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LETTER TO THE EDITOR

Phase transition in the two-dimensional disordered x–y model

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Abstract. It is pointed out that the disordered x–y model in two dimensions should undergo a phase transition of the same nature as the Kosterlitz–Thouless transition observed for a pure system in the limit of small disorder. Above a certain value of the disorder parameter x, mertices (which are bound pairs of merons) play the role attributed to vortices in the pure system.

The disordered x–y model of a spin glass is described by the Hamiltonian

$$\mathscr{H} = -\tfrac{1}{2} \sum_{i,\, j \in L} J_{ij} \boldsymbol{s}_i . \boldsymbol{s}_j \qquad |\boldsymbol{s}_i| = 1 \tag{1}$$

where $\boldsymbol{s}_i$ is a planar (x–y) spin, L is the lattice (we confine ourselves to two dimensions in this work) and the J_{ij} are the (nearest-neighbour) quenched random exchange couplings with a probability distribution

$$P(J_{ij}) = x\delta(J_{ij} + J) + (1 - x)\delta(J_{ij} - J). \tag{2}$$

It has recently been shown (Villain 1977, 1978) that this model is equivalent to a neutral, two-component Coulomb gas with integral and half-integral charges located on the dual lattice. In the limit of vanishing disorder ($x = 0$), the Coulomb gas reduces to the model considered by Kosterlitz and Thouless (1973, referred to as KT) involving only integral charges (which are equivalent to vortices). Non-trivial disorder manifests itself in the form of half-integral charges (merons) located at the frustrated plaquettes (Toulouse 1977). Vortices may be regarded as topological excitations of the medium and it is interesting that disorder forces distortions (merons) into the medium, which are closely related to vortices (Toulouse 1979).

A phase transition in the ordered x–y model is understood in terms of the dissociation of bound vortex–antivortex pairs (i.e. dipoles in the electrostatic analogy) as the temperature is increased and the ordered phase exhibits 'topological ordering' in place of the usual uniform magnetisation. This may be regarded as the transition of an electrolyte from an insulating, low-temperature phase to a conducting high-temperature phase in the electrostatic analogy. For the disordered x–y model the picture is not so clear since the existence of merons in the ground state suggested that there should always be a conducting phase (Villain 1977, 1978). In this work, we study the nature of the phase transition in the disordered x–y model using the electrostatic analogy. Only weak disorder (i.e. $x \ll 1$) is considered and the lowest-order terms in x are retained consistently.

The partition function of the disordered x–y model (divided by the non-singular spin-wave partition function) can be written in the electrostatic analogy as

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$$Z = \sum_{\{m_i = 0, \pm 1, \pm 2 \ldots\}} \exp(-\beta H_{\text{Coulomb}}) \delta_{0, \Sigma_i m_i} \tag{3}$$

where

$$H_{\text{Coulomb}} = -q^2 \sum_{\substack{i, j \in L \\ i \neq j}} \ln(|r_i - r_j|/a_0) \mu_i \mu_j + \mu_0 \sum_i \mu_i^2 \tag{4}$$

$$\mu_i = m_i + v_i/2 \qquad q^2 = \pi J \qquad \mu_0 = q^2(\gamma + \tfrac{3}{2} \ln 2) \tag{5}$$

where a_0 is the lattice constant and γ is Euler's constant. The integers v_i are obtained from the distribution of the exchange interactions J_{ij} as follows. For $J_{ij} > 0$ (< 0), we assign an integer $n = 0$ ($= 1$) and compute in a square surrounding the dual-lattice point $v_i = n_{\text{East}} + n_{\text{South}} - n_{\text{West}} - n_{\text{North}}$. It is clear that a frustrated plaquette corresponds to odd integral values of v_i and hence half-integral μ_i. It can also be seen that the net 'charge' $\sum v_i = 0$ by construction (every bond is to the east or south of the plaquette and to the west or north of the adjacent one). The electrical neutrality condition $\sum m_i = 0$ is thus equivalent to $\sum \mu_i = 0$. The chemical potential μ_0 is the cost of creating a unit charge and may be thought of as one half of the energy of a pair of charged particles kept at the distance of minimum approach. Here a_0 plays the role of a hard-disc cut-off. The cut-off a_0 is an important parameter in the problem since it sets the scale for the Coulomb interaction in two dimensions.

In order to produce a systematic expansion we arrange Z as an expansion in $\exp(-\mu_0 \beta)$ (analogous to the fugacity expansion of the grand canonical partition function). The expansion is facilitated by the observation that the composite nature of μ_i (as $m_i + v_i/2$) may be essentially neglected and we may sum over μ_i (subject to $\sum \mu_i = 0$). Clearly the μ_i run over positive and negative integers for the unfrustrated (flat) plaquettes and over positive and negative half-odd integers over the frustrated (curved) plaquettes. We designate F as the set of frustrated lattice points and R as the remainder ($F \cup R = L$). Also, we restrict ourselves to unit or half-unit charges at any lattice point as discussed by KT. We may thus write

$$Z = \exp(-\beta \mu_0 N_f/4) Z_m Z_v \tag{6}$$

where N_f is the number of frustrated lattice points

$$Z_m = \sum_{\{\mu_i = \pm 1/2\}} \delta_{0, \Sigma \mu_i} \exp\left(-\beta q^2 \sum_{i, j \in F} \mu_i \mu_j \ln(|r_i - r_j|/a_0) \right) \tag{7}$$

$$Z_v = 1 + \exp(-\mu_0 \beta) Z_1 + \exp(-2\mu_0 \beta) Z_2 + \ldots \tag{8}$$

$$Z_1 = \frac{2}{2 + N_f} \sum_{\substack{i \in F \\ m \in R}} \exp[-\beta q^2 \ln(|r_i - r_m|/a_0)] \left\langle \exp\left[4q^2 \beta \sum_{j \neq i} \mu_i \mu_j \ln\left(\frac{|r_m - r_j|}{|r_i - r_j|} \right) \right] \right\rangle \tag{9}$$

$$Z_2 = \sum_{m, n \in R} \exp[-2\beta q^2 \ln(|r_m - r_n|/a_0)] \left\langle \exp\left[2q^2 \beta \sum_j \mu_j \ln\left(\frac{|r_j - r_m|}{|r_j - r_n|} \right) \right] \right\rangle. \tag{10}$$

The averages in equations (8) and (9) are with respect to Z_m. Z_1 is obtained by putting one vortex in R and compensating the charge by flipping one of the μ_i in F. Z_2 is obtained by creating two vortices in R, etc. (The term Z_1 actually is irrelevant in the thermodynamic limit.) The physical significance of Z_2 is clear: we have two oppositely charged vortices interacting with each other in a 'polarisable medium' provided by the merons. Higher terms may be written down in a similar way. Z_m, the meron partition function is essentially an Ising model with zero magnetisation and a logarithmic exchange (Villain 1977).

We now impose the condition that x is sufficiently small for the theory to be tractable. We have $\sim 4xN$ frustrated lattice points (dropping higher-order terms in x) which occur in pairs (since reversing one J_{ij} creates two frustrated cells). To lowest order in x, we then have $2xN$ pairs of F lattice points distributed at random which may be considered as located on a (random) lattice F' (a lattice point of F' is the mid-point of F lattice pair). The mean separation between two meron pairs is

$$a_m = a_0/(2\pi x)^{1/2}. \tag{11}$$

At very low temperatures Z_v may be omitted. We expect Z_m to show dipolar behaviour since charge fluctuations are minimised by populating the F lattice pairs with opposite charges which form dipoles. These dipoles may be considered as bound pairs of merons and antimerons. Each dipole can point in two directions, E–W or N–S and consistute two-level systems (Villain 1977). The dipole–dipole interaction in two dimensions is $\sim 1/r^2$ and hence scaling arguments can be used (similar to the RKKY model in three dimensions) to give a specific heat density independent of x and a characteristic temperature $T_D = xJ/k_B$.

At higher temperatures there is the possibility of charge fluctuations occurring in the F' lattice. A 'mertex' may be created at a lattice point by populating the pair of lattice points with two like charges. A mertex corresponds to charge $+1$ (an antimertex to charge -1) and can be viewed as a bound pair of two merons and is similar to a vortex. It is clear that mertices can only be created in pairs because of neutrality. The interaction between two mertices is again logarithmic (corrections from the internal structure of the mertex are negligible at large distances compared with the logarithmic part). Thus we are led back to just the KT problem on the F' lattice. The fact that the F' lattice points are randomly located should be irrelevant for any phase transition since the transition is dominated by the long-range part of the Coulomb interaction (in fact KT used a continuum approximation for the lattice for just this reason).

In order to make the relationship precise, consider two oppositly charged mertices separated by the mean distance a_m. The energy of the pair is

$$E = 2q^2 \ln|a_m/a_0| = -q^2 \ln(2\pi x). \tag{12}$$

Hence the mertex system corresponds to a chemical potential

$$\mu_m = -\tfrac{1}{2}q^2 \ln(2\pi x). \tag{13}$$

(The same conclusion can be reached by rearranging the F' lattice as a regular array with lattice constant a_m and considering the energy of a $2n$ mertex configuration.) It should be emphasised that the mertex chemical potential is a genuine excitation energy; it is the characteristic energy for creating a pair of mertices in the F' lattice. Using the fact that the F' lattice has $2xN$ lattice points and taking into account the extensivness of the free energy, we can write

$$(1/N) \ln Z_m = -2x\beta f\{T, q^2 \ln[(2\pi x)^{-1/2}]\} \tag{14}$$

where f is the Kosterlitz–Thouless free energy per site.

As given by KT (1973) and Kosterlitz (1973) we note that the critical temperature is given by†

† It seems to us that equation (26) given by KT is missing a factor of $\sqrt{2}$. This factor of two has been lost between their equations 18 and 19. The coefficient in front of π should be 1·84 which we have replaced by 2 for simplicity.

$$q^2/2k_B T_c = 1 + 2\pi \exp(-\beta_c \mu_0). \tag{15}$$

From equation (14) we predict that Z_m undergoes a phase transition at T_m given by

$$k_B T_m = (\pi J/2)(1 - 4\pi^2 x). \tag{16}$$

The nature of the transition is clearly completely analogous to the KT transition with the mertices playing the role of vortices. The prefactor in equation 14 implies that the amplitudes of the various derivatives of the free energy are proportional to x.

The behaviour of Z is more complicated. Rather than use the expressions (8) and (9), we adopt a physical point of view and argue that we have a vortex gas in a polarisable medium provided by the merons. Clearly the phase transition in Z_m effects the behaviour of the vortex. In particular, if the mertex system has a transition 'sufficiently early', as the temperature is increased, the vortices would be driven towards a phase transition (since the mertices would screen out long-range Coulomb interaction in their conducting phase). If the mertices remain bound, however, the usual KT transition of the vortices will occur. A rough criterion for 'sufficiently early' may be obtained as follows. We note from equation (15) that the critical temperature is determined by the chemical potential: the larger is the chemical potential, the lesser is the ease with which the charged pairs are created and hence the higher is T_c. Thus a comparison of the chemical potentials of the two species should suffice. In order for the mertices to have an early phase transition, we require that μ_m, the chemical potential for a mertex, should be smaller than μ_0. This puts a restriction on x:

$$x > x_0 \qquad x_0 = \exp(-2\gamma)/16\pi = 0{\cdot}0063. \tag{17}$$

For $x < x_0$, the mertex chemical potential is larger than the vortex value and hence we expect the usual KT transition to occur first. The KT transition temperature should change with x in the range $x < x_0$. We can estimate this effect through the dielectric constant of the polarisable meron system. The dielectric constant of the F lattice at low temperature (neglecting mertices) is $\epsilon = 1 + \pi\beta x q^2$ and replacing $k_B T_c$ by $k_B T_c/\epsilon$ in equation (15) we find

$$k_B T_{KT}(x) = k_B T_{KT}(0) - \pi^2 J x. \tag{18}$$

In the range $x < x_0$, it is not clear whether a second transition of the mertex system will occur. The region $x \simeq x_0$ is also complicated since both mertices and vortices should be taken into account. We hope to return to these complex questions later.

In conclusion, the physical picture of the phase transition in the disordered x–y model that emerges from our analysis is as follows: for $x < x_0$, we have the usual Kosterlitz–Thouless transition (modified somewhat by disorder) and for $1 \gg x > x_0$ we should again have a KT transition with mertices playing the role of vortices. Equation (14) is interesting in that the disordered x–y model is related to a pure x–y model with a disorder-determined chemical potential. We should mention that our results differ from those obtained by Josè (1978), who predicted an abrupt departure from the KT picture for arbitrarily small x.

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