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FAST TRACK COMMUNICATION

Parameter-dependent commuting matrices, Plücker relations and related quantum glass models

B Sriram Shastry

Department of Physics, University of California, Santa Cruz, CA 95064, USA

E-mail: sriram@physics.ucsc.edu

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Abstract
Type-I matrices were introduced recently as finite-dimensional prototypes of quantum integrable systems. These matrices are linearly dependent on an 'interaction' type parameter, and possess interesting properties such as commuting partner matrices and generically violate the von Neumann Wigner noncrossing rule. The important role of Plücker relations in this construction is noted. Type-I matrices are given a transparent formulation in terms of Fermi or Bose-type particle operators; they represent a quantum glass model with either Fermi or Bose statistics, with several free parameters that may be chosen at will.

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1. Introduction

In [1], the author initiated a study of quantum integrable systems in finite dimensions, within the context of parameter-dependent commuting matrices. This in turn was motivated by several examples of specific integrable models, such as the Hubbard model [2–4] and the Heisenberg model [5][1]. In these examples, one studies the realizations of the general model in Fock space for particular sectors of quantum numbers, such as momentum, parity, total spin and number. This leads one to real symmetric matrices in various dimensions $N \geqslant 2$. These have the remarkable feature that the von Neumann–Wigner noncrossing rule [6] is violated. One ends up with several level crossings that are conventionally termed 'accidental'. This terminology is rather avoidable, since there is also a belief that there is nothing accidental in having such level crossings; the existence of several dynamical conservation laws (dependent on coupling constants) are believed to be causally implicated. Further, the statistics of energy levels of these integrable models are also known to be close to Poisson statistics, and hence consistent with the absence of level repulsion that generic systems are known to possess [7, 8].

[1] The dynamical conservation laws of the Heisenberg model are discussed in this reference and several works e.g., [8].

While the current general programs for the study of quantum integrable systems focus on properties such as factorizable S matrices, or the Yang–Baxter relation, the approach of [1] gets to the core of the issue of the matrix realizations of these models. As such, it is 'blind' to the specific physical details of the models. The main result of [1] is the identification of a class of matrices, termed Type-I matrices, discussed in detail below. Here the core property of multiple parameter-dependent conservation laws is made explicit, and one has an algorithm for generating such matrices as well as a count of the number of such matrices.

In an impressive work, [9] Owusu, Wagh and Yuzbashyan (OWY) have built on this initial advance, and produced several further results. A fundamental advance is the introduction of a basis of matrices, in terms of which the matrices of [1] can be expanded. OWY further show a link with an integrable model due to Michel Gaudin [10], that is currently enjoying popularity in the context of superconductivity of finite systems [10]. OWY also throw light on the 'mechanism' of the the level crossing, and give explicit formulas for the number of level crossings one finds in type-I matrices.

The objective of this paper is multifold. Firstly, a fundamental constraint equation in [1] for constructing type-I matrices is shown to be related to the so-called Plücker relations of Pfaffians. Since Pfaffians are basic to anticommuting objects such as Majorana fermions, one sees that Fermi statistics enters this program of describing integrable systems in a fundamental and unexpected fashion. From this analysis, the parameterization of the solutions of Type-I matrices by OWY in [9], arises as an elegant consequence, and the entire construction becomes more transparent.

Secondly, I show that the link with the Gaudin-type model [10] is made more naturally with Fermi (or canonical Bose) statistics. The connection made by OWY with the Gaudin model assumed hard core Bose statistics for the particles, and is confined to the sector of one spin wave, i.e. is confined to a specific sub manifold. The basic quantum operator underlying this class of problems is the permutation operator that has several possible realizations, leading to distinct models. The permutation operator has a Fermi representation: this is shown to be more natural than the (hard core) Bosonic one used in [9]. Once the commutation relations of a set of matrices is established, we can elevate these to operator relations with either Fermi or Bose statistics (see equation (19) below) and thus also have a Bosonic representation of these.

With either Fermi, or soft core (i.e. canonical) bosons, we construct a quantum glass model below, i.e. a particular type of Anderson model for disordered carriers. This model is akin to Gaudin's model with hard core spins, and depends on several parameters that may be chosen as one wishes, and has commuting partners in *all particle sectors*. These commuting partners may be thought of as local charges that are broadened out in a specific way.

Interestingly the Plücker relations arise in other aspects of integrable systems as well. these are central objects in Sato's work on classical solitonic theories (i.e. classically integrable systems) [11, 12], where the so-called τ functions satisfy these relations. For quantum integrable models, a connection has been shown to exist between the transfer matrices of and the bilinear identities of the τ functions [13] satisfying Plücker relations.

A few remarks are useful to put the current work and the related [1, 9], within the context of matrix theory as used in quantum mechanics. In order to keep things simple, let us specialize to finite dimensions[2]. Quantum observables lead to Hermitian, or real symmetric matrices, and simultaneous measurability of observable pairs translates to the theorem that commuting matrices of the above type are simultaneously diagonalizable. One simple result about two such commuting matrices a and b, is that one of them is expressible as a power series in the

[2] By sticking to lattice models in finite dimensions, we are following the Mark Kac dictum:
Be Wise,
Discretize.

other[3]. The current series of works differ from these in that the focus is on matrices that depend in a simple way (linearly, or possibly a polynomial of low degree) on a parameter, and one insists upon the commutation property for all values of the parameter. This problem is natural in the context of examples in quantum theory, such as the hydrogen atom, where the Laplace–Runge–Lenz vector depends linearly on the squared electric charge. It is also true in the structure of the *higher conservation laws* in models such as the Heisenberg [5] and Hubbard models [3, 4]. The notable results in the works [1, 9] follow from the detailed study of the simple parameter dependence of the commuting pair.

2. Summary of [1] and the introduction of a basis of commuting operators

In [1], we introduced a family of real symmetric matrices in N dimensions depending linearly on a parameter x. These were introduced as purely algebraic prototypes of integrable systems in finite dimensions and termed as type-I matrices. They are efficiently represented as

$$\alpha = \mathbf{a} + x\mathbf{A}, \qquad \text{with} \qquad \mathbf{A} = \mathbf{A}_d + [\mathbf{a}, \mathbf{S}], \tag{1}$$

with two generic diagonal matrices $\mathbf{a}$ and $\mathbf{A}_d$ having unequal entries, i.e. $\mathbf{a} = \text{DiagonalMatrix}\{u_1, \ldots, u_N\}$ with $u_i \neq u_j$[4], $\mathbf{A}_d = \text{DiagonalMatrix}\{A_1, \ldots, A_N\}$ with $A_i \neq A_j$, and a real antisymmetric matrix $S_{i,j} = -S_{j,i}$. One may think of $\mathbf{a}$ and $\mathbf{A}$ as the kinetic and potential energy matrices, and the parameter x as a perturbation parameter in typical quantum systems. In this notation, the role of the antisymmetric matrix S_{ij} is made explicit. There is no loss of generality since if we are given the matrix $\alpha(x)$ in an arbitrary basis as the sum of two noncommuting matrices, we can convert it to this form by performing an orthogonal transformation that diagonalizes the matrix $\alpha(x = 0)$.

In this way, we model integrable systems, without reference to their explicit origin in the physical world, as parameter-dependent matrices. This construction is inspired by the standard examples of the Hubbard and Heisenberg models. In these models, finite-dimensional matrices of the above type emerge on restricting the state space to various sectors of usual (parameter independent) conservation laws such as particle number, parity, spin and total momentum.

Since integrable systems are known to possess several parameter-dependent (i.e. dynamical) conservation laws, one wants to know if other matrices depending on x, possibly linearly, can be found. It was indeed shown that under certain conditions on S, summarized below, such commuting partners $\beta(x)$ can be found, i.e. $[\alpha, \beta] = 0$. The form of the dynamical conservation laws $\beta(x)$ was shown to be very similar to that of $\alpha(x)$:

$$\beta = \mathbf{b} + x\mathbf{B}, \qquad \text{with} \qquad \mathbf{B} = \mathbf{B}_d + [\mathbf{b}, \mathbf{S}], \tag{2}$$

where $\mathbf{b}$ and $\mathbf{B}_d$ are the diagonal matrices.

In [1], it was shown that the number of matrices of type-I in N dimensions is $\mathcal{V}_a = (3N-1)$, and for a given matrix $\alpha(x)$ out of this set there are a further $\mathcal{V}_b = N + 1$ matrices of the type $\beta(x)$.

The crucial condition on the antisymmetric matrix $\mathbf{S}$ was written in [1], in terms of its inverse elements $R_{i,j} \equiv \frac{1}{S_{ij}}$:

$$\phi_{(i,j,k,l)} \equiv R_{i,j}R_{k,l} - R_{i,k}R_{j,l} + R_{i,l}R_{j,k} = 0. \tag{3}$$

[3] If one of them a has distinct eigenvalues, say a_λ, then it is possible to express $b = \sum_{j=1}^{N} c_j a^j$ with suitable constants c_j, and N is the dimension of the matrices. This expansion is most easily seen in the basis where both matrices are diagonal. For the linear equations to be consistent, the Vandermonde determinant $\prod_{\lambda<\mu}(a_\lambda - a_\mu) \neq 0$ is required to be nonvanishing, leading to the requirement of nondegenerate eigenvalues. The case of degeneracy is obtained by taking suitable limits within this framework.

[4] We have changed the notation here from equation (1) and equation (2) with $a_r \to u_r$ and $b_r \to v_r$ in order to avoid a conflict with the notation of the Fermionic operators $a_r, a_r^\dagger$.

These equations are extensively discussed in mathematics literature as the Plücker relations [14], and their analysis is presented later. In our original work [1], we noted that these are greatly overdetermined equations, since there are ${}^{N}C_4$ quartets of indices and equations, but only ${}^{N}C_2$ matrix elements $R_{i,j}$ to be determined. In [1] it was shown, by using a consistency condition involving five indices (equation (I-15)), that this set has $\mathcal{V}_R = 2N - 3$ free parameters and hence independent solutions. For example one may choose at will the parameters $R_{1,j}$; $2 \leqslant j \leqslant N$ and $R_{2,k}$; $3 \leqslant k \leqslant N$, and the remaining R_{lm} are determined in terms of these with no conflicts.

In [9] OWY have shown that it is more efficient to introduce a basis of commuting operators in terms of which both the matrices α equation (1) and β equation (2) can be expanded linearly. Here a commuting basis guarantees the commutation of the matrices α and β. The basis of commuting operators $\{Z(r)\}$, with $1 \leqslant r \leqslant N$ may be written in terms of the Dirac projection operators $|i\rangle\langle j|$, as

$$Z(r) = |r\rangle\langle r| + x \sum_s [\rho_s(r)|s\rangle\langle s| + S_{r,s}(|r\rangle\langle s| + |s\rangle\langle r|)]. \tag{4}$$

Here $S_{r,s}$ will be seen below to be the same elements as in equation (1). The following commutator vanishes:

$$[Z(r), Z(s)] = 0, \tag{5}$$

provided the matrix elements $\rho_i(j)$ satisfy the conditions

$$\sum_r \rho_s(r) = 0, \tag{6}$$

$$\sum_s \rho_s(r) = 0, \tag{7}$$

$$\frac{S_{i,l}S_{l,j}}{S_{i,j}} = \Delta(i, j; l) = \rho_i(l) - \rho_j(l). \tag{8}$$

The constraint equation (7) guarantees that the trace of the operators vanishes, it is not necessary for the commutation of the Z's but is a convenient condition.

Now equation (8) can be rearranged in a way that eliminates the $\rho_i(j)$ variables as a four-index identity (where 'l' is a spectator index):

$$\Delta(i, j; l) + \Delta(j, k; l) + \Delta(k, i; l) = 0, \tag{9}$$

in this form it is identical to equation (I-10), and by using the inverse matrix elements $R_{i,j} \equiv \frac{1}{S_{ij}}$, it becomes precisely equation (3) above. The point of this construction is that we can now take sums of the basis operators in equation (1):

$$\alpha = \sum u_j Z(j), \quad \beta = \sum v_j Z(j)$$

and in this way recover the matrices found in [1]. By allowing some of the a_j to be pairwise equal, OWY obtain a somewhat greater freedom than that in [1], where all the u_j were chosen to be distinct in order to obtain generic matrices.

The number of independent parameters in the S's or equivalently in R's is $N(N-1)/2$, and as mentioned above, we showed in [1] that the constraints in equation (3) are mutually consistent, giving $2N - 3$ free parameters in S. OWY parameterize the solutions of equation (3) by a neat *ansatz*, namely

$$R_{i,j} = \frac{(\varepsilon_i - \varepsilon_j)}{(\gamma_i \gamma_j)}. \tag{10}$$

These are obtained in turn by performing a local gauge transformation equation (11), on a particular solution $R_{i,j} = (\varepsilon_i - \varepsilon_j)$ noted in [1]. This transformation consists of multiplying each matrix element by an arbitrary j-dependent factor:

$$R_{i,j} \to 1/(\gamma_i\gamma_j)R_{i,j}, \qquad \phi_{ijkl} \to \phi_{ijkl}\,/(\gamma_i\gamma_j\gamma_k\gamma_l), \tag{11}$$

and is clearly a way of generating further solutions from a given one.

Indeed equation (10) has the correct number of parameters $(2N - 3)$. To see this, we start with the N ε's, and the N γ's giving us $2N$ parameters. As discussed more fully below in equation (17), we subtract three parameters from $2N$, since we can shift all $\varepsilon's$ by a single constant and further scale all the ε's and all the γ's by two j-independent constants. Using equation (8), we may then infer the $\rho_j(i)$ from this parametrization of $R_{ij} = 1/S_{ij}$, and find

$$\begin{aligned}\rho_i(j) &= \frac{\gamma_j^2}{\varepsilon_i - \varepsilon_j}, \quad i \neq j \quad \text{and}\\ \rho_i(i) &= \gamma_i^2 \sum_{j\neq i} \frac{1}{\varepsilon_i - \varepsilon_j}.\end{aligned} \tag{12}$$

We will use this convenient parameterization in the rest of this work. For completeness, we note that the parameterization of equation (1) i.e. $\alpha = \sum u_j Z(j)$ in (I) translates to the new variables as follows:

$$\begin{aligned}S_{ij} &= \frac{\gamma_i\gamma_j}{\varepsilon_i - \varepsilon_j}\\ A_i &= \text{const} - \sum_{j\neq i} \gamma_j^2 \frac{u_i - u_j}{\varepsilon_i - \varepsilon_j}\\ Y_{ij} &= \frac{\gamma_i^2}{\varepsilon_i - \varepsilon_j} - \sum_{k\neq i} \frac{\gamma_k^2}{\varepsilon_i - \varepsilon_k},\end{aligned} \tag{13}$$

with Y_{ij} from equation (I-5), so that $\mu(i;\, jk)$ in equation (I-19) vanishes identically.

3. Plücker relations and the parametrization of the antisymmetric S matrix

Equation (3) was recognized belatedly by the author, as Plücker relations of the mathematical literature. Since these are central to the construction of this class of matrices we take a closer look at the $2N - 3$ solutions that were found in [1]. We explore the structure of the relations by using a more rigorous technique next, and see that the *ansatz* of OWY follows from the analysis as the unique solution. We show that these relations involve the so-called Plücker relations for Grassman variables, and hence presage the final form our presentation that involves Fermions in a fundamental way.

We begin by noting that equation (3) involves $\phi_{(ijkl)}$, which is a Pfaffian of a real 4×4 skew symmetric matrix $R_{i,j}$. The vanishing of $\phi_{(ijkl)}$ is a standard example of a Plücker relation [14]. The totality of these equations is expressed elegantly using exterior forms. Let us define an N-dimensional real vector space spanned by unit vectors e_j and define an antisymmetric wedge products $e_i \wedge e_j$. These provide a basis for the linear vector space $W(2)$ [14]. In this space, we define for a skew symmetric $R_{i,j}$ a 'two form':

$$\mathcal{R} = \sum_{i<j} R_{i,j}\; e_i \wedge e_j.$$

It is now easy to see that

$$\mathcal{R} \wedge \mathcal{R} = 2 \sum_{i<j<k<l} \phi_{i,j,k,l}\, e_i \wedge e_j \wedge e_k \wedge e_l,$$

and hence we recognize that the totality of relations in equation (3) are precisely equivalent to finding solutions of

$$\mathcal{R} \wedge \mathcal{R} = 0. \tag{14}$$

This condition defines [14] the 'decomposability' of the two-form $\mathcal{R}$. This problem can be resolved by noting that every skew symmetric matrix can be expressed in its real normal form involving $2n$ orthonormal vectors a^α and b^α with $1 \leqslant \alpha \leqslant n$, satisfying the conditions $\sum_j R_{i,j} a_j^\alpha = \lambda^\alpha b_i^\alpha$ and $\sum_j R_{i,j} b_j^\alpha = -\lambda^\alpha a_i^\alpha$. We may term these as the pseudo eigenvectors and pseudo eigenvalues, since the Hermitian matrix $i\,R$ has real eigenvalues $\pm\lambda^\alpha$ and real eigenfunctions $\frac{1}{\sqrt{2}}(a_j^\alpha \pm b_j^\alpha)$, and $n \leqslant N/2$ is the number of nonzero eigenvalues of $i\,R$. The normal form is expressed as

$$R_{i,j} = \sum_{\alpha=1,n} \lambda^\alpha a_i^\alpha b_j^\alpha.$$

With this decomposition, and with $v^\alpha = \sqrt{\lambda^\alpha} \sum_j a_j^\alpha e_j$ and $w^\alpha = \sqrt{\lambda^\alpha} \sum_j b_j^\alpha e_j$, we can rewrite the relation

$$\mathcal{R} = \frac{1}{2} \sum_{\alpha=1,n} v^\alpha \wedge w^\alpha.$$

We thus see that equation (14) is possible if and only if the number of vectors $n = 1$, i.e. there is only *one pseudo eigenvector* of R. This is known as the condition of decomposability, and provides us with a neat representation equation (15) with a single eigenvalue λ and a pair of orthonormal vectors x_j y_j:

$$R_{i,j} = \lambda(x_i y_j - y_i x_j). \tag{15}$$

Using the local gauge invariance in equation (11), we can drop the condition of orthonormality of x_j and y_j in equation (15), as far as generating solutions to the original problem equation (3) is concerned. We may also absorb the λ^α factor into the vectors, and it appears that we have $2N$-independent real parameters in the solution of equation (3). However, we observe that there is a redundancy in this counting, the vectors x_j and y_j can be transformed without changing $R_{i,j}$ if use three linear transformations with arbitrary parameters p, q, r as

$$\begin{aligned} (x_j, y_j) &\to (x_j + p\ y_j, y_j) \\ (x_j, y_j) &\to (x_j, y_j + q\ x_j) \\ (x_j, y_j) &\to \left(r\ x_j, \frac{1}{r}\ y_j\right). \end{aligned} \tag{16}$$

We thus see that the total number of real parameters available is exactly $2N - 3$ as known already from [1]. One convenient set of $2N - 3$ variables was given as $R_{1,j}$ with $2 \leqslant j \leqslant N$, and $R_{2,j}$ with $3 \leqslant j \leqslant N$, in terms of the x_j, y_j we may e.g. set $x_1 = 1, y_1 = 1, x_2 = 1$ and determine the remaining $2N - 3$ variables from the $R_{i,j}$'s. The parameterization equation (10) of OWY can be obtained from equation (15) by setting $\varepsilon_j = \frac{x_j}{y_j}$ and $\gamma_j = \frac{1}{y_j}$, and the symmetries of equation (16) are transformed into

$$\begin{aligned} (\varepsilon_j, \gamma_j) &\to (\varepsilon_j + p, \gamma_j) \\ (\varepsilon_j, \gamma_j) &\to \left(\frac{\varepsilon_j}{1 + q\varepsilon_j}, \frac{\gamma_j}{1 + q\varepsilon_j}\right) \\ (\varepsilon_j, \gamma_j) &\to (r^2\varepsilon_j, r\gamma_j). \end{aligned} \tag{17}$$

We may again reduce the apparent $2N$ parameters by (3) using these relations; it amounts to choosing three parameters, say $(\varepsilon_1, \gamma_1, \gamma_2)$ arbitrarily as e.g. (1, 1, 1) and the rest are fixed using the inverse of equation (10).

4. Fermionic representation of commuting operators

We next show that the matrices $Z(r)$ in equation (4) lead to a neat Fermionic representation, which may be thought of as a model for a Fermi glass with localized states. Let us define a Fermionic set of operators $a_j, a_j^\dagger$ and $n_j = a_j^\dagger a_j$, obeying the standard anticommutation relations

$$\left\{a_i, a_j^\dagger\right\} = \delta_{ij}, \tag{18}$$

with $1 \leqslant i \leqslant N$. It is elementary to see that two commuting matrices $[P, Q] = 0$ lead to a commuting set of Fermionic operators (e.g. see [15]), i.e.

$$\left[\sum_{ij} P_{ij} a_i^\dagger a_j, \sum_{ij} Q_{ij} a_i^\dagger a_j\right] = \sum_{lm} [P, Q]_{lm} a_l^\dagger a_m = 0, \tag{19}$$

where $[P, Q]$ is the matrix commutator of the two matrices P_{ij} and Q_{ij}. Thus, we obtain a set of N Fermionic operators:

$$\hat{Z}(r) = n_r + x \sum_s \left[\rho_s(r) n_s + S_{r,s}\left(a_r^\dagger a_s + a_s^\dagger a_r\right)\right]. \tag{20}$$

We see that these inherit the commutation property $[\hat{Z}(r), \hat{Z}(s)] = 0$ from equation (5). Using the parametrization equation (10) and equation (12), we write the basis set of commuting operators as

$$\hat{Z}(r) = n_r + x \sum_s{}' \frac{1}{\varepsilon_r - \varepsilon_s} \left[\gamma_r \gamma_s \left(a_r^\dagger a_s + a_s^\dagger a_r\right) - \gamma_r^2\, n_s - \gamma_s^2\, n_r\right], \tag{21}$$

where the prime indicates $s \neq r$.

Readers wishing to skip the earlier discussions, can directly verify that the commutator $[\hat{Z}(r), \hat{Z}(s)]$ vanishes, for arbitrary values of the given parameters by a straightforward calculation.

We also remark that the choice of the statistics of the canonical operators a_j is not the only one possible. The entire argument of this section can be repeated if we use canonical Bosonic operators instead, i.e. $a_j \to b_j$ where $\left[b_i, b_j^\dagger\right] = \delta_{ij}$. Thus, one can equally well consider a Bosonic glass rather than a Fermi glass model here.

Finally we note that the single particle sectors of the Bosonic, Fermionic and hard core Bosonic models are all identical and correspond to type-I matrices. For higher numbers of particles, these correspond to other classes of matrices depend on the statistics chosen, e.g. these are Kronecker products of type-I matrices in the case of canonical fermions and bosons.

4.1. Mapping to the Gaudin model

The mapping discussed by OWY views equation (4) as the $S^z_{\text{Total}} = N/2 - 1$ subspace representation of the Gaudin Hamiltonian [10, 16]:

$$Z_i^{\text{Gaudin}} = S_i^z + x \sum_j{}' \frac{1}{\varepsilon_i - \varepsilon_j} \vec{S}_i \cdot \vec{S}_j. \tag{22}$$

This model was first written down by Gaudin [10]. Gaudin actually wrote it without the first term S_i^z, this was supplied later by Sklyanin [16] from twisting the boundary conditions. To be exact equation (4) has an extra factor of $\gamma_i \gamma_j$ that OWY argue can be incorporated into the equations, and also their magnetic field term B is $\propto 1/x$. The Gaudin model is currently very

popular for describing the dynamics of Cooper pairs within the BCS theory for finite systems [17, 18]. Each spin flip represents a Cooper pair, from the Anderson mapping of the BCS theory to spin waves. Thus, $S_i^- = c_{k_i\downarrow}c_{-k_i\uparrow}$ and the label i is actually a momentum space label.

The point about the Fermionic representation equation (21) of equation (4) is that *it is true for all numbers of fermions*, and not restricted to a single particle sector. In this sense, the present Fermionic representation is much more powerful, and further the factors γ_i do not need any special treatment, they are automatically treated in the commutation relations. Thus, equation (4) are embedded without any further qualifications in the operator equation (21). We see below that this representation enables us to find applications of this model for fermions in a disordered potential, i.e. the Fermi glass problem. The same statement is also true if we use canonical Bosonic operators instead of fermions, as mentioned above. However, the Gaudin model is expressed in terms of hard core bosons, and the magnitude of the spin is related to the γ_j making the scheme somewhat cumbersome[5].

5. Diagonalizing the Fermi Hamiltonian and the density of states

We now turn to a study of equation (21) and a related Hamiltonian obtained by summing

$$H_R = \sum_r \varepsilon_r \hat{Z}(r) = \sum_r n_r \varepsilon_r + x \sum_{ij} \gamma_i \gamma_j a_i^\dagger a_j - x\ \hat{N}\ \sum_j \gamma_j^2, \tag{23}$$

where $\hat{N} = \sum_r n_r$ is the number operator. For fermions or canonical bosons, this Hamiltonian is the analog of the so-called Richardson [19] Hamiltonian in the theory of nuclear matter (the γ factors do not usually arise in the latter). The Richardson model and also the related BCS [17] problem for finite systems [18], are expressed in terms of hard core bosons (i.e. spin half objects) representing Cooper pairs $S_i^- = c_{k_i\downarrow}c_{-k_i\uparrow}$. These are in turn, obtained by taking sums over the Gaudin Z_j^{Gaudin} operators of equation (22). The Hamiltonian equation (23) is considerably simpler to solve for a general population of particles than the corresponding problem for hard core bosons, and is akin to a free gas of particles in a suitable one-body potential. Clearly our Hamiltonian equation (23) commutes with each of the $\hat{Z}(r)$, and plays a central role in the Fermi glass interpretation.

If we view the labels i, j as wave vector indices, then H describes a band model with an arbitrary dispersion ε_i. It is subject to a potential that scatters from every wave vector to each of the others, with a potential matrix element $x\gamma_i\gamma_j$. Since the γ_i are arbitrary, they may be chosen at random. We thus realize a band model with a separable random scattering potential. If on the other hand, we view i as site labels in a tight binding model, the energies ε_i may be chosen at random, and the kinetic energy hops between every pair of sites- i.e. realizing an infinite ranged random Fermionic Anderson model.

We now turn to the task of diagonalizing the Hamiltonian and all the $\hat{Z}(r)$ by an orthogonal transformation. This transformation for the single particle sector is essentially identical to the one in Richardson [19] , and many subsequent works, and hence we will be brief. Define a new canonical Fermion set

$$d_i^\dagger = \sum_j Q_{ij}\ a_j^\dagger, \qquad \text{with} \qquad \left\{d_i, d_j^\dagger\right\} = \delta_{ij}, \tag{24}$$

[5] It is possible to include the γ_i factors into an inhomogeneous 6-vertex model, provided we allow for horizontal and vertical electric fields. The only value of anisotropy that readily supports the inclusion of these fields is the Free Fermi point of the 6-vertex model, so that we end up with the Fermi representation reported here. It does not seem useful to dwell on the detailed construction in view of the simplicity of the alternate argument in equation (19).

through an orthogonal transformation generated by a real orthogonal matrix Q such that $Q^T \cdot Q = \mathbf{1}$, and

$$Q_{ij} = \frac{\phi_i\ \gamma_j}{\omega_i(x) - \varepsilon_j}, \tag{25}$$

$$\phi_i^{-2} = \sum_j \left(\frac{\gamma_j}{\omega_i(x) - \varepsilon_j}\right)^2, \tag{26}$$

$$\frac{1}{x} = \sum_j \frac{\gamma_j^2}{\omega_m(x) - \varepsilon_j}. \tag{27}$$

Here $\omega_m(x)$ in equation (27) are the N eigenvalues of the transformed Hamiltonian equation (23), we write the argument (x) to emphasize that these depend parametrically on x. A further short calculation gives

$$H_R = \sum_m \omega_m(x)\ d_m^\dagger d_m - x\ \hat{N}\ \sum_j \gamma_j^2, \tag{28}$$

$$\hat{Z}(j) = x\ \gamma_j^2\ \sum_m \frac{1}{\omega_m(x) - \varepsilon_j}\ d_m^\dagger d_m. \tag{29}$$

As $x \to 0^{\pm}$, one sees that $\omega_j \to \varepsilon_j \pm 0$, i.e the ω's are pinned to the ε's. The eigenvalues $\omega_m(x)$ are in 1–1 correspondence and evolve out of the numbers ε_m smoothly as x increases from zero. Thus, the eigenvalues ω_j of the Richardson Hamiltonian equation (23) interlace the numbers ε_j, with one extremal eigenvalue that grows linearly with x. For $x \gg 0$ ($x \ll 0$), the extremal eigenvalue $\omega_N \gg \varepsilon_N$ ($\omega_1 \ll \varepsilon_1$). The density of states of ω_j has a width that remains fixed with x if we ignore the exceptional extremal case. It is easy to see that the ω_m do not cross each other as x varies, but they *do satisfy* the von Neumann–Wigner noncrossing rule. The conserved quantities $\hat{Z}_j$ may be visualized as evolving continuously from the occupation numbers n_j as x increases from zero.

We could more generally consider the two operators formed from the sums

$$\hat{\alpha}(x) = \sum_r u_r\ \hat{Z}(r), \qquad \text{and} \qquad \hat{\beta}(x) = \sum_r v_r\ \hat{Z}(r), \tag{30}$$

with arbitrary u_r and v_r, and see immediately that these are the Fermi space representations of the operators introduced in equation (1) and equation (2):

$$\begin{aligned}
\hat{\alpha}(x) &= \sum_r u_r\ n_r + \frac{x}{2} {\sum_{r,s}}' \frac{u_r - u_s}{\varepsilon_r - \varepsilon_s} \left[\gamma_r \gamma_s \left(a_r^\dagger a_s + a_s^\dagger a_r\right) - \gamma_r^2\ n_s - \gamma_s^2\ n_r\right], \\
\hat{\beta}(x) &= \sum_r v_r\ n_r + \frac{x}{2} {\sum_{r,s}}' \frac{v_r - v_s}{\varepsilon_r - \varepsilon_s} \left[\gamma_r \gamma_s \left(a_r^\dagger a_s + a_s^\dagger a_r\right) - \gamma_r^2\ n_s - \gamma_s^2\ n_r\right].
\end{aligned} \tag{31}$$

These commute mutually for any choice of the parameters, including x, and also with the constants of motion Z_j in equation (21), and on diagonalization become

$$\begin{aligned}
\hat{\alpha}(x) &= \sum_m \alpha_m(x)\ d_m^\dagger d_m, \\
\alpha_m(x) &= x\ \sum_j \frac{\gamma_j^2\ u_j}{\omega_m(x) - \varepsilon_j}\ .
\end{aligned} \tag{32}$$

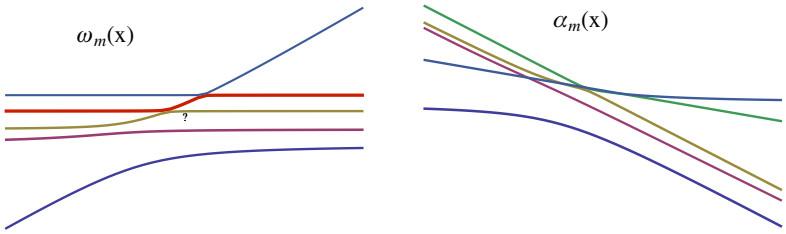

Figure 1. Left panel shows the eigenvalues $\omega_m(x)$ from equation (27) for the case of $N = 5$, where one observes narrowly avoided level crossings involving the top three levels. Right panel shows the effect of mixing levels through equation (30) and (32) with $u_j - \varepsilon_j$ chosen randomly with a small scale of variation. We see that the eigenvalues $\alpha_m(x)$ obtained from equation (30) and (32), cross each other profusely, thereby violating the Wigner von Neumann noncrossing rule.

A comment on the conservation laws equation (21) and their relationship with the 'Hamiltonian' equation (31) is useful here. At $x = 0$ the existence of N constants of motion of the Hamiltonian is obvious since the $Z's$ are just the number operators of the fermions. When we perturb the Hamiltonian from this 'free case' by adding *any* term proportional to x, we can always fix the conservation law to be valid to $O(x)$, but generally the terms do not commute to $O(x^2)$. This is familiar in the theory of integrability violating perturbations to integrable systems, as in the Kolmogorov–Arnold–Moser theory [20][6]; the conservation laws analogous to equation (21) *can be rescued* to linear order in the new perturbations, but not to higher orders. The speciality of the specific perturbation in equation (31) is that there are no corrections to $O(x^2)$ and the the conservation law equation (21) commute exactly.

The inevitability of level crossings for type-I matrices was noted empirically in [1], on the basis of several examples that were studied. However the theoretical explanation awaited the work of OWY, who showed that for a generic choice of u_r, the eigenvalues of $\hat{\alpha}(x)$, i.e. $\alpha_m(x)$ in equation (32), have at least one and at most ${}^{N-1}C_2$ level crossings as x varies over its range. These eigenvalues thus defy the von Neumann–Wigner noncrossing rule, unlike the eigenvalues of the Richardson Hamiltonian $\omega_m(x)$, which do obey the rule. The one exceptional case is $u_j = \varepsilon_j$ when the $\alpha(x)$ reduces to the Richardson Hamiltonian equation (23)[7]. One may understand the violations of the noncrossing rule by thinking of the eigenvalues of $\hat{\alpha}(x)$ as smeared versions of $\omega_m(x)$, and thereby less sharply governed by the rule. This is illustrated in figure 1, where we plot the energy levels for $N = 5$, and show that while $\omega_m(x)$ of equation (27) avoids level crossings, the derived eigenvalues $\alpha_m(x)$ from equation (30) and (32) *do display level crossings*. In this sense, there is a hidden generic model H_R satisfying the noncrossing rule, behind the violations of the same in the constructed matrices $\alpha(x)$.

Finally, we note that the Hamiltonians equation (31) with Fermionic a_j (Bosonic b_j) can be viewed as representing a class of localized states in the Fermi (Bose) glass problem of disordered noninteracting quantum particles. At $x = 0$, the model consists of localized states with energies u_r, and clearly has N conservation laws $Z(r)$ as in equation (21), corresponding

[6] The Kolmogorov–Arnold–Moser theorem is discussed in this reference.

[7] In OWY, the exception is accommodate with the help of a slightly different viewpoint. In their view equation (23) does have the requisite number of level crossings, provided we include the limiting cases $x \to \pm\infty$. The spectrum of the large $|x|$ limit of equation (23), namely $\sum_{ij} \gamma_i \gamma_j a_i^\dagger a_j - \hat{N} \sum_j \gamma_j^2$, has a single isolated eigenvalue, and $N - 1$ degenerate (null) eigenvalues. The null eigenvalues can be viewed as consisting of ${}^{N-1}C_2$ pairwise crossings. As $|x|$ reduces from ∞, some of the level crossings move towards smaller $|x|$, whereby all choices of u_j fall into a common description. Our view is a slightly different; at a qualitative level it seems useful to think of the level crossing as arising from a smearing of the avoided crossing.

to the occupation numbers of the different sites. As x varies from zero, the particles hop around as dictated by the Hamiltonian, but with generalized conserved occupancies at all sites given by equation (21). These are therefore localized to all orders in the perturbation x, despite hoppings that carry them far away. We can easily see that the energy level statistics of these systems follow the Poisson distribution for small separations, due to the level crossings that occur in these Hamiltonians. The absence of level repulsion what one expects from localized states in the Anderson model on general grounds.

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