_2} c^*_{i_3} c_{j_3} \rangle_0$$

and clearly i_3, j_3 always take index values already covered by $\{i_1, j_1, i_2, j_2\}$. Thus the bracket $\langle\,\rangle$ is zero by the Grassmann character of the c's.

Let us now calculate the first few orders of $\Delta(\beta)$. With g_i being a dummy propagator, we find

$$\Delta_1 = -\beta \sum_i L_{ii} g_i \tag{45a}$$

$$\Delta_2 = \tfrac{1}{2}\beta^2 \sum_{ij} (L_{ij} L_{ji} - L_{ii} L_{jj}) g_i g_j \tag{45b}$$

$$\Delta_3 = -\tfrac{1}{3!}\beta^3 \sum_{\substack{i_1,i_2,i_3,\\ j_1,j_2,j_3}} L_{i_1 j_1} L_{i_2 j_2} L_{i_3 j_3} \langle c^*_{i_1} c^*_{i_2} c^*_{i_3} c_{j_1} c_{j_2} c_{j_3} \rangle_0$$
$$= -\tfrac{1}{3!}\beta^3 \sum_{ijk} (L_{ik} L_{jj} L_{ki} - L_{ik} L_{ji} L_{kj} - L_{ij} L_{jk} L_{ki} + L_{ii} L_{jk} L_{kj}$$
$$+ L_{ij} L_{ji} L_{kk} - L_{ii} L_{jj} L_{kk}) g_i g_j g_k. \tag{45c}$$

Introducing the diagrammatic notation $i\bullet\!\longrightarrow j \equiv L_{ij}$, we can represent these expressions by their graphs as in figure 1. Note that only in equation (45c) do we need to worry about the ordering of the matrix products. If we ignore that ordering for the moment—the classical case—we have

$$\Delta_3 = -\tfrac{1}{3!}\beta^3 \sum_{ijk} [3L_{ii} L_{jk} L_{kj} - 2L_{ij} L_{jk} L_{ki} - L_{ii} L_{jj} L_{kk}] g_i g_j g_k. \tag{46}$$

We see that the second term in equation (46), representing the fully connected diagram, is actually just constant $\times$ $\mathrm{Tr}(L^3)$.

Let us see what these expressions tell us. (i) We see that in fact the Δ_n's are just the I_n's of CMR with $g_i = 1$. (ii) We observe that the fully connected diagrams give the traces of powers of L in the classical case as was expected from the well known matrix formula $\ln\det A = \mathrm{Tr}\ln A$. Thus these connected diagrams are the quantum analogue of the *classical* integrals of motion. (iii) The ordering of the matrix products becomes important for $n > 2$, thus necessitating order labelling of diagrams.

3.4. Constructing connected diagrams

We now want to rewrite the perturbation expansion (43) so that we only use fully connected diagrams, and we want to do this such that we can minimize the ordering problems coming

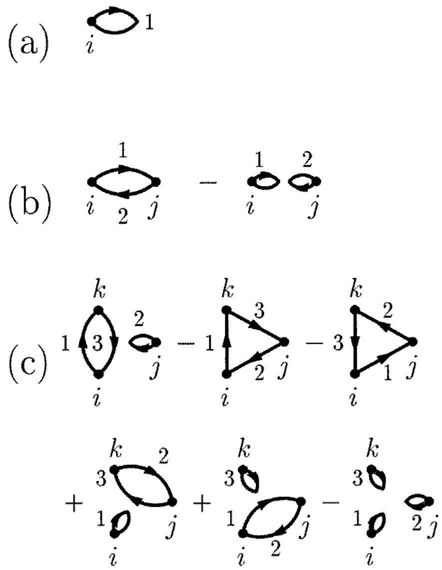

Figure 1. Diagrammatic representation of equation (45). Each line is labelled by a number indicating precedence in the corresponding matrix product. 1 corresponds to the right-most matrix.

from the quantum character of the Lax matrix. The basic program is due to Thiele and is known as the linked cluster theorem. It can be summarized as follows. We resum the series (44) as

$$\Delta(\beta) = \exp\left[-\sum_{n=1} \frac{\beta^n}{(n-1)!} T_n\right] \tag{47}$$

and use it to define the T_n's. Comparing equations (44) and (47), we find up to $n = 4$,

$$T_1 = I_1 \tag{48a}$$

$$= \sum_i p_i \tag{48b}$$

$$T_2 = I_1^2 - 2I_2 \tag{48c}$$

$$= \sum_i p_i^2 + {\sum_{ij}}' \alpha_{ij}^2 \tag{48d}$$

$$T_3 = I_1^3 - 3I_1I_2 + 3I_3 \tag{48e}$$

$$= \sum_i p_i^3 + 3{\sum_{ij}}' \alpha_{ij}^2 p_i \tag{48f}$$

$$T_4 = I_1^4 - 4I_2I_1^2 + 2I_2^2 + 4I_3I_1 - 4I_4 \tag{48g}$$

$$= \sum_i p_i^4 + 2{\sum_{ij}}' \alpha_{ij}^2[p_j^2 + p_i p_j + p_i^2] + \mathrm{Tr}\,\alpha^4 - 2{\sum_{ij}}' \alpha_{ij}''\alpha_{ij} - 2{\sum_{ij}}' (\alpha_{ij}')^2 - 4i{\sum_{ij}}' \alpha_{ij}'\alpha_{ij} p_i. \tag{48h}$$

Since CMR have already proven $[I_n, I_m] = 0$, there is no ordering problem for the I_n's in the construction of the T_n's and we, furthermore, have $[T_n, T_m] = 0$. Since T_2 is, up to a constant, the Hamiltonian, this implies both involution and invariance.

As expected, we find that each T_n corresponds to the fully connected diagrams of the series (44). We can now directly use the diagrammatic approach to construct the T_n's. However, for a given n, there are $(n-1)!$ different labelled diagrams and thus different matrix orderings. Each diagram itself is an ordered operator expression and it is quite tedious to get them into a form as in equation (48) with all momenta to the right. As an example, we give the diagrams for T_4 in figure 2.

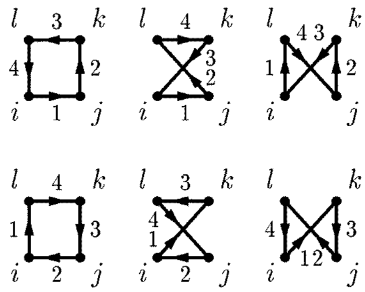

Figure 2. The six diagrams associated with T_4.

Ignoring matrix and quantum ordering, the resultant expressions for the T_n's are equal to the classically invariant traces of equation (8). Thus we may hope that due to the special form of the quantum Lax matrix L, the matrix product order somehow is unimportant and the T_n's are just the quantum traces $\mathrm{Tr}\, L^n$. Explicitly calculating the quantum traces, we find that indeed up to $n = 3$, we have $T_n = \mathrm{Tr}\, L^n$. However, for $n = 4$, the quantum trace includes the non-zero term $-2\sum_{ijk}{}' \alpha_{ij}\alpha_{jk}\alpha'_{ki}$. Note that this term has a factor $[\lambda(\lambda-1)]^{2/3}$†, but as shown in section 3.1, such a term does not arise in the I_n's and consequently also not in the T_n's. Therefore, the T_n's are not simply the quantum traces.

Note that these expressions are again very close to the ones obtained for J_n. However, as before in the classical case, we see that already for $n = 2, 3$, there are the same constants in the J expressions which do not appear in the T expressions.

3.5. Relation between invariants

We again would like to see if we can express Shastry's integrals in terms of our T_n's. As we have seen in section 2.3 for the classical case, we expect this relation to be algebraic. Fortunately, as shown in the last section, fractional powers of λ neither appear in the J_n's nor in the T_n's so that no *a priori* reasons forbid an algebraic relationship in the quantum case.

Furthermore, given two sets of integrals of motion $\{T_n\}$ and $\{J_n\}$, we know that commutators of integrals are themselves integrals of motion, and since asymptotically these integrals evaluate to zero, the two sets of integrals can be simultaneously diagonalized. A relationship between asymptotic integrals of the form $\mathcal{J}_n = A_n[\{\mathcal{T}_m\}]$ can always be found, since either set of integrals gives an algebraically complete set of symmetric polynomials of increasing degree. Suppose we have such a relationship. Then, the operators J_n and $A_n[\{T_m\}]$ have the same eigenvalues in the same basis, hence must be the same operator, and so there must exist a relationship $J_n = A_n[\{T_m\}]$ between the operators themselves.

† This term seems to be missing in [6].

Replacing α_{ij} and χ_{ij} by their appropriate definitions, using the explicit form for the derivatives as given in the appendix and counting powers of λ and p's, we then have

$$J_1 = T_1 \tag{49a}$$

$$J_2 = \Phi^2\lambda[3+\lambda(N-2)]N(N-1)/3 + T_2 \tag{49b}$$

$$J_3 = \Phi^2\lambda[3+\lambda(N-2)](N-1)T_1 + T_3 \tag{49c}$$

$$\begin{aligned} J_4 = {} & [-75+120\lambda-48\lambda^2+(55-110\lambda+52\lambda^2)N+(-5+25\lambda-18\lambda^2)N^2+2\lambda^2N^3] \\ & \times\lambda^2N(N-1)\Phi^4/15+[2+\lambda(N-2)]\lambda\Phi^2T_1^2 \\ & +[10-12\lambda+11(\lambda-1)N+(2\lambda-1)N^2]\lambda\Phi^2T_2+T_4. \end{aligned} \tag{49d}$$

Hence we have succeeded in writing Shastry's quantum integrals in terms of the T_n's which in turn are derived from Calogero's quantum integrals for up to $n = 4$. Again, as in the classical case, this relationship is not linear, since we observe the T_1^2 term in equation (49*d*), and only if we restrict ourselves to the potential $V_0(x) = g^2/x^2$, by taking the limit $\Phi \to 0$, do we find that both sets of integrals of motion are identical.

With Ψ_0 the N-particle ground state as before, we may use equation (36) and hence relate the expectation values of various T_n's. For example,

$$\Psi_0^\dagger T_2\Psi_0 = \Phi^2\lambda[3+\lambda(N-2)]N(N-1)/3$$

and

$$\Psi_0^\dagger T_3\Psi_0 = -\Phi^2\lambda[3+\lambda(N-2)](N-1)\Psi_0^\dagger T_1\Psi_0 \sim \Psi_0^\dagger P\Psi_0 = 0.$$

We further note that as in the classical case, we may define new non-trivial constants of motion $K_n = J_n - T_n$ for $\Phi \neq 0$. We then have $\Psi_0^\dagger K_n\Psi_0 = -\Psi_0^\dagger T_n\Psi_0$. Unfortunately, we can not give a simple formula directly in terms of the Lax matrices for the construction of the K_n's analogous to equation (21).

4. The asymptotic limit

In the asymptotic $t \to \infty$ limit—equivalent to the low-density limit—the elements of the Lax matrix are no longer operators, but numbers. Thus explicit calculations are much easier and we hope that we can study the connection between asymptotic Calogero and Shastry integrals of motion in more detail than in the last section.

4.1. Asymptotics of Calogero's integrals

The asymptotic form for the Lax matrix L gives a corresponding asymptotic form for the Calogero integrals $I_n \to \mathcal{I}_n$. We define a generating function of the asymptotic Calogero integrals,

$$\mathcal{D}[z,\lambda] = \tfrac{1}{2}\left[\prod_{j=1}^{N}(1-zp_j-\mathrm{i}\lambda z)+\prod_{j=1}^{N}(1-zp_j+\mathrm{i}\lambda z)\right] \tag{50}$$

and then have

$$\det[1-zL^\infty] = \mathcal{D}[z,\sqrt{\lambda(\lambda-1)}]. \tag{51}$$

We also define the elementary symmetric functions of the N variables p_j as

$$a_r[p] = \sum_{1\leqslant j_1<\ldots<j_r}^{N} p_{j_1}\cdots p_{j_r} \tag{52}$$

and for convenience $a_0 = 1$ and $a_{-r} = 0$. Then

$$\mathcal{D}[z,\lambda] = \sum_{r=0}^{\infty} (-\lambda^2 z^2)^r a_{N-2r}[1 - zp]. \tag{53}$$

In particular,

$$\mathcal{I}_N = a_N[p] - \lambda(\lambda-1)a_{N-2}[p] + \lambda^2(\lambda-1)^2 a_{N-4}[p] - \cdots \tag{54}$$

and the other integrals can be constructed via

$$\left[\sum_{k=1}^{N} \frac{\partial}{\partial p_k}, \mathcal{I}_j\right] = (N - j + 1)\mathcal{I}_{j-1}. \tag{55}$$

Since the elementary symmetric functions a_j obey the same relationship, this also gives us the expression for $\mathcal{I}_j$ as a linear combination of a_j, a_{j-2}, a_{j-4}, ... The general expression for $\mathcal{I}_j$ in terms of the $a_r[p]$ can be obtained using

$$a_r[1 - zp] = \sum_{j=0}^{N} (-z)^{r-j} \binom{N + j - r}{j} a_{r-j}[p] \tag{56}$$

and we find

$$\mathcal{I}_j = \sum_{j=0}^{N} [-\lambda(\lambda-1)]^r \binom{N + 2r - j}{2r} a_{j-2r}[p]. \tag{57}$$

4.2. Some generating functions

Let us define the quantity

$$\mathcal{Z}[z,\lambda] = \prod_{j=1}^{N} [1 - z(p_j + \mathrm{i}\lambda)] \equiv \mathcal{D}[z,\lambda] - \mathrm{i}\lambda z\mathcal{N}[z,\lambda]. \tag{58}$$

The symmetric part of $\mathcal{Z}$ is $\mathcal{D} = (\mathcal{Z}[z,\lambda] + \mathcal{Z}[z,-\lambda])/2$ as in the previous section, while the antisymmetric part $(\mathcal{Z}[z,\lambda] - \mathcal{Z}[z,-\lambda])/2$ is given by

$$\mathrm{i}\lambda z\mathcal{N}[z,\lambda] = \mathrm{i}\lambda z \sum_{r=0}^{\infty} (-\lambda^2 z^2)^r a_{N-2r-1}[1 - zp]. \tag{59}$$

For z real, these are the real and imaginary parts of $\mathcal{Z}$, respectively. This expression for $\mathcal{Z}$ is the standard generating function for the elementary symmetric functions, so that

$$\mathcal{Z}[z,\lambda] = \sum_{j=0}^{N} (-z)^j a_j[p + i\lambda]. \tag{60}$$

Clearly, then, we have $\mathcal{I}_j = \mathrm{Re}\{a_j[p + \mathrm{i}\sqrt{\lambda(\lambda-1)}]\}$.Taking the logarithm of $\mathcal{Z}$,

$$\ln \mathcal{Z}[z,\lambda] = \sum_{j=1}^{N} \ln[1 - z(p + j + i\lambda)] \tag{61}$$

one advantage of using the generating function $\mathcal{Z}$ is clear when one anticipates the thermodynamic limit. We now consider

$$\frac{\partial}{\partial z} \ln \mathcal{Z} = \sum_{j=1}^{N} -\frac{p_j + \mathrm{i}\lambda}{1 - z(p_j + \mathrm{i}\lambda)}. \tag{62}$$

This, however, is the generating function for the symmetric power sums

$$b_r[p] = \sum_{j=1}^{N} p_j^r \tag{63}$$

since

$$P[z] = \sum_{r=0}^{\infty} z^r b_{r+1}[p] = \sum_{j=1}^{N} \frac{p_j}{1 - z p_j} \tag{64}$$

and we see that

$$\frac{\partial}{\partial z} \ln \mathcal{Z}[z, \lambda] = \sum_{r=0}^{\infty} z^r b_{r+1}[p + \mathrm{i}\lambda] \equiv P[z, \lambda]. \tag{65}$$

4.3. Asymptotics of Shastry's integrals

Shastry's integrals also approach an asymptotic form $J_j \to \mathcal{J}_j$, and similarly we define an asymptotic generating function

$$\mathcal{G}[z, \lambda] \equiv \frac{\mathcal{N}[z, \lambda]}{\mathcal{D}[z, \lambda]} \tag{66}$$

then, we see that $\mathcal{N}[z, \lambda] = \mathcal{G}[z, \lambda]\mathcal{D}[z, \lambda]$, so expanding, we have

$$\sum_{k=1}^{N} (-z)^k \operatorname{Im}\{a_k[p + \mathrm{i}\lambda]\} = \lambda \sum_{k=1}^{\infty} \sum_{j=1}^{\infty} (-1)^{j-1} \mathcal{J}_{j-1} \operatorname{Re}\{a_{k-j}[p + \mathrm{i}\lambda]\}. \tag{67}$$

Equating powers of z, this gives

$$\operatorname{Im}\{a_k[p + \mathrm{i}\lambda]\} = \lambda \sum_{j=1}^{k} (-1)^{j-1} \mathcal{J}_{j-1} \operatorname{Re}\{a_{k-j}[p + \mathrm{i}\lambda]\}. \tag{68}$$

More explicitly,

$$\lambda^{-1} \operatorname{Im}\{a_k[p + \mathrm{i}\lambda]\} = N \operatorname{Re}\{a_k[p + \mathrm{i}\lambda]\} - \mathcal{J}_1 \operatorname{Re}\{a_{k-1}[p + \mathrm{i}\lambda]\} + \cdots + (-1)^k \mathcal{J}_k. \tag{69}$$

This allows us to iterate and find $\mathcal{J}_k$ in terms of the other $\mathcal{J}_j$,

$$\mathcal{J}_k = \mathcal{J}_{k-1} \operatorname{Re}\{a_1[p + \mathrm{i}\lambda]\} - \cdots - (-1)^k N \operatorname{Re}\{a_k[p + \mathrm{i}\lambda]\} + (-1)^k \lambda^{-1} \operatorname{Im}\{a_{k+1}[p + \mathrm{i}\lambda]\}. \tag{70}$$

Since $\mathcal{I}_j = \operatorname{Re}\{a_j[p + \mathrm{i}\lambda]\}$, this recursion relation also relates the asymptotic forms of the CMR and SS integrals. However, this relation is between asymptotic integrals with the same parameter, and such integrals do not even commute.

5. Conclusions

Our original intent and hope at taking up the present work was to give a simple connection between the integrals of motion of CMR and the recently discovered integrals of SS. In fact, we were speculating that due to the special structure of the quantum Lax matrix, we would simply find $T_n \sim J_n$.

Quite the opposite has happened. In the quantum case, although we have succeeded in rewriting the I_n integrals of CMR into extensive quantities T_n, we have, however, failed to give a simple formula for the connection of these T_n's to the J_n integrals of SS for all but the simplest potential $V_0(x) = g^2/x^2$. In general, we do find a complicated algebraic relationship which gives rise to yet another set of non-trivial integrals of motion K_n.

In the classical case, we show that the quantum definition of SS may also be used to construct classical integrals of motion. Hence here the situation now is just as in the quantum case and again we show that we may re-express the integrals of SS in terms of the classical integrals of Moser. Again, the difference of these two sets of integrals vanishes only for the potential V_0 and otherwise may be used to define new constants and this time, we can give an explicit formula for the K_n's directly in terms on the Lax L matrix. Indeed, we may even define a one-parameter family of integrals of motion in the classical case, for example,

$$R_n(\delta) = \mathrm{Tr}[L^n(1 + \delta\Delta')] \tag{71}$$

with $\Delta'_{ij} = 1$ if $i \neq j$ and zero otherwise. Then R_n interpolates between Moser's integrals $R_n(0) = T_n$ and the integrals of SS $R_n(1) = J_n$.

Most of the previous formulae are given in terms of α's and χ's and are thus valid not only for the hyperbolic pair potential V_{h}, but also for the trigonometric V_{t} and the rational V_0 after taking the appropriate limits as mentioned at the beginning of section 2. However, we have found an up-boost only for the classical and the quantum many-body system with algebraic potential $V_0(r) = g^2/r^2$. Also, the elliptic potential $V_{\mathrm{e}} = 1/\mathrm{sn}^2$, is not included in this study. Although the classical integrals of Moser and the quantum integrals of CMR are valid for this potential, the proof of SS no longer holds both for the classical and the quantum case. This is due to the fact that the row and column sums of the elliptic Lax M matrix are no longer zero. The ansatz $J'_n = \mathrm{Tr}\, L^n g[\{x_j\}]$ gives an equation for the coordinate dependent matrix g as $\partial/(\partial t)g = Mg - gM$. Unfortunately, we have not found a non-trivial g such that it gives the SS integrals of motion in the elliptic case.

Acknowledgment

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Appendix A. Derivatives

We have defined $\gamma_{ij} = \coth[\Phi(x_i - x_j)]$ and so $\chi_{ij} = \Phi\lambda\gamma_{ij}$ and $\alpha_{ij} = \Phi\sqrt{\lambda(\lambda-1)}\gamma_{ij}$. Then the derivatives are

$$\frac{\partial\chi_{ij}}{\partial x_i} = \chi'_{ij} = \Phi^2\lambda[1 - \gamma_{ij}^2] \tag{A1a}$$

$$\chi''_{ij} = -2\Phi^3\lambda\gamma_{ij}(1 - \gamma_{ij}^2) \tag{A1b}$$

$$\chi'''_{ij} = -2\Phi^4\lambda(1 - 4\gamma_{ij}^2 + 3\gamma_{ij}^4). \tag{A1c}$$

The same relations hold for α_{ij}, i.e.

$$\alpha'_{ij} = \Phi^2\sqrt{\lambda(\lambda-1)}[1 - \gamma_{ij}^2] \qquad \alpha''_{ij} = -2\Phi^3\sqrt{\lambda(\lambda-1)}\gamma_{ij}(1 - \gamma_{ij}^2)$$
$$\alpha'''_{ij} = -2\Phi^4\sqrt{\lambda(\lambda-1)}(1 - 4\gamma_{ij}^2 + 3\gamma_{ij}^4)$$

and, lastly,

$$\gamma'_{ij} = \Phi[1 - \gamma_{ij}^2], \gamma''_{ij} = -2\Phi\gamma_{ij}(1 - \gamma_{ij}^2) \qquad \gamma'''_{ij} = -2\Phi(1 - 4\gamma_{ij}^2 + 3\gamma_{ij}^4).$$

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