Functionally independent conservations laws in a quantum integrable model

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Abstract

We study a recently proposed quantum integrable model defined on a lattice with N sites, with fermions or bosons populating each site, as a close relative of the well known spin-$\frac{1}{2}$ Gaudin model. This model has $2N$ arbitrary parameters, a linear dependence on an interaction type parameter x, and can be solved exactly. It has N known constants of motion that are linear in x. We display further constants of motion with higher fermion content that are linearly independent of the known conservation laws. Our main result is that despite the existence of the higher conservation laws, the model has only *N functionally independent conservation laws*. Therefore we propose that N can be viewed as the number of degrees of freedom, in parallel to the classical definition of integrability.

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Introduction

Quantum integrable systems have emerged from an esoteric beginning in the pioneering works of Bethe, Onsager, Yang, Baxter [1] and others, and in recent years have been of interest to a much wider community. Optical lattices where quantum quenches can be realized to study dynamics away from equilibrium [2–4], transport theory [5, 6] and considerations of entanglement entropy [7] are some areas where the non-standard models of quantum integrability, such as the Ising model in a transverse field, the anisotropic Heisenberg or XXZ model, and the 1D Hubbard model find a wide range of applications unanticipated in the pioneering studies.

In this renewed era of interest, some basic questions about quantum integrability remained unanswered. While the Yang–Baxter equations provide a deep mathematical underpinning to this field and indeed are the source of most quantum integrable models, there are good reasons to look more broadly at the field. For instance, there are other families of quantum integrable models that do not fit naturally into this scheme, such as the Calogero–Sutherland systems. Also, classical integrable systems have a well evolved language [8] of 'degrees of freedom'

and matching 'functionally independent conservation laws', that are not easily translated to the quantum arena. The meaning of the phrase *degrees of freedom* is not always clear in many models of current interest. The number of conservations laws, even of well established models, such as the XXZ model, is not quite a settled matter, since new conservation laws have been found in very recent studies [5], with important implications for the transport behaviour of this model [6]. Discussions of the meaning of quantum integrability are therefore of interest and have been addressed in recent literature [9–12].

In this work, we study a recently proposed quantum lattice model [9] that is defined on a lattice of N sites, with either fermions or bosons at each site, which is similar to the well known Gaudin spin-$\frac{1}{2}$ model [1, 13]. For the fermion model, we show that N, the number of lattice sites, can be viewed as the number of degrees of freedom in close analogy to the classical definition. To do this, we are able to enumerate the basic N conservation laws from the one-particle sector. These basic conservation laws are *linear in a parameter x*; this linearity (or simple polynomial dependence) has been highlighted in a recent work [12]. This reference also gives the interested reader an extended discussion of the relationship between the present model and its parameters, and the well known families of models described by the Yang–Baxter equations. Our most important result is as follows. We find that while other non-trivial (i.e., four-Fermi and six-Fermi) conservation laws do exist in the higher particle sector, even linearly independent from the basic ones, these are *functionally dependent* on the basic ones. We display the generating functions for the higher conservation laws, and the non-trivial functional relation between these. In the case of bosons and the Gaudin model, using an interesting algorithm, we have checked that up to at least $N = 20$, there are *no other non-trivial conservation laws* that are polynomial in the parameter x, and hence the N conservation laws available from the single particle sector are again the entire set.

To summarize the models studied here, let us note that a prototypical quantum integrable model can be built up in two steps.

Step I. We identify a family of real symmetric commuting matrices $H(x)$ and $\widetilde{H}(x)$ in the N dimension, that depend linearly on a parameter x:

$$[H(x), \widetilde{H}(x)] = 0, \quad \text{for all values of } x. \tag{1}$$

This problem leads to several classes of solutions that differ in the number of independent Hamiltonians $\widetilde{H}(x)$ that can be found for a given $H(x)$. These are referred to as type M, with various values for M as in [14, 15]. Type 1 represents a maximal set of such matrices with at most N commuting partners (including the identity matrix), where all such matrices can be expressed in terms of the projection operators $\pi_{ij} = |i\rangle\langle j|$ and $3N$ arbitrary real constants $\{\gamma_i, \varepsilon_i, d_i\}$ as:

$$H(x) = \sum_i d_i Z_i, \tag{2}$$

$$Z_i = \pi_{ii} + x \sum_j{}' \frac{\gamma_i \gamma_j (\pi_{ij} + \pi_{ji}) - \gamma_i^2 \pi_{jj} - \gamma_j^2 \pi_{ii}}{\varepsilon_i - \varepsilon_j}. \tag{3}$$

The prime indicates the exclusion of the summed index with the fixed index. The commutation of the N basis operators $[Z_i, Z_j] = 0$ is at the heart of this construction, and provide the N 'higher constants of motion' or dynamical symmetries, which depend parametrically on x. This is the maximal set of commuting operators, whose number equals the dimension of the defining space N.

Step II. The next step is to embed the operators of equation (3) into Fock space, thus $Z_i \to \hat{Z}_i$ by writing these in terms of quantum field operators acting upon suitable spaces. This procedure thereby gives rise to a quantum integrable model with Hamiltonians

$H \rightarrow \hat{H} = \sum_i d_i \hat{Z}_i$. We may use spins (i.e. hard core bosons), canonical bosons or fermions to get models that coincide in the single particle subspace, but that in higher subspaces are essentially different models. Let us note the Fermi model:

$$\hat{Z}_r \equiv n_r + x \sum_s{}' \frac{\gamma_r \gamma_s (a_r^\dagger a_s + a_s^\dagger a_r) - \gamma_r^2 n_s - \gamma_s^2 n_r}{\varepsilon_r - \varepsilon_s}, \tag{4}$$

where $\{a_i, a_j^\dagger\} = \delta_{ij}$, $n_r = a_r^\dagger a_r$, and $[\hat{Z}_r, \hat{Z}_s] = 0$ for all $r, s = 1, 2, \ldots, N$ and for all values of x. Different commuting operators, linearly dependent on the parameter x, are expressible as in equation (2). With (canonical) bosons we get yet another model. Also, with spins in place of fermions and with coefficients $\gamma_i \rightarrow 1$, this model reduces to the well known Gaudin model. In all higher particle number sectors, these operators for any choice of the embedding Fock space have obvious descendent representatives that commute with each other and with $\hat{H}$. For instance, in the two-particle sector for fermions or the spin-$\frac{1}{2}$ Gaudin model, the Hilbert space has a dimension of $\binom{N}{2}$, where the above operators provide N commuting operators. These are of limited interest to us. Our aim is to seek out *operators that are linearly independent* of these descendent operators. In this larger space we could, for instance, independently construct type 1 matrices that are $\binom{N}{2}$ in number. Ostensibly there is room for additional, as yet undiscovered conservation quantities that act in the two-particle sector but are null in the one-particle sector.

In fact, our main goal in this work is to examine the proposal that N is the maximum number of *functionally independent commuting operators* for $\hat{H}$. Our search for functional independence starts with the more restrictive but technically feasible search for linear independence, followed by the test of functional dependence.

Algorithm for finding additional conservation laws

To uncover any such additional conservation laws, one can use an algorithm that we next describe. This algorithm can be implemented numerically for a relatively small size of the Hilbert space $\mathcal{N}$, as an illustration for fermions $\mathcal{N} \sim \binom{N}{r}$ where r is the number of particles. We used $\mathcal{N} \leqslant \binom{10}{3}$ in our studies of the Fermi, Bose and spin-$\frac{1}{2}$ Gaudin models, and the results are described below.

The algorithm uses the matrix decomposition outlined in [9, 14] and the current-constructing algorithm outlined in [15]. In $\mathcal{N}$ dimensions, if $[H(x), \widetilde{H}(x)] = 0$, it can be shown that there exists an antisymmetric matrix S such that

$$\begin{aligned} H(x) &= T + x([T, S] + W) \\ \widetilde{H}(x) &= \widetilde{T} + x([\widetilde{T}, S] + \widetilde{W}) \end{aligned} \tag{5}$$

where $T, \widetilde{T}, W, \widetilde{W}$ all mutually commute and, without loss of generality, we take them to be diagonal matrices. Given this decomposition equation (1) is satisfied if

$$[[T, S], [\widetilde{T}, S]] + [W, [\widetilde{T}, S]] - [\widetilde{W}, [T, S]] = 0. \tag{6}$$

For a given $H(x)$, S, T and W can be determined. Finding a conserved current $\widetilde{H}(u)$, i.e. finding a solution for the non-zero elements of diagonal matrices $\widetilde{T}$ and $\widetilde{W}$, is then a matter of solving $\binom{\mathcal{N}}{2}$ simultaneous linear equations in $2\mathcal{N}$ unknowns, where the number of linearly independent solutions is the number of independent conservation laws[1].

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[1] It should be emphasized that this algorithm yields trivial $\widetilde{H}(x)$ for general $H(x)$. The algorithm will only yield non-trivial results for an integrable matrix $H(x)$.

Fermions: the four-Fermi conserved current

We applied the above algorithm to the matrices arising from the action of the type 1 fermionic Hamiltonian $\hat{H} = \sum_j d_j \hat{Z}_j$ equation (4), in the two-particle sector and found that there is a $2N - 3$ member family of mutually commuting currents. N of these are just the action of the original currents $\hat{Z}_r$ (see equation (4)) in that sector. The $N - 3$ other conserved quantities correspond to the particle action of *four*-Fermi operators with a generating function of the form:

$$\hat{Q}(\alpha) = \frac{1}{2} \sum_{ij}{}' \frac{n_i n_j}{(\alpha - \varepsilon_i)(\alpha - \varepsilon_j)} + \frac{1}{2} x \sum_{i,j,k}{}' \frac{\phi_{ijk}}{(\alpha - \varepsilon_i)(\alpha - \varepsilon_j)(\alpha - \varepsilon_k)}, \qquad (7)$$

where $\phi_{ijk} \equiv \omega_{ij} n_k + n_i n_j \gamma_k^2$ and $\omega_{ij} \equiv \gamma_i \gamma_j (a_i^\dagger a_j + a_j^\dagger a_i) - n_i \gamma_j^2 - n_j \gamma_i^2$. Operators analogous to equation (4) that are indexed by 'r' can be found by taking residues

$$\hat{Q}_r = \lim_{\alpha \to \varepsilon_r} (\alpha - \varepsilon_r) Q(\alpha), \qquad (8)$$

these are seen to be (i) linear in x, (ii) are $N - 3$ in number, and (iii) linearly independent of $\hat{Z}_r$. Result (ii) requires an elaborate proof of three relationships between the apparently independent $\hat{Q}_r$. In particular one can show that $\sum_i \hat{Q}_i = 0$, and we skip the remaining two for brevity.

Analytical proof of commutation

In what follows we will prove that the generating function $\hat{Q}(\alpha)$ is conserved i.e.

$$[\hat{Z}_r, \hat{Q}(\alpha)] = 0 \qquad (9)$$

for all $r = 1, 2, \ldots, N$ and all values of x and α. To this end it is useful to define the quantity $\beta_{ij} \equiv a_i^\dagger a_j - a_j^\dagger a_i$.

The commutator has terms of $O(x^n)$ with $n = 0, 1, 2$ and each must vanish identically. The $O(x^0)$ terms vanish as all number operators mutually commute.

To satisfy equation (9) to $O(x)$ we must show[2] that

$$\sum_{jk}{}' \left[\frac{w_{ij}}{\varepsilon_i - \varepsilon_j}, \frac{n_i n_k}{(\alpha - \varepsilon_i)(\alpha - \varepsilon_k)} + \frac{n_j n_k}{(\alpha - \varepsilon_j)(\alpha - \varepsilon_k)} \right]$$

$$= \sum_{j,k}{}' \left[\frac{\phi_{ijk}}{(\alpha - \varepsilon_i)(\alpha - \varepsilon_j)(\alpha - \varepsilon_k)}, n_i \right], \qquad (10)$$

where we will subsequently follow the convention in which the prime on summation implies that all indicated indices are distinct. The RHS evaluates to

$$- \sum_{j,k}{}' \frac{\gamma_i \gamma_j \beta_{ij} n_k}{(\alpha - \varepsilon_i)(\alpha - \varepsilon_j)(\alpha - \varepsilon_k)}.$$

In the LHS, we note that $[w_{ij}, n_i n_k] = -[w_{ij}, n_j n_k] = -\gamma_i \gamma_j \beta_{ij} n_k$ and using the partial fraction identity

$$\frac{1}{\varepsilon_i - \varepsilon_j} \left(\frac{1}{\alpha - \varepsilon_i} - \frac{1}{\alpha - \varepsilon_j} \right) = \frac{1}{\alpha - \varepsilon_i} \times \frac{1}{\alpha - \varepsilon_j}, \qquad (11)$$

we see that equation (10) is satisfied.

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[2] We have used the identity $[w_{ij}, n_i n_j] = 0$ and also noted that on the right-hand side, the density n_i commutes with the variables labelled by k in $\phi_{i,j,k}$.

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To $O(x^2)$ we need to show

$$\sum_{j}^{'} \frac{[w_{ij}, \hat{Q}'(\alpha)]}{\varepsilon_i - \varepsilon_j} = 0, \tag{12}$$

where $\hat{Q}'(\alpha) = \frac{d}{dx}\hat{Q}(\alpha)$. We see from equation (7) that $\hat{Q}'$ depends on three indices and so for the fixed pair in w_{ij} in the equation (12), we may organize $\hat{Q}'$ as follows. The terms not involving either i or j commute and can be neglected. We may have both these indices in a class of terms $\hat{Q}'$, or only one index in another class of terms. Let us look at them separately.

Two indices common

The relevant part of $\hat{Q}'$ in equation (12) may then be written as

$$\sum_{k}^{'} \frac{1}{(\alpha - \varepsilon_i)(\alpha - \varepsilon_j)(\alpha - \varepsilon_k)} (\phi_{ijk} + \phi_{kij} + \phi_{jki}).$$

Now w_{ij} commutes with the first of these terms and for the others we calculate

$$[w_{ij}, \phi_{kij}] = -(\gamma_i^2 + \gamma_j^2)\beta_{ij}n_k + \gamma_i\gamma_k\beta_{jk}n_i - \gamma_j\gamma_k\beta_{ik}n_j, \tag{13}$$

and the term $[w_{ij}, \phi_{jki}]$ is obtained by exchanging $i \leftrightarrow j$ in equation (13), giving the negative of the first so that the sum vanishes identically.

One index common

The relevant terms in equation (12) may be written as

$$\sum_{j,k<l}^{'} \left[\frac{w_{ij}}{\varepsilon_i - \varepsilon_j}, \left(\frac{\phi_{ikl} + \phi_{ilk} + \phi_{kli}}{(\alpha - \varepsilon_i)(\alpha - \varepsilon_k)(\alpha - \varepsilon_l)} + (i \to j) \right) \right], \tag{14}$$

where we have used the symmetry of ϕ_{ijk} in the first two indices to isolate all terms with a triple of indices ilk that have one index i. It is easy to see that

$$[w_{ij}, \phi_{kli}] = -\gamma_i\gamma_j \, \beta_{ij}w_{kl} \tag{15}$$

$$[w_{ij}, \phi_{ikl} + \phi_{ilk}] = -\gamma_i\gamma_j\gamma_k(\beta_{ik}\gamma_j - \beta_{jk}\gamma_i)n_l - (k \leftrightarrow l) \tag{16}$$

$$[w_{ij}, \phi_{kli}\phi_{ikl} + \phi_{ilk}] = \gamma_i\gamma_j M_{ijkl}. \tag{17}$$

Here $\gamma_i\gamma_j M_{ijkl}$ is defined as the sum of the two terms equations (15) and (16). It is clearly symmetric in $k \leftrightarrow l$ and also antisymmetric in exchanging $i \leftrightarrow j$, so that using equation (11) we can rewrite the LHS of equation (12) as

$$\sum_{j,k<l}^{'} \frac{\gamma_i\gamma_j M_{ijkl}}{(\alpha - \varepsilon_i)(\alpha - \varepsilon_j)(\alpha - \varepsilon_k)(\alpha - \varepsilon_l)}. \tag{18}$$

We note an identity satisfied by cyclically permuting the indices jkl:

$$\gamma_j M_{ijkl} + \gamma_k M_{iklj} + \gamma_l M_{iljk} = 0, \tag{19}$$

whereby the commutator vanishes identically.

Functional relationship between the generating functionals

Let us note the generating function for the two-Fermi currents:

$$\hat{Z}(\alpha) = \sum_i \frac{n_i}{\alpha - \varepsilon_i} + \frac{1}{2}x \sum_{ij}' \frac{w_{ij}}{(\alpha - \varepsilon_i)(\alpha - \varepsilon_j)}, \tag{20}$$

so that $[\hat{Z}(\alpha), \hat{Z}(\beta)] = 0$ for any α, β, and

$$\hat{Z}_i = \lim_{\alpha \to \varepsilon_i} (\alpha - \varepsilon_i) Z(\alpha). \tag{21}$$

This is similar to the generating function for the four-Fermi currents in equation (7). For completeness we note that from equation (8)

$$\hat{Q}_i = \sum_j' \frac{n_i n_j}{\varepsilon_i - \varepsilon_j} + x \sum_{j,k}' \frac{\phi_{ijk} + \phi_{kji}}{(\varepsilon_i - \varepsilon_j)(\varepsilon_i - \varepsilon_k)}. \tag{22}$$

To summarize, we computed a bilinear generating function in equation (20) and showed that it has a four-fermionic set of constants of motion $\hat{Q}_r$ in equation (8), with a generating function equation (7), which are linear in x and also linearly independent of the $\hat{Z}_r$ in equation (21). Remarkably these four-Fermi currents, while linearly independent of the $\hat{Z}$ are not *functionally* independent of them. Indeed a brief calculation shows that

$$\hat{Q}(\alpha) = \frac{1}{2} \left[\frac{\hat{Z}(\alpha)^2}{1 - \frac{x}{\tilde{x}(\alpha)}} + \left(1 - \frac{x}{\tilde{x}(\alpha)} \right) \frac{\mathrm{d}}{\mathrm{d}\alpha} \left(\frac{\hat{Z}(\alpha)}{1 - \frac{x}{\tilde{x}(\alpha)}} \right) \right] \tag{23}$$

where $\tilde{x}(\alpha) \equiv \left(\sum_m \gamma_m^2/(\alpha - \varepsilon_m) \right)^{-1}$.

To gain a feel for these generating functions, it is useful to transform to the diagonal representation for the quadratic fermion model. In [9] one finds modes that diagonalize all the $\hat{Z}_i$ and, by extension $\hat{Z}(\alpha)$ by constructing the x-dependent canonical fermion creation/annhilation operator set:

$$d_i^\dagger = \sum_j \frac{\phi_i \gamma_j}{\lambda_i(x) - \varepsilon_j} a_j^\dagger, \quad \text{with} \quad \{d_i, d_j^\dagger\} = \delta_{ij}, \tag{24}$$

where we require that $\phi_i^{-2} = \sum_j \left(\gamma_j/(\lambda_i(x) - \varepsilon_j) \right)^2$ and $\sum_m \gamma_m^2/(\lambda_i(x) - \varepsilon_m) = 1/x$. It follows that the two-Fermi generating function can expressed as

$$\hat{Z}(\alpha) = \left(1 - \frac{x}{\tilde{x}(\alpha)} \right) \sum_m \frac{\tilde{n}_m}{\alpha - \lambda_m(x)}, \tag{25}$$

where $\tilde{n}_m \equiv d_m^\dagger d_m$, x-dependent fermionic number operator. Equation (23) can be rewritten in terms of these operators as:

$$\hat{Q}(\alpha) = \left(1 - \frac{x}{\tilde{x}(\alpha)} \right) \sum_{i<j} \frac{\tilde{n}_i \tilde{n}_j}{(\alpha - \lambda_i(x))(\alpha - \lambda_j(x))}. \tag{26}$$

In this representation, since both generating functions consist of the number operators in the diagonal basis, their mutual commutation is evident. However, the known linear dependence of the two generating functions on the parameter x is now hidden.

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Higher order fermionic conservation laws

We can continue the logic of the above section to higher particle sectors. While a more complete discussion is not possible, we shall content ourselves with a short presentation of our results. The type 1 fermionic model appears to possess a hierarchy of conservation laws, e.g. four-, six-, eight-Fermi terms, etc which are linear in x. We find from the above algorithm that a sequence of generating functions exist with $m = 1, 2, 3, \ldots$, where

$$\hat{C}^{(m)}(\alpha) \equiv \frac{1}{m!} \left(1 - \frac{x}{\tilde{x}(\alpha)} \right) \sideset{}{'}\sum_{k_1, k_2, \ldots, k_m} \prod_j^m \frac{\tilde{n}_{k_j}}{\alpha - \lambda_{k_j}(x)}, \tag{27}$$

is a $2m$-Fermi conserved current also linear in x, for $m < N$. Thus $\hat{Z}$ and $\hat{Q}$ discussed above correspond to $m = 1, 2$. If we denote reduced generating functions by $\hat{C}^{(m)}(\alpha) = (1 - \frac{x}{\tilde{x}(\alpha)}) \hat{C}^{(m)}_{\text{red}}(\alpha)$, then these satisfy the differential equation

$$\hat{C}^{(m)}_{\text{red}}(\alpha) = \frac{1}{m} \left(\frac{\mathrm{d}}{\mathrm{d}\alpha} + \hat{C}^{(1)}_{\text{red}}(\alpha) \right) \hat{C}^{(m-1)}_{\text{red}}(\alpha). \tag{28}$$

Though the commutativity of the generating functions is a straightforward consequence of equation (27), deriving the linearity in x of $\hat{C}^{(m)}(\alpha)$ is a bit more involved. In the (unrotated) original representation the linear x-dependence can be made explicit, i.e.

$$\hat{C}^{(m)}(\alpha) = \sideset{}{'}\sum_{k_1, k_2, \ldots, k_m} \left[\underbrace{\frac{1}{m!} \prod_j^m \frac{n_{k_j}}{\alpha - \varepsilon_{k_j}} \left(1 - x \sum_{s>m} \frac{\gamma_{k_s}^2}{\alpha - \varepsilon_{k_s}} \right)}_{c_1^{(m)}} \right.$$

$$\left. + x \underbrace{\frac{1}{2(m-1)!} \sum_{s>m} \gamma_{k_s} \gamma_{k_m} \frac{a_{k_s}^\dagger a_{k_m} + a_{k_m}^\dagger a_{k_s}}{(\alpha - \varepsilon_{k_s})(\alpha - \varepsilon_{k_m})} \prod_j^{m-1} \frac{n_{k_j}}{\alpha - \varepsilon_{k_j}}}_{c_2^{(m)}} \right], \tag{29}$$

where $k_1, k_2, \ldots, k_N$ is an arbitrary permutation of the list of integers from 1 to N. For example, the six-Fermi conserved current is:

$$\hat{C}^{(3)}(\alpha) = \sideset{}{'}\sum_{k_1, k_2, k_3} \left[\frac{1}{6} \frac{n_{k_1} n_{k_2} n_{k_3}}{(\alpha - \varepsilon_{k_1})(\alpha - \varepsilon_{k_2})(\alpha - \varepsilon_{k_3})} \left(1 - x \sum_{s>3} \frac{\gamma_{k_s}^2}{\alpha - \varepsilon_{k_s}} \right) \right.$$

$$\left. + x \frac{1}{4} \sum_{s>3} \frac{n_{k_1} n_{k_2} (a_{k_3}^\dagger a_{k_s} + a_{k_s}^\dagger a_{k_3})}{(\alpha - \varepsilon_{k_1})(\alpha - \varepsilon_{k_2})(\alpha - \varepsilon_{k_3})(\alpha - \varepsilon_{k_s})} \right]. \tag{30}$$

The proof that this expression satisfies equation (28) proceeds by induction. Assume equation (29) to be true for $\hat{C}^{(m-1)}(\alpha)$. Note that the product $(1 - x/\tilde{x}(\alpha))^{-1} \hat{C}^{(1)}(\alpha) \cdot \hat{C}^{(m-1)}(\alpha)$ contains $(m-1)$-Fermi terms, as well as m-Fermi ones—we are concerned with the latter, as the former are exactly cancelled in equation (28) by the $\frac{\mathrm{d}}{\mathrm{d}\alpha} \hat{C}^{(m-1)}(\alpha)$ term. The m-Fermi terms are given by

$$\left[c_1^{(1)} c_1^{(m-1)} \right]_m = \sideset{}{'}\sum_{k_1, k_2, \ldots, k_m} \left[m \left(1 - \frac{x}{\tilde{x}(\alpha)} \right)^2 + x \sum_i^m \frac{\gamma_{k_i}^2}{\alpha - \varepsilon_{k_i}} \left(1 - \frac{x}{\tilde{x}(\alpha)} \right) \right.$$

$$\left. + 2x^2 \sideset{}{'}\sum_{i,j} \frac{\gamma_{k_i}^2 \gamma_{k_j}^2}{(\alpha - \varepsilon_{k_i})(\alpha - \varepsilon_{k_j})} \right] \frac{1}{m!} \prod_j^m \frac{n_{k_j}}{\alpha - \varepsilon_{k_j}} \tag{31}$$

$$[c_1^{(1)} c_2^{(m-1)} + c_2^{(1)} c_1^{(m-1)}]_m = \sum_{k_1,k_2,\ldots,k_m}' \left[m \left(1 - \frac{x}{\tilde{x}(\alpha)} \right) + 2x \sum_i^m \frac{\gamma_{k_i}^2}{\alpha - \varepsilon_{k_i}} \right]$$
$$\cdot x \frac{1}{2(m-1)!} \sum_{s>m} \gamma_{k_s} \gamma_{k_m} \frac{a_{k_s}^\dagger a_{k_m} + a_{k_m}^\dagger a_{k_s}}{(\alpha - \varepsilon_{k_s})(\alpha - \varepsilon_{k_m})} \prod_j^{m-1} \frac{n_{k_j}}{\alpha - \varepsilon_{k_j}} \tag{32}$$

$$[c_2^{(1)} c_2^{(m-1)}]_m = -2x^2 \sum_{k_1,k_2,\ldots,k_m}' \left[\sum_{i,j}' \frac{\gamma_{k_i}^2 \gamma_{k_j}^2}{(\alpha - \varepsilon_{k_i})(\alpha - \varepsilon_{k_j})} \frac{1}{m!} \prod_j^m \frac{n_{k_j}}{\alpha - \varepsilon_{k_j}} \right.$$
$$\left. + d_m \sum_i^m \frac{\gamma_{k_i}^2}{\alpha - \varepsilon_{k_i}} \sum_{s>m} \gamma_{k_s} \gamma_{k_m} \frac{a_{k_s}^\dagger a_{k_m} + a_{k_m}^\dagger a_{k_s}}{(\alpha - \varepsilon_{k_s})(\alpha - \varepsilon_{k_m})} \prod_j^{m-1} \frac{n_{k_j}}{\alpha - \varepsilon_{k_j}} \right] \tag{33}$$

where $d_m = (2(m-1)!)^{-1}$. Summing these terms yields a $(1 - x/\tilde{x}(\alpha))^{-1} \hat{C}^{(1)}(\alpha) \cdot \hat{C}^{(m-1)}(\alpha)$ linear in x and, thus, $\hat{C}^{(m)}(\alpha)$ is also linear in x.

Bosonic and spin models

We have used the above algorithm to study the higher particle number sectors for bosons on a lattice with $N \leqslant 6$ and the number of particles $2 \leqslant n \leqslant 6$ and find that *there are no linear conservation laws beyond the $\hat{Z}_j$ that are linear in x*. For the spin-$\frac{1}{2}$ Gaudin model, we have studied higher particle number sectors, where the operators analogous to Z_j do exist. In that case we find that again *there are no other conservation laws that are linear in x*.

Discussion

The type 1 fermionic model appears to be unique in possessing these 'extra' conservation laws linear in the parameter x. Not only do the type 1 bosonic and Gaudin models not have them, but numerical experiments with randomly generated type M—$M > 1$—fermionic and bosonic models similarly exhibit no such additional conservation laws. As we have shown, however, these extra type 1 conservation laws are functionally dependent on the N 'original' ones—the $\hat{Z}_j$.

In conclusion, we have provided a non-trivial example of the enumeration of dynamical symmetries in the model considered. It appears to be the first case where one can establish the *functional dependence* between a 'higher set' of conservation laws on a more fundamental set, both surprisingly being linear in the parameter x. With this, we have a close parallel to the classical idea of integrability, and a well defined enumeration of *the degrees of freedom* that have so far been missing.

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