Theory of Extremely Correlated Fermions (I-II)

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Brainwashed by Feynman.
& Tyranny of Feynman diagrams

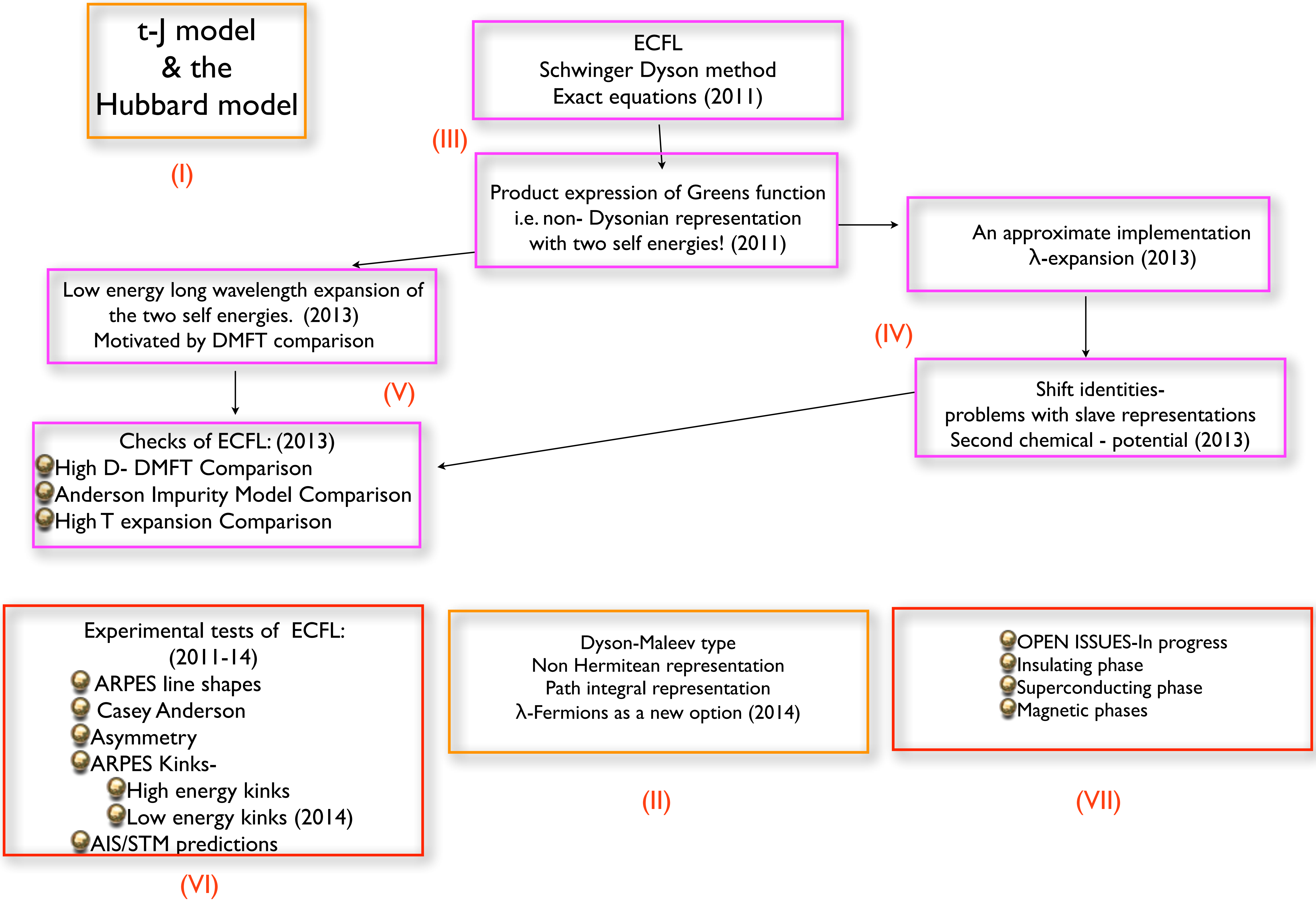
Physics Today
2000. (PWAnderson)
in describing the state
of many body physics

Brainwashed by Schwinger.
& Tyranny of Schwingers equations

Early versions of ECFL
may be appear to be
dominated by Schwinger
"sourcery"

Middle way possible
Dyson (Maleev) type theory
& Inconvenience of non Hermitean QFT

We need
a middle way to
enhance clairity
Our presentation here follows
this middle way


t-J model
& the
Hubbard model
(I)
ECFL
Schwinger Dyson method
Exact equations (2011)
(III)
Product expression of Greens function
i.e. non- Dysonian representation
with two self energies! (2011)
An approximate implementation
λ-expansion (2013)
Low energy long wavelength expansion of
the two self energies. (2013)
Motivated by DMFT comparison
(IV)
(V)
Shift identities-
problems with slave representations
Second chemical - potential (2013)
Checks of ECFL: (2013)
High D- DMFT Comparison
Anderson Impurity Model Comparison
High T expansion Comparison
Experimental tests of ECFL:
(2011-14)
ARPES line shapes
Casey Anderson
Asymmetry
ARPES Kinks-
High energy kinks
Low energy kinks (2014)
AIS/STM predictions
(VI)
Dyson-Maleev type
Non Hermitean representation
Path integral representation
λ-Fermions as a new option (2014)
(II)
OPEN ISSUES-In progress
Insulating phase
Superconducting phase
Magnetic phases
(VII)

Formalism papers

- `` Extremely Correlated Fermi Liquids '', B. S. Shastry, arXiv:1102.2858 (2011),Phys. Rev. Letts. 107, 056403 (2011); 108, 029702 (2012).

  `Anatomy of the Self Energy'', B. S. Shastry, arXiv:1104.2633; Phys. Rev. B 84, 165112 (2011); Phys. Rev. B 86, 079911(E) (2012).

  ``Extremely Correlated Fermi Liquids: Self consistent solution of the second order theory'', D. Hansen and B. S. Shastry, arXiv:1211.0594, (2012), Phys. Rev. B 87, 245101 (2013).

  ``Extremely Correlated Fermi Liquids: The Formalism'', B. S. Shastry, arXiv:1207.6826 (2012); Phys. Rev. B 87, 125124 (2013).

  ``ECFL in the limit of infinite dimensions'', E. Perepelitsky and B. S. Shastry, arXiv:1309.5373 (2013), Anns. Phys. 338, 283-301 (2013).

  Theory of extreme correlations using canonical Fermions and path integrals'', B. S. Shastry, arXiv:1312.1892 (2013), Ann. Phys. 343, 164-199 (2014).

Benchmarking papers

DMFT and AIM related papers

- ``Extremely correlated Fermi liquid theory meets Dynamical mean-field theory: Analytical insights into the doping-driven Mott transition'', R. Zitko, D. Hansen, E. Perepelitsky, J. Mravlje, A. Georges and B. S. Shastry, arXiv:1309.5284 (2013), Phys. Rev. B 88, 235132 (2013).
- ``Extremely Correlated Fermi Liquid study of the $U=\infty$ Anderson Impurity Model'', B. S. Shastry, E. Perepelitsky and A. C. Hewson, arXiv:1307.3492 [cond-mat.str-el], Phys. Rev. B 88, 205108 (2013).

High T Expansion comparison papers

- ` Linked-Cluster Expansion for the Green's function of the Infinite-U Hubbard Model'', E. Khatami, E. Perepelitsky, M. Rigol and B. S. Shastry, arXiv: 1310.8029 (2013).
- ``Electronic spectral properties of the two-dimensional infinite-U Hubbard model'', E. Khatami, D. Hansen, E. Perepelitsky, M. Rigol, B. S. Shastry, arXiv:1303.2657 [cond-mat.str-el] (2013), Phys. Rev. B 87, 161120 (R) (2013).

Experimentally relevant papers

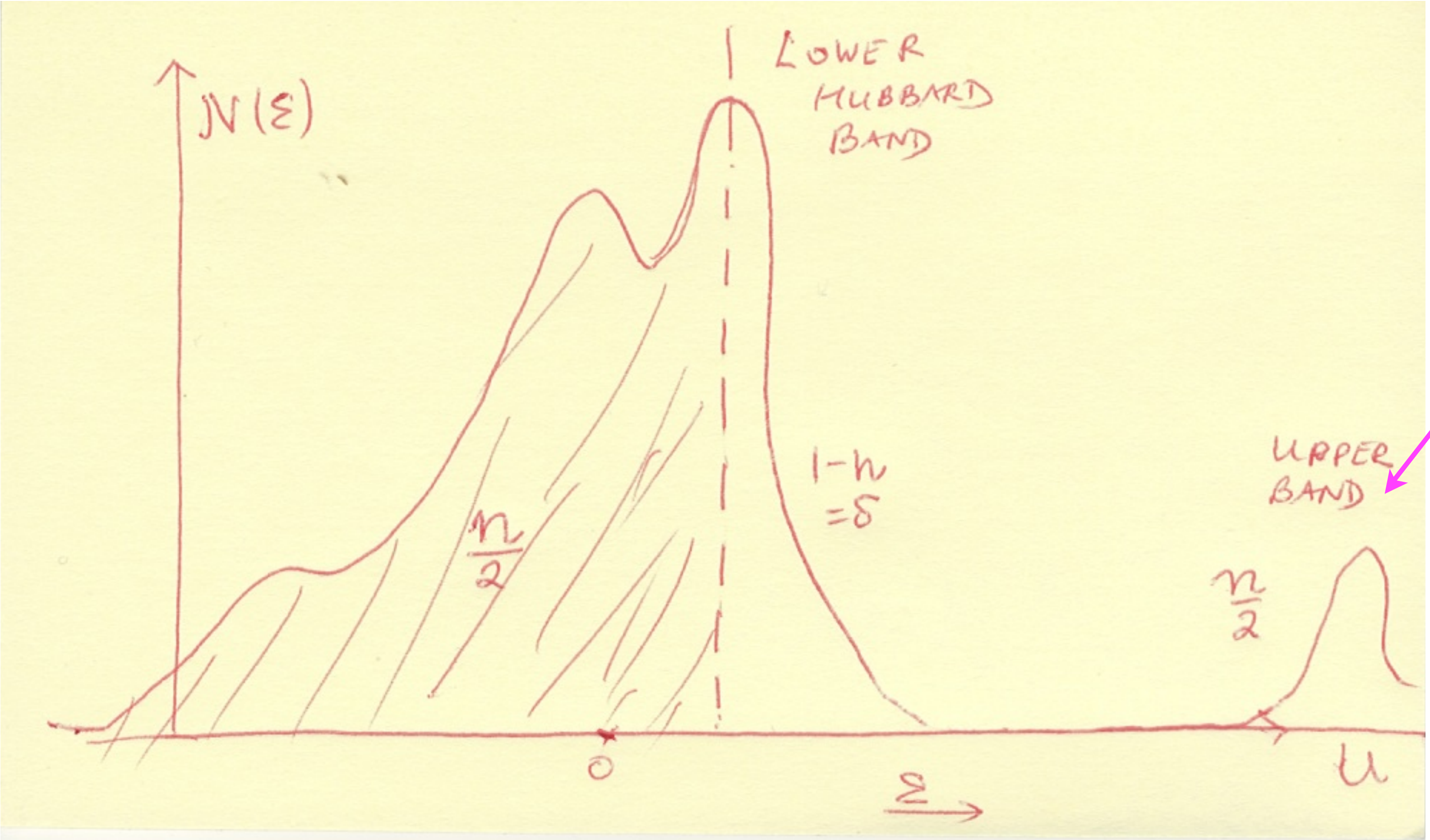
- `` Dynamical Particle Hole Asymmetry in Cuprate Superconductors", B. S. Shastry, arXiv:1110.1032 (2011), Phys. Rev. Letts. 109, 067004 (2012).
- ``Extremely Correlated Fermi Liquid Description of Normal State ARPES in Cuprates", G.-H. Gweon, B. S. Shastry and G. D. Gu, arXiv:1104.2631 (2011), Phys. Rev. Letts. 107, 056404 (2011).
- Phenomenological Model for the Normal-State Angle-Resolved Photoemission Spectroscopy Line Shapes of High-Temperature Superconductors Kazue Matsuyama and G.-H. Gweon Phys. Rev. Lett. 111, 246401 (2013).

- Low energy Kinks and ECFL (Shastry 2014 Annals of Physics)
- Line Shapes: P.A. Casey, P. W. Anderson, Phys. Rev. Letts. 106, 097002, (2011), Nature Physics, 4, 210 (2008) { Hidden Fermi Liquid Theory}-
- S Doniach and M Sunjic (1970), P. Noizeres and C. De Dominicis (1969) X-ray edge singularity.

The Setting

- T=0: Q=0=ω (Relatively Simpler problem than the excitations problem)
- Ground states: Superconductivity from "repulsive" interactions
  - t J model, Hubbard model, 3 band model,..
  - RVB of P W Anderson & friends (Plain Vanilla is almost ideal),
  - Gossamer SC of R B Laughlin,..
- Competing phases near half filling- CDW's, AFM, Spin Glass,.. RBL recent

- T> 0, Finite Q, ω (Remaining agenda)
  - Need to understand transport, ARPES, NMR,..
  - Hubbard U may be "finite", but U/Z large near half filling where $Z \sim (1-n)$



Eliminate using Gutzwiller Projection

Theoretical setting of the ECFL methodology for systematically studying the t J model.

$$|a> \quad a=\uparrow, a=\downarrow, a=0$$

$$a \neq \uparrow\downarrow$$

X's are Fermions with built in projection ops.

The hatted Fermions are equivalent to X's

$$X_i^{ab} = |a><b|$$

$$\hat{C}_\sigma^\dagger = (1-n_{-\sigma})C_\sigma^\dagger$$

$$\hat{C}_\sigma = (1-n_{-\sigma})C_\sigma$$

$$\left\{C_a, C_b^\dagger\right\} = \delta_{ab}$$

$$\begin{aligned} H &= -\sum_{i,j,\sigma} t_{ij} X_i^{\sigma 0} X_j^{0\sigma} - \boldsymbol{\mu} \sum_{i,\sigma} X_i^{\sigma\sigma} + \frac{1}{2}\sum_{i,j} J_{ij}\{\vec{S}_i.\vec{S}_j - \frac{1}{4} n_i n_j\}, \\ &= -\sum_{i,j,\sigma} t_{ij} X_i^{\sigma 0} X_j^{0\sigma} - \boldsymbol{\mu} \sum_{i,\sigma} X_i^{\sigma\sigma} + \frac{1}{4}\sum_{ij,\sigma} J_{ij}\left(X_i^{\sigma\bar{\sigma}} X_j^{\bar{\sigma}\sigma} - X_i^{\sigma\sigma} X_j^{\bar{\sigma}\bar{\sigma}}\right) \end{aligned}$$

X's satisfy a Lie algebra (with anticommuting objects i.e. grading) as opposed to simple canonical Fermi operators.

$$\{X_i^{0\sigma}, X_j^{\sigma' 0}\} = \delta_{ij}(\delta_{\sigma\sigma'} - \sigma\sigma' X_i^{\bar{\sigma}'\bar{\sigma}})$$

$$\bar{\sigma} = -\sigma$$

What does extreme correlations mean?

$$H = -\sum_{i,j} t_{ij} c^\dagger_{i\sigma} c_{j\sigma} + U \sum_i n_{i\uparrow} n_{i\downarrow}$$

Hubbard model (t,U)

(1) Weak Correlations $U \ll t$

Semiconductors

(2) Intermediate Correlations $U \leq t$

DFT (Band theory), Wide band free electron like metals

(3) Strong Correlations $U \geq t$

Transition metal magnetism, Dense Kondo Heavy Fermi systems, Iron arsenide superconductors etc

(4) Extreme Correlations $U \gg t$

High Tc systems, cobaltates, Anderson Impurity Model, some Heavy Fermi systems.

t J model

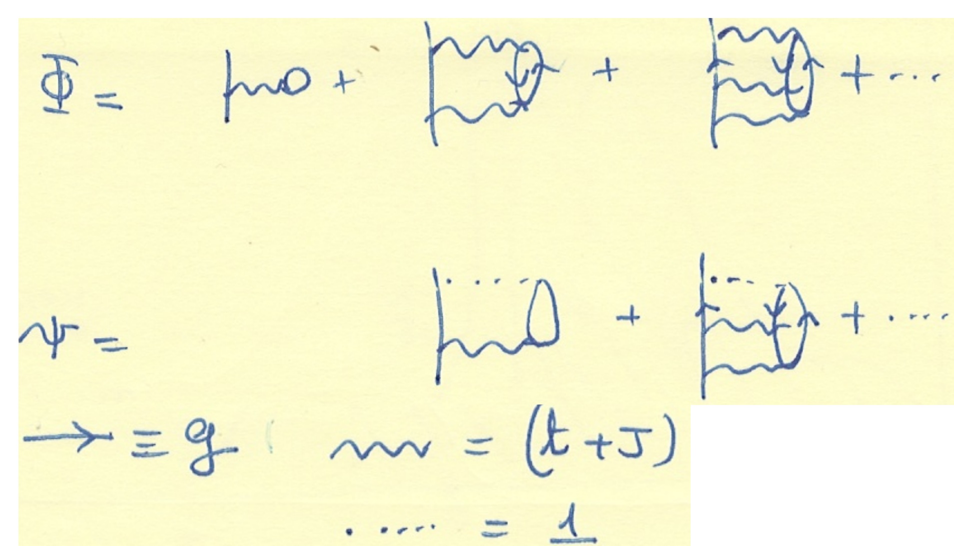
$$H = P_{d=0} \; (-\sum t_{ij} \; c^\dagger_i c_j) \; P_{d=0} + \frac{1}{2} \sum J_{ij} \vec{S}_i . \vec{S}_j$$

Useful to summarize one important Idea in ECFL:
Non Dysonian representation of Greens functions
are **Natural** and **Fundamental**

Weak Coupling theories	Strong Coupling theories $\Sigma'(\omega,\vec{k}) \sim \omega,\ for\ 0 \ll \omega \ll U$	
$G(k) = \frac{1}{i\omega - \varepsilon_k - \Sigma(k)}$	$\mathcal{G}(k) = \frac{1-\frac{n}{2}}{i\omega - c\ \varepsilon_k - \Sigma_{DM}(k)}$	$\mathcal{G}(k) = \frac{(1-\frac{n}{2}+\Psi(k))}{i\omega - c\ \varepsilon_k - \Phi(k)} = \mathbf{g}(k) \times \mu(k)$
Standard Dyson representation	Dyson-Mori representation	ECFL representation twin self energies

Reconstruction of Σ_{DM} possible but Physics better captured by ECFL pair. $\{\Phi,\Psi\} \to \Sigma_{DM}$

$\{\Phi,\Psi\}$ are BOTH generically ideal Fermi liquid like, but not so with Σ_{DM}



Why is the extreme correlation problem (t J model) so difficult?

- Non canonical field theory- Cannot consult existing text books!
    - Absence of Wicks theorem and Feynman series
    - Absence of any obvious small parameter.
- Gutzwiller projection is a ``singular perturbation”, hence a major stumbling block for the dynamics.
- ECFL approach uses an adaptation of Schwinger’s method.
    - Bypass Wicks theorem.
    - Uses extra time dependent potentials and magnetic fields to generate exact equations of motion (EOM).
- Freedom intrinsic to the Schwinger Dyson method + shift identities+ insights from spectral sum rules helps us to make progress.
- Connects with with Dyson Maleev approach invented for the spin problem
- ECFL describes a new **framework** for calculation with twin self energies and vertices.
- Obtain analytical results that are useful-novel and have experimental consequences. Also helpful in building bridges with DMFT and other approaches

A quick overview of "why things are so". Work in the liquid state (no broken symmetry)

$$\mathcal{G}_{\sigma\sigma'}(i\tau_i, f\tau_f) = -\frac{\langle T_\tau e^{-\hat{A}_S}\left(X_i^{0\sigma}(\tau_i) X_f^{\sigma' 0}(\tau_f)\right)\rangle}{\langle T_\tau e^{-\hat{A}_S}\rangle}$$

Added time dependent potentials, finally set to zero.

$$A = \sum_i \int_{\tau'} \mathcal{V}_i^{\sigma\sigma'}(\tau')\ \hat{C}_{i\sigma}^\dagger(\tau')\hat{C}_{i\sigma'}(\tau')$$

Double Hat Theorem: (1963)

$$X_i^{\sigma 0} = \tilde{C}_\sigma^\dagger = (1 - n_{-\sigma})C_\sigma^\dagger$$

$$X_i^{0\sigma} = \tilde{C}_\sigma = (1 - n_{-\sigma})C_\sigma$$

$$X_i^{\sigma\sigma'} = C_{i\sigma}^\dagger C_{i\sigma'}$$

Hat Removal Theorem: (2013)

Provided Gutzwiller projector supplied at initial time

$$X_i^{0\sigma} \to C_{i\sigma},\ X_i^{\sigma 0} \to \widetilde{C}_{i\sigma}^\dagger = C_{i\sigma}^\dagger(1 - N_{i\bar{\sigma}}),\ X_i^{\sigma\sigma'} \to C_{i\sigma}^\dagger C_{i\sigma'}.$$

$$\mathcal{G}_{\sigma_i\sigma_f}(i\tau_i, f\tau_f) = -\langle\langle C_{i\sigma_i}(\tau_i)\widetilde{C}_{f\sigma_f}^\dagger(\tau_f)\rangle\rangle$$

$$\langle\langle A(\tau_1)B(\tau_2)\ldots\rangle\rangle = \frac{1}{Z}\mathrm{Tr}\ e^{-\beta\hat{H}_{eff}}T_\tau\left(e^{-\hat{A}_S}A(\tau_1)B(\tau_2)\ldots\hat{P}_G(0^-)\right)$$

Leads to straightforward path integral rep, with Gutzwiller projector ONLY at initial time.

$$\hat{A}_S = T_{eff} + JS.S + A$$

$$\hat{T}_{eff} = -\sum_{ij\sigma} t_{ij}\widetilde{C}_{i\sigma}^\dagger C_{j\sigma}$$

Proof of Theorem: Upper Triangular Representation

t-J basis of states

$$[\psi]'_{final} = Q'_M \dots Q'_2.Q'_1.[\psi]'_{initial} \qquad Q'_j \sim e^{-it_j H_{tJ}}$$

Next we study the Canonical basis of states

$$[\psi] = \begin{bmatrix} \psi^{ph} \\ \psi^{un} \end{bmatrix} \; ; \; \hat{P}_G = \begin{bmatrix} \mathbb{1}^{ph} & 0 \\ 0 & 0 \end{bmatrix}$$

In the canonical basis, we can express the operators of interest, and end up with block structure-

$$Q_j = \begin{bmatrix} Q_j^{pp} & Q_j^{pu} \\ Q_j^{up} & Q_j^{uu} \end{bmatrix}$$

Problem:
How to retain time evolution in the physical space

$$[\psi]_{final} = Q_M \dots Q_2.Q_1.\hat{P}_G.[\psi]_{initial}$$

Requiring that the final state remains in the physical subspace. This has two classes of sufficient conditions:

(A) First sufficiency condition (-e.g. slave Bosons) too restrictive

$$[Q_j, \hat{P}_G] = 0$$ Requiring: two vanishings $Q_j^{pu} = 0, \; Q_j^{up} = 0$

(B) Much better sufficiency condition (least constrainted!)

$$[Q_j, \hat{P}_G].\hat{P}_G \; = \; 0;$$ Requiring: only one vanishing $Q_j^{up} = 0$

Thus upper triangular representation of Q

$$Q_j = \begin{bmatrix} Q_j^{pp} & Q_j^{pu} \\ 0 & Q_j^{uu} \end{bmatrix}$$

Theorem: Product of upper triangular matrices remains upper triangular. QED

$$[\psi]_{final} = \begin{bmatrix} Q_M^{pp} \dots Q_2^{pp}.Q_1^{pp}. \; \psi_{initial}^{ph} \\ 0 \end{bmatrix}$$

$$\mathcal{G}_{\sigma_i\sigma_f}(i\tau_i, f\tau_f) = -\langle\langle C_{i\sigma_i}(\tau_i)\widetilde{C}^{\dagger}_{f\sigma_f}(\tau_f)\rangle\rangle$$

Reminder from last page: definition of hatted operators

$$\tilde{C}^{\dagger}_{\sigma} = C^{\dagger}_{\sigma}(1 - N_{\bar{\sigma}})$$

$$\mathcal{G}_{\sigma_i\sigma_f}(i\tau_i, f\tau_f) = -\langle\langle C_{i\sigma_i}(\tau_i)C^{\dagger}_{f\sigma_f}(\tau_f)\rangle\rangle + \langle\langle C_{i\sigma_i}(\tau_i)C^{\dagger}_{f\sigma_f}(\tau_f)N_{f\bar{\sigma}_f}(\tau_f)\rangle\rangle.$$

Thus the G splits into two parts, the second term is from the definition of the creation operators with a hat

$$\mathbf{g}_{\sigma_i\sigma_j}(i\tau_i, j\tau_j) = -\langle\langle C_{i\sigma_i}(\tau_i)C^{\dagger}_{j\sigma}(\tau_j)\rangle\rangle$$

Firstly the auxiliary Greens defined.
Notice the <<>> are still non trivial

Next set $< CC^{\dagger}N >=< CC^{\dagger} >< N > +g \times \Psi$

$$\Psi_{\sigma_i\sigma_f}(i\tau_i, f\tau_f) = \mathbf{g}^{-1}_{\sigma_i\sigma_\mathbf{k}}(i\tau_i, \mathbf{k}\tau_\mathbf{k}) \times \langle\langle C_{\mathbf{k}\sigma_\mathbf{k}}(\tau_\mathbf{k})C^{\dagger}_{f\sigma_f}(\tau_f)N_{f\bar{\sigma}_f}(\tau_f)\rangle\rangle_c,$$

Second self energy by analogy with HM self energy defined through analogous ratio. (Integration of bold symbols is implied)

$$\mathcal{G}_{\sigma_i\sigma_f}(i\tau_i, f\tau_f) = \mathbf{g}_{\sigma_i\sigma_\mathbf{k}}(i\tau_i, \mathbf{k}\tau_\mathbf{k})\mu_{\sigma_\mathbf{k}\sigma_f}(\mathbf{k}\tau_\mathbf{k}, f\tau_f),$$
$$\mu_{\sigma_i\sigma_f}(i\tau_i, f\tau_f) = \delta(if)\,(1 - \langle N_{\bar{\sigma}_i}(\tau_i)\rangle) + \Psi_{\sigma_i\sigma_f}(i\tau_i, f\tau_f)$$

“The Notorious”
Caparison function
$\mu = (1 - \langle N \rangle + \psi$

This “explains” how the Product form arises

Caparison= elaborate decoration (*Ye Olde English*).

Idea is that the auxiliary “g(k,ω)” is already dressed by Fermi liquid renormalization, G requires a second layer of decoration!!

$$S_i^+ = (2s)\ b_i^\dagger\ (1 - \frac{n_i}{2s}\ )$$
$$S_i^- = b_i$$
$$S_i^z + s = n_i,$$
$$n_i = b_i^\dagger b_i$$

Dyson Maleev

Harris, Kumar, Halperin and Hohenberg

	Spins: The Dyson–Maleev mapping		Fermions: The non-Hermitian mapping	
Destruction operator	S_i^-	b_i	$X_i^{0\sigma}$	$C_{i\sigma}$
Creation operator	S_i^+	$(2s)\ b_i^\dagger(1 - \frac{n_i}{2s})$	$X_i^{\sigma 0}$	$C_{i\sigma}^\dagger(1 - \lambda N_{i\bar{\sigma}})$
Density operator(s)	$S_i^z + s$	$n_i = b_i^\dagger b_i$	$X_i^{\sigma\sigma'}$	$C_{i\sigma}^\dagger C_{i\sigma'}$
Projection operator	$\hat{P}_D$	$\prod_i\{\sum_{m=0}^{2s}\delta_{n_i,m}\}$	$\hat{P}_G$	$\prod_i(1 - N_{i\uparrow}N_{i\downarrow})$, for $\lambda = 1$
Vacuum	$\vert\downarrow\downarrow\ldots\downarrow\rangle$	$\vert 00\ldots 0\rangle$	$\vert Vac\rangle$	$\vert 00\ldots 0\rangle$
Small parameter & its range	$\frac{1}{2s}$	$\frac{1}{2s} \in [0, 1]$	λ	$\lambda \in [0, 1]$
Auxiliary Green's function		$\mathbf{g}(i,j) = -\langle\langle b_i b_j^\dagger\rangle\rangle$		$\mathbf{g}(i,j) = -\langle\langle C_{i\sigma} C_{j\sigma}^\dagger\rangle\rangle$
Caparison function		$\mu(i,j) = \delta_{ij}(1 - \frac{1}{2s}\langle n_j\rangle) + \frac{1}{2s}\Psi(i,j)$		$\mu(i,j) = \delta_{ij}(1-\lambda\gamma)+\lambda\Psi(i,j)$
Second Self energy Ψ		$\Psi(i,j) = \mathbf{g}^{-1}(i,\mathbf{a})\langle\langle b_\mathbf{a} b_j^\dagger n_j\rangle\rangle_c$		$\Psi(i,j) = \mathbf{g}^{-1}(i,\mathbf{a})\langle\langle C_{\mathbf{a}\sigma} C_{j\sigma}^\dagger N_{j\bar{\sigma}}\rangle\rangle_c$

This table summarizes the parallel between spins and extreme Fermions.
We have anticipated the parameter λ in analogy to the semiclassical parameter 1/(2 S) in D-M

Summary: With Fourier transforms, and with auxuliary "g" having its own self energy, and expand the caparison function μ

$$g^{-1}(k,\omega) = g_0^{-1}(k,\omega) - \Phi(k,\omega)$$

$$\mu(k,\omega) = 1 - \frac{n}{2}\Psi(k,\omega)$$

$$\mathcal{G}(\vec{k}, i\omega) = \frac{1 - \frac{n}{2} + \Psi(\vec{k}, i\omega)}{\mathbf{g}_0^{(-1)}(\vec{k}, i\omega) - \Phi(\vec{k}, i\omega)}$$

Novel non-Dysonian

The two self energies can be pursued in different ways

- Expansion in the λ parameter (λ ε [0,1])
- Low k,ω expansion
- In high dimensions we can show that these are further related through

$$\Psi(k) = \Psi(i\omega_k),$$
$$\Phi(k) = \chi(i\omega_k) + \epsilon_k \Psi(i\omega_k).$$

1.This relation implies that the Dyson (or Dyson Mori) self energy is momentum independent.

2. Proof is independent of Wicks theorem and is consistent with the momentum independence in the Hubbard model foundational to DMFT.

3. Thus the two limits of infinite U and infinite D mutually commute.

$G(k) = \frac{1}{i\omega - \varepsilon_k - \Sigma(k)}$	$\mathcal{G}(k) = \frac{1-\frac{n}{2}}{i\omega - c\,\varepsilon_k - \Sigma_{DM}(k)}$	$\mathcal{G}(k) = \frac{(1-\frac{n}{2}+\Psi(k))}{i\omega - c\,\varepsilon_k - \Phi(k)} = \mathbf{g}(k)\times\mu(k)$

A little algebra gives
the explicit relation

$$\Sigma_{DM}(k) = \Sigma_{DM}(i\omega_k) = \frac{(i\omega_k + \boldsymbol{\mu})\Psi(i\omega_k) + \left(1 - \frac{n}{2}\right)\chi(i\omega_k)}{1 - \frac{n}{2} + \Psi(i\omega_k)},$$

We also obtain a mapping, simliar in spirit to that of Georges- Kotliar, where the tJ model in infinite dimensions is identical to the AIM with self consistently chosen band structure of the conductions electrons.

tJ --> AIM map

$$H_{tJ}^{D=\infty} = -\sum_{ij} t_{ij} \hat{C}^{\dagger}_{i\sigma} \hat{C}_{\sigma} - \mu \sum_i N_{i\sigma}$$

$$t_{ij} \sim \frac{1}{\sqrt{D}}$$

$$H = \sum_{\sigma} \epsilon_d X^{\sigma\sigma} + \sum_{k\sigma} \widetilde{\epsilon}_k n_{k\sigma} + \sum_{k\sigma} (V_k\, X^{\sigma 0}\, c_{k\sigma} + V_k^*\, c^{\dagger}_{k\sigma}\, X^{0\sigma}),$$

$$\mathbf{g}_{i,i}(i\omega_n) \longrightarrow \mathbf{g}(i\omega_n)$$

$$\mu_{i,i}(i\omega_n) \longrightarrow \mu(i\omega_n)$$

$$\mu \longrightarrow \epsilon_d$$

$$\sum_{\vec{k}} \epsilon_{\vec{k}} \mathbf{g}(k) = \sum_{\vec{k}} \frac{|V_{\vec{k}}|^2}{i\omega_n - \widetilde{\epsilon_{\vec{k}}}} \mathbf{g}(i\omega_k),$$

We also obtain an independent solution of the tJ as well as the AIM model as an expansion in λ. Here λ is related to the density of particles or the filling of the d-level in the AIM. Will discuss the explicit solution to 2nd order later.

Simplest approach is to expand both the self energies at small (k, ω) assuming a Fermi liquid structure.

(DMFT-ECFL comparison paper generalized for k dependence. The FL nature is justified by the λ- expansion of these objects- explained later.)

Long wavelength expansion

$$1-\frac{n}{2}+\Psi(\vec{k},\omega)=\alpha_0+c_\Psi(\omega+\nu_\Psi\,\hat{k}\,v_f)+i\mathcal{R}/\gamma_\Psi+O(\omega^3)$$

$$\omega+\boldsymbol{\mu}-\left(1-\frac{n}{2}\right)\varepsilon_k-\Phi(k,\omega)=(1+c_\Phi)\left(\omega-\nu_\Phi\,\hat{k}\,v_f+i\mathcal{R}/\Omega_\Phi+O(\omega^3)\right)$$

$$\alpha_0=1-\frac{n}{2}+\Psi_0\to(1-n)$$

$$\mathcal{R}=\pi\{\omega^2+(\pi k_BT)^2\}$$

$$\hat{k}\;=\;(\vec{k}-\vec{k}_F).\vec{k}_F/|\vec{k}_F|$$

$v_f\;=\;(\partial_k\varepsilon_k)_{k_F}$ is the *bare* Fermi velocity

Spectral function with 5 parameters

$$A(\vec{k},\omega)=\frac{z_0}{\pi}\frac{\Gamma_0}{(\omega-\nu_\Phi\,\hat{k}\,v_f)^2+\Gamma_0^2}\times\mu(k,\omega)$$

$$\Gamma_0(\hat{k},\omega)=\eta+\frac{\pi(\omega^2+(\pi k_BT)^2)}{\Omega_\Phi}$$

$$\mu(\hat{k},\omega)=1-\frac{\omega}{\Delta_0}+\frac{\nu_0\,\hat{k}\,v_f}{\Delta_0}$$

Scaling property near half filling

$z_0\to\overline{z}_0\times\delta;\quad \Delta_0\to\overline{\Delta}_0\times\delta;\quad \Omega_\Phi\to\overline{\Omega}_\Phi\times\delta$

$\nu_0\to\overline{\nu}_0\times\delta;\quad \nu_\Phi\to\overline{\nu}_\Phi\times\delta;$

$$A(\hat{k},\omega|T,\delta)\sim A\left(\hat{k},\,\omega\frac{\delta_0}{\delta}\right|\left.T\frac{\delta_0}{\delta},\,\delta_0\right)$$

- Next we review the exact EOM for the G's obtained using Schwinger's method of functional derivatives and functional integration.
- It is pretty rough going, if you see it for the first time
- However!!

I recently discovered* an alternate universe, with other meanings of functional integration!

*Singapore (NTU October 2013)

The Schwinger method
Calculation in brief: liquid state
(sans broken symmetry)

$$X_i^{\sigma 0} = \tilde{C}_\sigma^\dagger = (1 - n_{-\sigma}) C_\sigma^\dagger$$

$$X_i^{0\sigma} = \tilde{C}_\sigma = (1 - n_{-\sigma}) C_\sigma$$

Added time dependent (Bosonic) potentials and spin densities. These are finally set to zero.

$$\langle\langle Q(\tau_1, \tau_2, \ldots)\rangle\rangle = \frac{\mathrm{Tr}[e^{-\beta H} T_\tau e^{-\mathcal{A}} \; Q(\tau_1, \tau_2, ..)]}{\mathrm{Tr}[e^{-\beta H} T_\tau (e^{-\mathcal{A}})]},$$

$$\mathcal{A} = \sum_i \int_{\tau'} \mathcal{V}_i^{\sigma\sigma'} \; X_i^{\sigma\sigma'}(\tau')$$

$$\{X_i^{0\sigma_1}, X_j^{\sigma_2 0}\} = \delta_{ij}\left(\delta_{\sigma_1\sigma_2} - (\sigma_1\sigma_2) X_i^{\bar{\sigma}_1\bar{\sigma}_2}\right)$$

$$\frac{\delta}{\delta \mathcal{V}_j^{\sigma_1\sigma_2}(\tau_1)} \langle\langle Q(\tau_2)\rangle\rangle = \langle\langle Q(\tau_2)\rangle\rangle \langle\langle X_j^{\sigma_1\sigma_2}(\tau_1)\rangle\rangle - \langle\langle X_j^{\sigma_1\sigma_2}(\tau_1) Q(\tau_2)\rangle\rangle$$

$$\mathcal{G}_{\sigma_i\sigma_f}[i, f] = -\langle\langle X_i^{0\sigma_i} \; X_f^{\sigma_f 0}\rangle\rangle. \qquad i \equiv (R_i, \tau_i)$$

$$(\partial_{\tau_i} - \boldsymbol{\mu}) \mathcal{G}(i, f) = -\delta(i, f)\{1 - \gamma(i)\} - \mathcal{V}_i \cdot \mathcal{G}(i, f) - X(i, \mathbf{\bar{j}}) \cdot \mathcal{G}(\mathbf{\bar{j}}, f) - Y(i, \mathbf{\bar{j}}) \cdot \mathcal{G}(\mathbf{\bar{j}}, f),$$

Bold indices are summed over

Local Greens function and (k) *stands for time reversal*

$$\gamma(i) = \mathcal{G}^{(k)}[i^-, i], \qquad \mathcal{G}^{(k)}_{\sigma_1\sigma_2} = \sigma_1\sigma_2 \; \mathcal{G}_{\bar{\sigma}_2\bar{\sigma}_1}$$

$$\begin{aligned} X[i, j] &= -t[i, j] \; (D[i^+] + D[j^+]) + \frac{1}{2} J[i, k] \; (D[i^+] + D[k^+]) \delta[i, j] \\ Y[i, j] &= -t[i, j] \; (\mathbb{1} - \gamma[i] - \gamma[j]) + \frac{1}{2} J[i, k] \; (\mathbb{1} - \gamma[i] - \gamma[k]) \delta[i, j]. \end{aligned}$$

$$D = \xi^* \frac{\delta}{\delta \mathcal{V}^*} \quad (^* \text{ represents spin indices})$$

Turning off sources, γ(i) -> n/2
(n= density)

$$\mathcal{G}(a, b) = \mathbf{g}(a, \mathbf{\bar{b}}) \cdot \mu(\mathbf{\bar{b}}, b),$$

The caparison function appears here. Motivation is to get rid of a crucial non canonical term γ(i).

Next we use the chain rule for functional derivatives

Using chain rule for functional derivatives get exact equation

$$[\mathbf{g}_0^{-1}(i,\mathbf{j}) - \mu\delta_{i\mathbf{j}} - t_{i\mathbf{j}} - \Phi(i,\mathbf{j})].\mathbf{g}(\mathbf{j},\mathbf{k}).\mu(\mathbf{k},\mathbf{f}) = \delta(\mathbf{i},\mathbf{f})(\mathbf{1} - \gamma(\mathbf{i})) + \mathbf{\Psi}(\mathbf{i},\mathbf{f})$$

$$\Phi(i,m) = \mathbf{L}(i,\bar{\mathbf{i}}) \cdot \mathbf{g}^{-1}(\bar{\mathbf{i}},m)$$

$$\Psi(i,m) = -\mathbf{L}(i,\bar{\mathbf{i}}) \cdot \mu(\bar{\mathbf{i}},m).$$

$$\mathbf{L}(i,f) = \left(t(i,\bar{\mathbf{j}})\xi^* \cdot \mathbf{g}(\bar{\mathbf{j}},f) - \frac{1}{2}J(i,\bar{\mathbf{j}})\xi^* \cdot \mathbf{g}(i,f)\right) \times \left(\frac{\delta}{\delta\mathcal{V}_i^*} + \frac{\delta}{\delta\mathcal{V}_{\bar{\mathbf{j}}}^*}\right),$$

Falls apart into two a pair of coupled exact equations with a canonical greens function if we write

$$\mu(i,f) = \delta(i,f)(1 - \gamma(i)) + \Psi(i,f)$$

$$[\mathbf{g}_0^{-1}(i,\mathbf{j}) - \mu\delta_{i\mathbf{j}} - t_{i\mathbf{j}} - \Phi(i,\mathbf{j})].\mathbf{g}(\mathbf{j},f) = \delta(i,f)$$

Turning off the sources, we restore translation invariance and can take Ft's. Number sum rules are obvious- both G and g satisfy the same number sum rule.

$$k \equiv (i\omega_n, \vec{k})$$

$$\mathbf{g}^{-1}(k) = i\omega_n + \mu - \frac{n}{2}\,\varepsilon_k - \Phi(k)$$

$$\mu(k) = 1 - \frac{n}{2} + \Psi(k)$$

$$\sum_k \mu(k)\mathbf{g}(k) = \frac{n}{2}$$

$$\sum_k \mathbf{g}(k) = \frac{n}{2}$$

Exact so far.
It is a formal solution and how do we make sense of it?
Can we devise an approximate solution, valid in some limit?

- Devise a small parameter λ between 0,1 related to particle density and also to double occupancy.

- At λ=0, get the Fermi gas, and at λ=1 get the full tJ model.

- We will require a good understanding of the shift invariance of the tJ model, this important invariance leads to a second chemical potential in the problem.

We could take the game one step further and do a Perturbation theory in small (fake) parameter λ $\lambda \in [0,1]$

$$X_i^{\sigma 0}(\lambda) \to C_{i\sigma}^\dagger (1 - \lambda C_{i\bar{\sigma}}^\dagger C_{i\bar{\sigma}})$$
$$X_i^{0\sigma}(\lambda) \to C_{i\sigma}$$
$$X_i^{\sigma\sigma'}(\lambda) \to C_{i\sigma}^\dagger C_{i\sigma'} \ .$$

Theory of λ Fermions
Lot of promise- but in infancy.
Hence will not purse in these lectures

Symbolic notation makes things simpler

Y represents the hopping matrix element broken into a static and dynamic parts.

$$Y \to (-t + \frac{J}{2}) + Y_1$$

$$\begin{cases} X = [-t + \frac{1}{2}J]\ D \\ Y_1 = -[-t + \frac{1}{2}J]\ \gamma \\ \hat{G}_0^{-1}(\boldsymbol{\mu}) \equiv (\boldsymbol{\mu} - \partial_\tau - \mathcal{V})\mathbb{1} - [-t + \frac{1}{2}J] \end{cases}$$

Fermi gas (non interacting) Greens function

Symbolic EOM for t J model

$$\mathcal{G} = (\hat{G}_0^{-1}(\boldsymbol{\mu}) - \lambda\ Y_1 - \lambda\ X)^{-1}.\ (\mathbb{1} - \lambda\ \gamma).$$

Parameter λ introduced here

Set λ=1 at the end.
At λ=0 it reduces a Fermi gas and provides continuity between Fermi gas and tJ model.
It plays the role of double occupancy- see this explicitly in atomic limit.

$$G = (\hat{G}_0^{-1} - UG - U\frac{\delta}{\delta v})^{-1}\ .\ \mathbf{1}$$

Inspiration comes from the symbolic EOM for Hubbard model (Canonical theory. Notice how the Hubbard U enters this eqn.

Start from exact EOM

$$\mathcal{G} = (\hat{G}_0^{-1}(\boldsymbol{\mu}) - \lambda\ Y_1 - \lambda\ X)^{-1}.\ (\mathbb{1} - \lambda\ \gamma)$$

- *We can set up Schwinger Dyson equations by taking successive functional derivatives.*
- *Generates the analog of the skeleton graph expansion in powers of λ.*
- *We will take terms up to O(λ^2) and study this "second order theory".*

EOM continues to bifurcate exactly defining the auxiliary FL etc at a given λ. At λ=0 Fermi gas, while at λ=1 the exact tJ eqns.

$$(\hat{G}_0^{-1} - \lambda\ Y_1 - \lambda\ \Phi).\ \mathbf{g} = \mathbf{1}$$ *Auxiliary Fermi liquid*

$$\mu = (\mathbf{1} - \lambda\gamma) + \lambda\Psi$$ *Caparison factor*

Comment: With some caveats, it might be useful to think of a mapping

$$\lambda \sim \frac{U}{U + z|t|}$$

Hence low order theory in λ is expected to be a VERY GOOD start. (since unlike U, the range of λ is [0,1].)

COMMENTS

- Since t is both the propagator and interaction term, we need a watchdog theorem to make sense.
- Shifting the center of gravity of the band should not change the physics.
- Exact Shift identities Simple but powerful. Analogs of pure Gauge transformations.
- Shift identities help us formulate a rigorous theory to each order in λ.

$$\begin{aligned} H &= -\sum_{i,j,\sigma} t_{ij} X_i^{\sigma 0} X_j^{0\sigma} - \boldsymbol{\mu} \sum_{i,\sigma} X_i^{\sigma\sigma} + \frac{1}{2}\sum_{i,j} J_{ij}\{\vec{S}_i.\vec{S}_j - \frac{1}{4} n_i n_j\}, \\ &= -\sum_{i,j,\sigma} t_{ij} X_i^{\sigma 0} X_j^{0\sigma} - \boldsymbol{\mu} \sum_{i,\sigma} X_i^{\sigma\sigma} + \frac{1}{4}\sum_{ij,\sigma} J_{ij}\left(X_i^{\sigma\bar{\sigma}} X_j^{\bar{\sigma}\sigma} - X_i^{\sigma\sigma} X_j^{\bar{\sigma}\bar{\sigma}}\right) \end{aligned}$$

Shift invariance: Under the shifts

$$t_{ij} \to t_{ij} - u_t\ \delta_{ij}, \quad J_{ij} \to J_{ij} + u_J\ \delta_{ij},$$

$$H \to H + \left(u_t + \frac{1}{4} u_J\right)\ \hat{N}$$

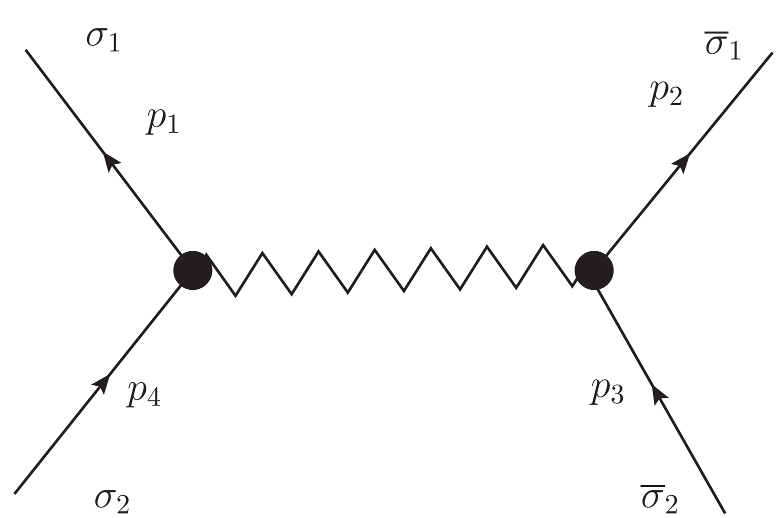
- *Shift theorem-(I):* A shift of either t or J can be absorbed into suitable parameters, leaving the physics unchanged.
- *Shift theorem-(II):* The two shifts of t and J cancel each other when $u_J = -4 \times u_t$.

- The auxiliary Fermi problem leading to g(k) is canonical in all respects.
- It has a Hamiltonian with canonical electrons, and one can use the Feynman series to each order in λ.
- The procedure outlined here "stitches" together this Feynman theory with the caparison factor to give the exact physical Greens function.
- Exact shift identities help us formulate a rigorous theory to each order in λ.

$$H_{eff} \;=\; -\sum_{ij} t_{ij} f^\dagger_{i\sigma} f_{j\sigma} + \sum_i (\frac{1}{4}J_0 - \mu) f^\dagger_{i\sigma} f_{i\sigma} + V_{eff},$$

$$V_{eff} \;=\lambda \frac{1}{4}\sum_{ij} t_{ij}(\sigma_1\sigma_2)\ \left[\left(f^\dagger_{i\sigma_1} f^\dagger_{i\bar{\sigma}_1} + f^\dagger_{j\sigma_1} f^\dagger_{j\bar{\sigma}_1}\right) f_{i\bar{\sigma}_2} f_{j\sigma_2} + (h.c.)\right] - \frac{1}{4}\sum_{ij} J_{ij}(\sigma_1\sigma_2)\ f^\dagger_{i\sigma_1} f^\dagger_{j\bar{\sigma}_1} f_{j\bar{\sigma}_2} f_{i\sigma_2}$$

$$+\frac{1}{4}\sum_i u_0(\sigma_1\sigma_2)\ f^\dagger_{i\sigma_1} f^\dagger_{i\bar{\sigma}_1} f_{i\bar{\sigma}_2} f_{i\sigma_2}.$$



$$W_{eff} = -\delta_{p_1+p_2,p_3+p_4}\ \ \{\Sigma_j\,\varepsilon_j + J_{p_2-p_3} - u_0\ \}$$

SUMMARY OF THE $O(\lambda^2)$ THEORY

$$\mathcal{G}(k) = \mathbf{g}(k)\ \mu(k) = \frac{\mu(k)}{i\omega_n + \boldsymbol{\mu} - \overline{\varepsilon}_k - \overline{\Phi}(k)}.$$

$$\mu(k) = 1 - \gamma + \lambda\ \Psi(k)$$

$$\frac{n}{2} = \sum_k \mathcal{G}(k)\ e^{i\omega_n 0^+}, \quad \frac{n}{2} = \sum_k \mathbf{g}(k)\ e^{i\omega_n 0^+}$$

$$\begin{aligned}
\mu(k) &= 1 - \lambda\frac{n}{2} + \lambda^2\ \frac{n^2}{4} + \lambda^2\Psi(k), \quad \text{where } (k) \equiv (\vec{k}, i\omega_k), \\
\Psi(k) &= -\sum_{p,q} (\varepsilon_p + \varepsilon_{k+q-p} + \varepsilon_k + \varepsilon_q + J_{k-p} - u_0)\ \mathbf{g}(p)\ \mathbf{g}(q)\ \mathbf{g}(q+k-p) \\
\mathbf{g}^{-1}(k) &= i\omega_n + \boldsymbol{\mu} - \overline{\varepsilon}_k - \lambda^2\ \overline{\Phi}(k) \\
\overline{\varepsilon}_k &= \left(1 - \lambda\ n + \lambda^2\ \frac{3n^2}{8}\right)\varepsilon_k + \lambda\ \sum_q \frac{1}{2} J_{k-q}\ \mathbf{g}(q) \\
\overline{\Phi}(k) &= -\sum_{q,p} \mathbf{g}(q)\ \mathbf{g}(p)\ \mathbf{g}(k+q-p) \\
&\quad \times (\varepsilon_k + \varepsilon_p + \varepsilon_q + \varepsilon_{k+q-p} + J_{k-p} - u_0\ )\ \{\varepsilon_k + \varepsilon_p + \varepsilon_q + \varepsilon_{k+q-p} + \frac{1}{2}(J_{k-p} + J_{p-q}) - u_0\ \}.
\end{aligned}$$

- Domain of validity of second order theory n < .7 (from the high frequency sum rule).
- Can be pushed to higher densities by going to higher order in lambda
- Already leads to interesting answers at the second order.
- Leads to insight about Fermi liquid behaviour of the two self energies ~ g g g. This is as in 2nd order perturbation of Fermi liquids.

End Lectures 1&2