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Extremely correlated Fermi liquids in the limit of infinite dimensions

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HIGHLIGHTS

- Infinite-dimensional t–J model ($J = 0$) studied within new ECFL theory.
- Mapping to the infinite U Anderson model with self consistent hybridization.
- Single particle Green's function determined by two local self energies.
- Partial projection through control variable λ.
- Expansion carried out to $O(\lambda^2)$ explicitly.

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ABSTRACT

We study the infinite spatial dimensionality limit ($d \to \infty$) of the recently developed Extremely Correlated Fermi Liquid (ECFL) theory (Shastry 2011, 2013) [17,18] for the t–J model at $J = 0$. We directly analyze the Schwinger equations of motion for the Gutzwiller projected (i.e. $U = \infty$) electron Green's function $\mathcal{G}$. From simplifications arising in this limit $d \to \infty$, we are able to make several exact statements about the theory. The ECFL Green's function is shown to have a momentum independent Dyson (Mori) self energy. For practical calculations we introduce a partial projection parameter λ, and obtain the complete set of ECFL integral equations to $O(\lambda^2)$. In a related publication (Zitko et al. 2013) [23], these equations are compared in detail with the dynamical mean field theory for the large U Hubbard model. Paralleling the well known mapping for the Hubbard model, we find that the infinite dimensional t–J model (with $J = 0$) can be mapped to the infinite-U Anderson impurity model with a self-consistently determined set of parameters. This mapping extends individually to the auxiliary Green's function $\mathbf{g}$ and the caparison

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factor μ. Additionally, the optical conductivity is shown to be obtainable from $\mathcal{G}$ with negligibly small vertex corrections. These results are shown to hold to each order in λ.

1. Introduction

1.1. Motivation

The Hubbard model (HM) with the Hamiltonian:

$$H = -\sum_{ij\sigma} t_{ij} c^{\dagger}_{i\sigma} c_{j\sigma} + U \sum_i n_{i\uparrow} n_{i\downarrow} - \boldsymbol{\mu} \sum_i n_i, \tag{1}$$

has attracted great theoretical interest in condensed matter physics, and is also a fairly realistic model of strongly correlated materials such as the cuprates. While the small $\frac{U}{t}$ limit is well described by standard Fermi-Liquid theory [1,2], the large and intermediate $\frac{U}{t}$ (strongly correlated) cases are much less well understood. Considerable progress has been made by considering the HM in the limit of infinite dimensions [3–11]. One important result is that the Dyson self energy, defined by inverting the expression for the electron Green's function $\mathcal{G}$:

$$\mathcal{G}(k) = \frac{1}{i\omega_k + \boldsymbol{\mu} - \epsilon_k - \Sigma_D(k)}, \tag{2}$$

becomes momentum independent in this limit [3–6]. Two other important results are the self-consistent mapping of the infinite dimensional HM onto the Anderson Impurity model (AIM), detailed in [8] (Dynamical Mean Field Theory), and the vanishing of the vertex corrections in the optical conductivity [10,11], so that the two particle response is obtainable from the single particle Green's function. The Dynamical Mean Field Theory (DMFT) provides a means for doing reliable numerical calculations for the Hubbard model, at any value of U and has continued to provide new, and interesting results [12,13].

A different approach to understanding strong correlations is to consider the extreme correlation limit, where on sets $U \to \infty$ at the outset. In this case, the Hilbert space is Gutzwiller projected so that only single occupancy is allowed on each lattice site. One such extremely correlated model, the t–J model, consists of taking the $U \to \infty$ limit of the Hubbard model (the t part of the model) and adding on a nearest neighbor anti-ferromagnetic coupling term (the J part of the model). The t model studied here, is obtained by dropping the J term and thus is identical to the $U = \infty$ limit of the HM. It has been argued by Anderson [14] that the t–J model describes the physics of the cuprates, thereby providing an impetus for its detailed study. The Hamiltonian for this model can be written in terms of the Hubbard X operators as [15]

$$H = -\sum_{ij\sigma} t_{ij} X_i^{\sigma 0} X_j^{0\sigma} - \boldsymbol{\mu} \sum_{i\sigma} X_i^{\sigma\sigma} + \frac{1}{2} \sum_{ij\sigma} J_{ij} X_i^{\sigma\sigma} + \frac{1}{4} \sum_{ij\sigma_1\sigma_2} J_{ij} \{X_i^{\sigma_1\sigma_2} X_j^{\sigma_2\sigma_1} - X_i^{\sigma_1\sigma_1} X_j^{\sigma_2\sigma_2}\}. \tag{3}$$

The operator $X_i^{ab} = |a\rangle\langle b|$ takes the electron at site i from the state $|b\rangle$ to the state $|a\rangle$, where $|a\rangle$ and $|b\rangle$ are one of the three allowed states $|\uparrow\rangle$, $|\downarrow\rangle$, or $|-\rangle$. Our present goal is to obtain a formally exact solution of the above t model in the limit of large dimensions by studying its equations of motion. This is designed to be methodologically independent of the available DMFT solution of the HM with $U = \infty$, and can be compared with it.

Our object of study is the Green's function written as

$$\mathcal{G}_{\sigma_1\sigma_2}(i, f) = -\langle T_\tau X_i^{0\sigma_1}(\tau_i) X_f^{\sigma_2 0}(\tau_f)\rangle, \tag{4}$$

where the angular brackets indicate the usual thermal average. Due to the non-canonical commutation relations of the X operators, the high frequency limit of Green's function is $\frac{1-\frac{n}{2}}{i\omega_n}$ rather than $\frac{1}{i\omega_n}$ as in the canonical case. To avoid linear growth of the self-energy in the high frequency limit [15], the Dyson self-energy must be redefined to the Dyson–Mori self energy [16] as in:

$$\mathcal{G}(k) = \frac{1-\frac{n}{2}}{i\omega_k + \boldsymbol{\mu} - \epsilon_k\left(1-\frac{n}{2}\right) - \Sigma_{DM}(k)}. \tag{5}$$

Just as is the case for Σ_D in the finite-U Hubbard model, Σ_{DM} is finite as $i\omega \to \infty$ in the t–J model.

Shastry has recently introduced a novel and promising approach for calculating correlation functions within the t–J model based on Schwinger's formulation of field theory [15,17,18]. This has culminated in the theory of the Extremely Correlated Fermi Liquid (ECFL) [17,18]. This theory has been successfully benchmarked against: line shapes from (ARPES) experiments [19,20], high-temperature series [21] and the numerical renormalization group (NRG) calculations for the Anderson impurity model [22]. A recent theoretical benchmarking related to this work is the comparison with DMFT calculations for the large U Hubbard model in a concurrent publication [23], with the formulas found here. Indeed the main motivation of the present paper is to obtain results in the limit of large d for the same model, the t–J model (at $J = 0$) or equivalently the $U = \infty$ Hubbard model by two different methods, the ECFL and the DMFT, allowing such a comparison.

In the ECFL theory, the physical Green's function $\mathcal{G}(k)$ is factored into a canonical auxiliary Green's function $\mathbf{g}(k)$ and an adaptive spectral $\mu(k)$, where $k = (\vec{k}, i\omega_k)$:

$$\mathcal{G}(k) = \mathbf{g}(k) \times \mu(k). \tag{6}$$

These two factors are in turn written in terms of two self-energies, $\Phi(k)$ and $\Psi(k)$.

$$\mathbf{g}^{-1}(k) = i\omega_k + \boldsymbol{\mu} - (1-n/2)\epsilon_k - \Phi(k), \tag{7}$$

$$\mu(k) = 1 - \frac{n}{2} + \Psi(k). \tag{8}$$

Here $\Phi(k)$ plays the role of a Dyson self-energy for the canonical Green's function $\mathbf{g}(k)$, and $\Psi(k)$ is a frequency-dependent correction to $\mu(k)$ from its high frequency value of $1-\frac{n}{2}$. Φ and Ψ are then given in terms of the vertices (i.e. functional derivatives w.r.t. the source of the $\mathbf{g}^{-1}$ and μ) as will be described below, leading to a closed set of Schwinger differential equations (the ECFL equations of motion). These equations are in general intractable since there is no obvious small parameter, and therefore to enable practical calculations, an expansion is carried out in a partial projection parameter λ. Here λ interpolates between the free Fermi gas and the t–J model. The meaning of λ as a partial projection parameter is detailed in [18], and may be summarized in the mapping $X_i^{\sigma 0} \to f_{i\sigma}^{\dagger}(1-\lambda\, n_{i\bar{\sigma}})$, where $f_{i\sigma}$ is a canonical electron operator. Thus at $\lambda = 0$ we have canonical electrons, whereas at $\lambda = 1$ we have the fully projected electrons.

In this work, our aim is to combine the two approaches, namely to consider the ECFL in the limit of infinite spatial dimensions. In this limit, $J \to 0$, and the infinite-dimensional t–J model becomes the infinite dimensional infinite U Hubbard model (see Section 6A of Ref. [23] for a brief discussion of this). It is not clear a priori, whether or not the aforementioned results, valid for the infinite dimensional finite-U Hubbard model, carry over to the infinite dimensional t–J model. The possible conflict arises from the fact that in the case of the former, the ratio $\frac{U}{d} \to 0$, while in the case of the latter, $\frac{U}{d} \to \infty$. This question was raised in Ref. [24], pointing to the ECFL solution of the infinite dimensional t–J model as a source of resolution. Working directly with the infinite-U Hamiltonian (Eq. (3) with $J = 0$), and using the corresponding ECFL equations of motion, we are able to address this challenging task and to show that the two limits $U \to \infty$ and $d \to \infty$ do in fact commute.

Moreover, we are able to determine the structure of the ECFL objects $\Phi(k)$ and $\Psi(k)$ in the limit of infinite dimensions. Such structural information has already been used to fit numerical results obtained through DMFT calculations to a convenient and flexible functional form [23]. Finally, we are able to elucidate the nature of the λ expansion in the large d limit. For readers who might be more interested in the results than the methodology, we provide a detailed summary of our results at the outset.

1.2. Results in the limit of infinite dimensions

We show that in the large d limit, the two self energies $\Phi(k)$ and $\Psi(k)$ simplify in the following way.

$$\Psi(k) = \Psi(i\omega_k), \tag{9}$$

$$\Phi(k) = \chi(i\omega_k) + \epsilon_k \Psi(i\omega_k). \tag{10}$$

These in turn show that the Dyson–Mori self energy behaves as

$$\Sigma_{DM}(k) = \Sigma_{DM}(i\omega_k) = \frac{(i\omega_k + \boldsymbol{\mu})\Psi(i\omega_k) + \left(1 - \frac{n}{2}\right)\chi(i\omega_k)}{1 - \frac{n}{2} + \Psi(i\omega_k)}, \tag{11}$$

and is therefore local in the limit of infinite dimensions. We show that to each order in the λ expansion, $\Psi(i\omega_k)$ and $\chi(i\omega_k)$ are each a product of an arbitrary number of factors, each of which take on the form $\sum_{\vec{p}} g(\vec{p}, i\omega_p)\epsilon_{\vec{p}}^m$, with m equal to zero or one, and with arbitrarily complex frequency dependence of the individual factors.

We show that just as in the finite U case [10,11], the optical conductivity is given by the expression

$$\sigma^{\alpha\beta}(\omega) = \frac{2}{i\omega} \sum_{\vec{p}, i\omega_p} \mathcal{g}(\vec{p}, i\omega_p) v_{\vec{p}}^{\alpha} v_{\vec{p}}^{\beta} [\mathcal{g}(\vec{p}, \omega + i\eta + i\omega_p) - \mathcal{g}(\vec{p}, i\eta + i\omega_p)], \tag{12}$$

where $v_{\vec{p}}^{\alpha}$ is the component of the velocity in the α direction (Eq. (39)). We show that this formula can be applied at each order of the λ expansion.

We show that there is a self consistent mapping between the ECFL theory of the infinite-dimensional t–J model and the ECFL theory of the infinite-U Anderson impurity model (AIM) [22]. This mapping is similar in spirit to the mapping first discussed by Georges and Kotliar for the Hubbard model [8], but is made directly in the infinite U limit here. In this mapping, $\mathbf{g}_{i,i}[\tau_i, \tau_f]$ and $\mu_{i,i}[\tau_i, \tau_f]$ of the t–J model are mapped to the objects $\mathbf{g}[\tau_i, \tau_f]$ and $\mu[\tau_i, \tau_f]$ of the Anderson model, written with the same symbols, but without the spatial or momentum labels. This mapping is valid under the self-consistency condition

$$\sum_{\vec{k}} \epsilon_{\vec{k}} \mathbf{g}(k) = \sum_{\vec{k}} \frac{|V_{\vec{k}}|^2}{i\omega_n - \widetilde{\epsilon}_{\vec{k}}} \mathbf{g}(i\omega_k), \tag{13}$$

where $\epsilon_{\vec{k}}$ is the dispersion of the lattice in the t–J model, and $V_{\vec{k}}$ and $\widetilde{\epsilon}_{\vec{k}}$ are the hybridization and dispersion of the bath respectively in the Anderson impurity model. This self-consistency condition is shown to be equivalent to the standard self-consistency condition from DMFT [8,9]. We also show that the mapping holds to each order in λ under the same self-consistency condition. We note that this implies that ECFL computations for the infinite-dimensional t–J model can be done with a DMFT-like self-consistency loop involving ECFL computations for the AIM. However, since the λ expansion provides integral equations which are relatively straightforward to solve numerically, this is not necessary as the t–J model equations can be solved directly.

1.3. Outline of the paper

The paper is structured as follows. In Section 2, some basic facts about lattice sums in the limit of large dimensions and the ECFL equations of motion as well as the λ expansion are reviewed. Additionally, the spatial dependence of various standard and ECFL specific objects in the limit of large dimensions is stated. Finally, we introduce a class of local functions denoted as class-L functions; these turn out to play a central role in the ECFL in the limit of large dimensions. In Sections 3.1 and 3.2, Eqs. (9) and (10) are proven in general and to each order in λ, and the locality of the Dyson–Mori self energy is shown as a consequence. In Section 3.3, Eq. (12) is shown to hold in general and to each order in λ. In Section 3.4, the ECFL self-consistent integral equations are derived to $O(\lambda^2)$ in the large-d limit. Finally, in Section 4, the ECFL of the infinite dimensional t–J model is mapped onto the ECFL of the infinite-U AIM under the self-consistency condition (Eq. (13)). This is done in general and to each order in λ.

2. Preliminaries

2.1. Spatial dependence of lattice sums in large d dimensions

We take the hopping to nearest neighbor sites on the d-dimensional hypercube. In this case, it is well known [4] that $t_{ij} \to \frac{1}{\sqrt{2d}} t_0$ with t_0 of $O(1)$. We would like to exploit the smallness of individual t_{ij}'s, these can only contribute (after multiplying with another like object), if one of the indices is summed over the d-neighbors as in the simplest example $\sum_j t_{ij}^2 = t_0^2$. Extending this argument further, for a pair of sites (i, m) located at a (Manhattan metric) distance r_{im} on the hypercube, suppose there are two objects $W_{i,m}$ and $V_{i,m}$ who both have the dependence on $r_{im} : V_{i,m}; W_{i,m} \sim O\left(\frac{1}{(\sqrt{d})^{r_{im}}}\right)$. Then it follows that

$$W_{i,\mathbf{n}} V_{\mathbf{n},m} \sim O\left(\frac{1}{\left(\sqrt{d}\right)^{r_{im}}}\right). \tag{14}$$

Here, and in the rest of the paper, bold and repeated indices are summed and/or integrated over. This relation can be understood by first considering the case that the site $\mathbf{n}$ is on one of the shortest paths between i and m. In this case, $r_{i\mathbf{n}} + r_{\mathbf{n}m} = r_{im}$ proving the relation. If, $\mathbf{n}$ is a certain distance r_o off of a shortest path, then $r_{i\mathbf{n}} + r_{\mathbf{n}m} = r_{im} + 2r_o$. This introduces an extra factor of $\frac{1}{d^{r_o}}$ into the lattice sum in Eq. (14). However, this factor is exactly cancelled by the d^{r_0} choices for the site $\mathbf{n}$. In this argument, the number of shortest paths between i and m is taken to be $O(1)$.

2.2. ECFL equations of motion and the λ expansion

The ECFL equations of motion for the finite dimensional t–J model can be found in Ref. [18]. There is some freedom in how these equations are written because one may add terms to them which vanish identically in the exact solution, but play a non-trivial role when implementing approximations (such as the λ expansion). We denote the version of these equations with no added terms the minimal theory, and the version containing the added terms the symmetrized theory (since the added terms make the resulting expressions symmetric in a certain sense). In Ref. [18], the ECFL equations of motion for the symmetrized theory are derived, and the added terms required to go from the minimal theory to the symmetrized theory are singled out. The ECFL equations for the minimal theory, which are the ones used in this paper and in Ref. [23], can therefore be obtained from those in Ref. [18] by dropping these extra terms.

Setting $J \to 0$ (as discussed in Section 1.1), we write the minimal theory ECFL equations of motion in expanded form:

$$\begin{aligned} \mathbf{g}^{-1}[i,m] &= (\boldsymbol{\mu} - \partial_{\tau_i} - \mathcal{V}_i)\delta[i,m] + t[i,m]\,(1-\lambda\gamma[i]) \\ &\quad + \lambda t[i,\mathbf{j}]\,\xi^* \cdot \mathbf{g}[\mathbf{j},\mathbf{n}] \cdot \Lambda_*[\mathbf{n},m;i], \\ \mu[i,m] &= (1-\lambda\gamma[i])\delta[i,m] - \lambda t[i,\mathbf{j}]\,\xi^* \cdot \mathbf{g}[\mathbf{j},\mathbf{n}] \cdot \mathcal{U}_*[\mathbf{n},m;i], \end{aligned} \tag{15}$$

where $\mathcal{V}_i \equiv \mathcal{V}_i(\tau_i)$ is the Bosonic Schwinger source function, and we have used the notation $\delta[i,m] = \delta_{i,m}\delta(\tau_i - \tau_m)$ and $t[i,m] = t_{i,m}\delta(\tau_i - \tau_m)$. These exact relations give the required objects $\mathbf{g}$ and μ in terms of the vertex functions. Here we also note that the local (in space and time) Green's function $\gamma[i]$, and the vertices $\Lambda[n,m;i]$ and $\mathcal{U}[n,m;i]$, are defined as

$$\begin{aligned} &\gamma[i] = \mu^{(k)}[\mathbf{n}, i^+] \cdot \mathbf{g}^{(k)}[i,\mathbf{n}]; \qquad \Lambda[n,m;i] = -\frac{\delta}{\delta\mathcal{V}_i}\mathbf{g}^{-1}[n,m]; \\ &\mathcal{U}[n,m;i] = \frac{\delta}{\delta\mathcal{V}_i}\mu[n,m], \end{aligned} \tag{16}$$

where we have used the notation $M^{(k)}_{\sigma_1,\sigma_2} = \sigma_1\sigma_2 M_{\bar{\sigma}_2,\bar{\sigma}_1}$ to denote the time reversed matrix $M^{(k)}$ of an arbitrary matrix M. These exact relations give the vertex functions in terms of the objects $\mathbf{g}$ and μ. The vertices defined above (Λ and $\mathcal{U}$) have four spin indices, those of the object being differentiated and those of the source. For example, $\mathcal{U}^{\sigma_1\sigma_2}_{\sigma_a\sigma_b}[n, m; i] = \frac{\delta}{\delta \mathcal{V}_i^{\sigma_a\sigma_b}} \mu_{\sigma_1\sigma_2}[n, m]$. In Eq. (15), $\xi_{\sigma_a\sigma_b} = \sigma_a\sigma_b$, and * indicates that these spin indices should also be carried over (after being flipped) to the bottom indices of the vertex, which is also marked with a $_*$. The top indices of the vertex are given by the usual matrix multiplication. An illustrative example is useful here: $(\xi^* \cdot \mathbf{g}[j, \mathbf{n}] \cdot \mathcal{U}_*[\mathbf{n}, m; i])_{\sigma_1\sigma_2} = \sigma_1\sigma_{\mathbf{a}}\mathbf{g}_{\sigma_{\mathbf{a}},\sigma_{\mathbf{b}}}[j, \mathbf{n}] \frac{\delta}{\delta \mathcal{V}_i^{\bar{\sigma}_1\bar{\sigma}_{\mathbf{a}}}} \mu_{\sigma_{\mathbf{b}},\sigma_2}[\mathbf{n}, m]$. Finally, in order to ensure that the shift identities (Ref. [18]) are satisfied, the substitution $t_{ij} \to t_{ij} + \frac{u_0}{2}\delta_{ij}$ is made, where u_0 is the second chemical potential. For the sake of clarity, this substitution will be ignored in the proofs given below, although they are easily generalized to account for it. This generalization is discussed at the end of Section 3.1.

The λ expansion is obtained by expanding Eqs. (15) and (16) iteratively in the continuity parameter λ. The $\lambda = 0$ limit of these equations is the free Fermi gas. Therefore, a direct expansion in λ will lead to a series in λ in which each term is made up of the hopping t_{ij} and the free Fermi gas Green's function $\mathbf{g}_0[i, f]$. As is the case in the Feynman series, this can be reorganized into a skeleton expansion in which only the skeleton graphs are kept and $\mathbf{g}_0[i, f] \to \mathbf{g}[i, f]$. However, one can also obtain the skeleton expansion directly by expanding Eqs. (15) and (16) in λ, but treating $\mathbf{g}[i, f]$ as a zeroth order (i.e. unexpanded) object in the expansion. This expansion is carried out to second order for the finite-dimensional case in Ref. [18]. In doing this expansion, one must evaluate the functional derivative $\frac{\delta \mathbf{g}}{\delta \mathcal{V}}$. This is done with the help of the following useful formula which stems from the product rule for functional derivatives:

$$\frac{\delta \mathbf{g}[i, m]}{\delta \mathcal{V}_r} = \mathbf{g}[i, \mathbf{x}] \cdot \Lambda[\mathbf{x}, \mathbf{y}, r] \cdot \mathbf{g}[\mathbf{y}, m]. \tag{17}$$

This is an exact formula and will be used extensively in the arguments given below. Within the λ expansion, the LHS is evaluated to a certain order in λ by taking the vertex Λ on the RHS to be of that order in λ.

2.3. Leading order spatial dependence of various objects

All objects may be expanded in the inverse square root of the number of spatial dimensions d. The lowest order term in the physical Green's function $\mathcal{G}[i, f]$ must be at least $O\left(\frac{1}{(\sqrt{d})^{r_{if}}}\right)$. This must be so because it takes at least r_{if} hops to get from the site i to the site f. Any terms that contribute to $\mathcal{G}[i, f]$ at higher order than $O\left(\frac{1}{(\sqrt{d})^{r_{if}}}\right)$ are neglected in the large d limit. In a similar vein, the lowest order term in $\mathbf{g}[i, f]$, $\mathbf{g}^{-1}[i, f]$, $\mu[i, f]$, $\Lambda[i, f; r]$, and $\mathcal{U}[i, f; r]$ must be at least $O\left(\frac{1}{(\sqrt{d})^{r_{if}}}\right)$. Furthermore, using the real space version of Eqs. (6) and (14), we see that any terms of higher order than this in $\mathbf{g}[i, f]$ and $\mu[i, f]$ will result in a higher order term in $\mathcal{G}[i, f]$ and may therefore be neglected as well. Finally, using matrix inversion in the space–time indices, we see that higher order terms may also be dropped from $\mathbf{g}^{-1}[i, f]$ as these will lead to higher order terms in $\mathbf{g}[i, f]$, and using Eq. (15), higher order terms may be dropped from $\Lambda[i, f; r]$, and $\mathcal{U}[i, f; r]$ as these will lead to higher order terms in $\mathbf{g}^{-1}[i, f]$ and $\mu[i, f]$ respectively. In summary, in all objects: $\mathcal{G}[i, f]$, $\mathbf{g}[i, f]$, $\mathbf{g}^{-1}[i, f]$, $\mu[i, f]$, $\Lambda[i, f; r]$, and $\mathcal{U}[i, f; r]$, terms of higher order than $O\left(\frac{1}{(\sqrt{d})^{r_{if}}}\right)$ may be neglected in the large d limit.

We also note that the correlation function $\Pi_{\alpha\beta}[i, f]$ appearing in Eq. (40) must be at least $O\left(\frac{1}{d^{r_{if}}}\right)$. This is due to the fact that unlike the creation and destruction operators which appear in the Green's function, the current operators appearing in this correlation function conserve particle number. Hence, one must hop from site i to site f and back, which takes $2 \times r_{if}$ hops. Any terms that contribute to $\Pi_{\alpha\beta}[i, f]$ at higher order than $O\left(\frac{1}{d^{r_{if}}}\right)$ are neglected in the large d limit.

2.4. Class L functions

For the arguments given below, we need to define a class of *localized functions*, denoted as class L functions. A class L function L_i has three properties.

- (a) $L_i \sim O\left(\frac{1}{d^0}\right)$.
- (b) L_i is a function of only one site i, and an arbitrary number of time variables. Upon turning off the sources, it becomes translationally invariant, but an arbitrary function of frequencies.
- (c) The $\mathcal{V}$ source derivative of L_i is also localized:

$$\frac{\delta}{\delta \mathcal{V}_i} L_j = \delta_{ij} L_i', \tag{18}$$

with L_i' again a class-L function.

Our proofs deal with functions that turn out to be of this class. Iterating property (c), the following equation must hold for any positive integer s.

$$\frac{\delta}{\delta \mathcal{V}_{r_1}} \cdots \frac{\delta}{\delta \mathcal{V}_{r_s}} L_i = \delta_{ir_1} \ldots \delta_{ir_s} \frac{\delta}{\delta \mathcal{V}_i(\tau_{r_1})} \cdots \frac{\delta}{\delta \mathcal{V}_i(\tau_{r_s})} L_i. \tag{19}$$

In the presence of the current source κ (Eq. (42)), class L functions acquire one additional property (d): consider a typical contribution to $\Pi_{\alpha\beta}[i, f]$ (Eq. (44)) denoted by O_{if}

$$O_{if} = W_{f,\mathbf{x}} \frac{\delta}{\delta \kappa_i^{\alpha}} (L_{\mathbf{x}}) V_{\mathbf{x},f}, \tag{20}$$

where the functions $V_{\mathbf{x},f}, W_{f,\mathbf{x}} \sim O\left(\frac{1}{(\sqrt{d})^{r_{\mathbf{x}f}}}\right)$. Then, neglecting terms of higher order than $O\left(\frac{1}{d^{r_{if}}}\right)$ in O_{if}, $\sum_{i-f} O_{if} \to 0$ as $\mathcal{A} \to 0$. Again iterating property (c) and using property (d), the following must hold for any nonnegative integer s:

$$\sum_{i-f} \left(W_{f,\mathbf{x}} \frac{\delta}{\delta \kappa_i^{\alpha}} \frac{\delta}{\delta \mathcal{V}_{\mathbf{x}}(\tau_{r_1})} \cdots \frac{\delta}{\delta \mathcal{V}_{\mathbf{x}}(\tau_{r_s})} (L_{\mathbf{x}}) V_{\mathbf{x},f} \right)_{\mathcal{A}\to 0} = 0. \tag{21}$$

3. Limit of large dimensionality through the ECFL equations of motion

3.1. Simplification of the ECFL self energies

We use notation in which we indicate spatial dependence by subscripts, so that $\mathbf{g}[i,j] \to \mathbf{g}_{i,j}[\tau_i, \tau_j]$, and recall that $t[i,j] = t_{i,j}\, \delta(\tau_i - \tau_j)$, $\delta[i,j] = \delta_{i,j}\, \delta(\tau_i - \tau_j)$, $\delta[\tau_i, \tau_j] = \delta(\tau_i - \tau_j)$ etc. After some inspection of Eqs. (15) and (16) in the limit of high dimension, we make an Ansatz – to be proven below – namely

$$\begin{aligned} \mathbf{g}^{-1}[i,m] \;&=\; (\boldsymbol{\mu} - \partial_{\tau_i} - \mathcal{V}_i)\, \delta[i,m] + t[i,m]\, (1 - \lambda\gamma[i]) - \lambda\, \delta_{i,m}\, \chi_i[\tau_i, \tau_m] \\ &\quad + \lambda\, t_{i,m}\, \Psi_i[\tau_i, \tau_m], \\ \mu[i,m] &= \delta[i,m](1 - \lambda\gamma[i]) + \lambda\, \delta_{i,m}\, \Psi_i[\tau_i, \tau_m], \end{aligned} \tag{22}$$

where $\Psi_i[\tau_i, \tau_m]$, $\chi_i[\tau_i, \tau_m]$, and $\gamma[i]$ are class L functions. We will prove Eq. (22) by assuming that it is true, and then showing that this assumption is consistent with the equations of motion (Eqs. (15) and (16)). This argument will consist of a loop which begins with Eq. (22). Then, substituting this equation into Eq. (16), we will derive a certain form for Λ, $\mathcal{U}$, and γ. Finally, substituting these objects into Eq. (15), and using simplifications which occur in the large d limit, we will complete the loop and arrive back at Eq. (22).

Substituting Eq. (22) into Eq. (16), we find that the vertices and $\gamma[i]$ have the following form.

$$\begin{aligned}
\Lambda[n,m;i] &= \delta_{i,n}\delta_{i,m}\, A_i[\tau_n,\tau_m;\tau_i] + \delta_{i,n}t_{n,m}\, B_i[\tau_n,\tau_m;\tau_i],\\
\mathcal{U}[n,m;i] &= -\delta_{i,n}\delta_{i,m}\, B_i[\tau_n,\tau_m;\tau_i],\\
\gamma[i] &= \left(1-\lambda\gamma^{(k)}[i]\right)\mathbf{g}^{(k)}[i,i] + \lambda\, \Psi_i^{(k)}[\tau_{\mathbf{j}},\tau_i]\mathbf{g}_{ii}^{(k)}[\tau_i,\tau_{\mathbf{j}}],
\end{aligned} \tag{23}$$

where we defined two new functions:

$$\begin{aligned}
A_i[\tau_n,\tau_m;\tau_i] &= \delta[\tau_i,\tau_n]\delta[\tau_i,\tau_m]\,\mathbb{1} + \lambda\,\frac{\delta}{\delta\mathcal{V}_i}\chi_i[\tau_n,\tau_m],\\
B_i[\tau_n,\tau_m;\tau_i] &= \lambda\,\delta[\tau_n,\tau_m]\frac{\delta}{\delta\mathcal{V}_i}\gamma_i[\tau_n] - \lambda\,\frac{\delta}{\delta\mathcal{V}_i}\Psi_i[\tau_n,\tau_m].
\end{aligned} \tag{24}$$

Here A_i and B_i are class L functions since they inherit this property from Ψ_i, χ_i, and $\gamma[i]$ by functional differentiation. Substituting Eq. (23) into Eq. (15) and comparing with Eq. (22),

$$\begin{aligned}
\chi_i[\tau_i,\tau_m] &= -t_{i,\mathbf{j}}\,\xi^*\cdot\mathbf{g}_{\mathbf{j},i}[\tau_i,\tau_{\mathbf{n}}]\cdot A_{i,*}[\tau_{\mathbf{n}},\tau_m;\tau_i],\\
\Psi_i[\tau_i,\tau_m] &= t_{i,\mathbf{j}}\,\xi^*\cdot\mathbf{g}_{\mathbf{j},i}[\tau_i,\tau_{\mathbf{n}}]\cdot B_{i,*}[\tau_{\mathbf{n}},\tau_m;\tau_i].
\end{aligned} \tag{25}$$

If we can now show that χ_i, Ψ_i, and $\gamma[i]$ as defined in Eqs. (23) and (25) are Class L functions, we will have justified our Ansatz and therefore we will have proven all of the above equations. To do this, we must show that $\mathbf{g}_{ii}[\tau_i,\tau_m]$ and $t_{i,\mathbf{j}}\,\mathbf{g}_{\mathbf{j},i}[\tau_i,\tau_m]$ are Class L functions. Taking their functional derivatives we obtain:

$$\begin{aligned}
\frac{\delta}{\delta\mathcal{V}_r}t_{i,\mathbf{j}}\,\mathbf{g}_{\mathbf{j},i}[\tau_i,\tau_m] = {}& t_{i,\mathbf{j}}\mathbf{g}_{\mathbf{j},r}[\tau_i,\tau_{\mathbf{k}}]\,A_r[\tau_{\mathbf{k}},\tau_{\mathbf{l}};\tau_r]\mathbf{g}_{r,i}[\tau_{\mathbf{l}},\tau_m]\\
&+ t_{i,\mathbf{j}}\mathbf{g}_{\mathbf{j},r}[\tau_i,\tau_{\mathbf{k}}]\,B_r[\tau_{\mathbf{k}},\tau_{\mathbf{l}};\tau_r]t_{r,\mathbf{l}}\mathbf{g}_{\mathbf{l},i}[\tau_{\mathbf{l}},\tau_m],
\end{aligned} \tag{26}$$

and

$$\begin{aligned}
\frac{\delta}{\delta\mathcal{V}_r}\mathbf{g}_{i,i}[\tau_i,\tau_m] = {}& \mathbf{g}_{i,r}[\tau_i,\tau_{\mathbf{k}}]\,A_r[\tau_{\mathbf{k}},\tau_{\mathbf{l}};\tau_r]\mathbf{g}_{r,i}[\tau_{\mathbf{l}},\tau_m]\\
&+ \mathbf{g}_{i,r}[\tau_i,\tau_{\mathbf{k}}]\,B_r[\tau_{\mathbf{k}},\tau_{\mathbf{l}};\tau_r]t_{r,\mathbf{l}}\mathbf{g}_{\mathbf{l},i}[\tau_{\mathbf{l}},\tau_m].
\end{aligned} \tag{27}$$

Using Eq. (14), the terms on the RHS of Eqs. (26) and (27) survive the large d limit if and only if $r=i$. Moreover, upon making the substitution $r\to i$, we see that the RHS is made up of the same objects that appear on the LHS of the equations (as well as the class L functions A and B). Therefore, this argument can be iterated to any number of derivatives acting on $t_{i,\mathbf{j}}\;\mathbf{g}_{\mathbf{j},i}[\tau_i,\tau_m]$ or $\mathbf{g}_{i,i}[\tau_i,\tau_m]$ (as required by Eq. (19)), which are therefore class L functions. Thus, we have shown the self-consistency of our ansatz Eq. (22).

The above results hold for any value of λ, since the proof was done with λ present in all of the equations. In the bare expansion, this would imply that they also hold to each order in λ. However, this line of reasoning is not as straightforward in the skeleton expansion because each order in the skeleton expansion contains contributions from all orders in the bare expansion. Nonetheless, the above results do hold to each order in λ in the skeleton expansion. In proving this, we shall shed more light on the nature of the objects Ψ_i, χ_i, $\gamma[i]$, A_i, and B_i. In particular, we will show that they satisfy a certain explicit form stated below in Eq. (28). We will do this using an inductive argument, in which we will assume that they have this form through a certain order in λ, and then substituting this form into the equations of motion, will show that it must hold for the next order.

We now use the symbol R_i *as a proxy* for either of the two functions $\mathbf{g}_{i,i}[\tau_n,\tau_m]$ or $t_{i,\mathbf{j}}\mathbf{g}_{\mathbf{j},i}[\tau_n,\tau_m]$ where the time indices are arbitrary. *Inductive hypothesis*: through nth order in λ, Eqs. (22) and (23) hold. Through $n-1$st order in λ, the objects Ψ_i, χ_i, and $\gamma[i]$, and through nth order, the objects A_i and B_i, (all denoted below by the generic object L_i) can be written as the following product (multiplied by some delta functions in time variables):

$$(L_i)^{(n)} = \lambda^n (R_i)^m, \tag{28}$$

where m is arbitrary. We first examine the base case of zeroth order. In this case,

$$A_i^{(0)}[\tau_n, \tau_m; \tau_i] = \delta[\tau_i, \tau_n]\delta[\tau_i, \tau_m]; \qquad B_i^{(0)}[\tau_n, \tau_m; \tau_i] = 0. \tag{29}$$

Clearly the hypothesis is satisfied. Now, we prove the inductive step. Explicitly displaying the order in λ of all objects, the equations for χ, Ψ, and γ (Eqs. (25) and (23)) become

$$\begin{aligned}
\chi_i^{(n)}[\tau_i, \tau_m] &= -t_{i,\mathbf{j}}\, \xi^* \cdot \mathbf{g}_{\mathbf{j},i}[\tau_i, \tau_\mathbf{n}] \cdot A_{i,*}^{(n)}[\tau_\mathbf{n}, \tau_m; \tau_i], \\
\Psi_i^{(n)}[\tau_i, \tau_m] &= \ t_{i,\mathbf{j}}\, \xi^* \cdot \mathbf{g}_{\mathbf{j},i}[\tau_i, \tau_\mathbf{n}] \cdot B_{i,*}^{(n)}[\tau_\mathbf{n}, \tau_m; \tau_i], \\
\gamma^{(n)}[i] &= -\lambda\, \gamma^{(k)(n-1)}[i]\mathbf{g}^{(k)}[i, i] + \lambda\, \Psi_i^{(k)(n-1)}[\tau_\mathbf{j}, \tau_i]\mathbf{g}_{ii}^{(k)}[\tau_i, \tau_\mathbf{j}].
\end{aligned} \tag{30}$$

By the inductive hypothesis, $\chi_i^{(n)}$, $\Psi_i^{(n)}$, and $\gamma^{(n)}[i]$ have the required form. The equations for A and B (Eq. (24)) become

$$\begin{aligned}
A_i^{(n+1)}[\tau_n, \tau_m; \tau_i] &= \lambda \left(\sum_{r\le n} \frac{\delta}{\delta \mathcal{V}_i} \chi_i^{(r)}[\tau_n, \tau_m] \right)^{(n)}, \\
B_i^{(n+1)}[\tau_n, \tau_m; \tau_i] &= \lambda\, \delta[\tau_n, \tau_m] \left(\sum_{r\le n} \frac{\delta}{\delta \mathcal{V}_i} \gamma_i^{(r)}[\tau_n] \right)^{(n)} - \lambda \left(\sum_{r\le n} \frac{\delta}{\delta \mathcal{V}_i} \Psi_i^{(r)}[\tau_n, \tau_m] \right)^{(n)}.
\end{aligned} \tag{31}$$

To see that $A^{(n+1)}$ and $B^{(n+1)}$ have the required form we note that for all $l \le n$,

$$\begin{aligned}
\left(\frac{\delta}{\delta \mathcal{V}_r} t_{i,\mathbf{j}}\, \mathbf{g}_{\mathbf{j},i}[\tau_i, \tau_m] \right)^{(l)} &= t_{i,\mathbf{j}} \mathbf{g}_{\mathbf{j},r}[\tau_i, \tau_\mathbf{k}]\, A_r^{(l)}[\tau_\mathbf{k}, \tau_\mathbf{l}; \tau_r] \mathbf{g}_{r,i}[\tau_\mathbf{l}, \tau_m] \\
&\quad + t_{i,\mathbf{j}} \mathbf{g}_{\mathbf{j},r}[\tau_i, \tau_\mathbf{k}]\, B_r^{(l)}[\tau_\mathbf{k}, \tau_\mathbf{l}; \tau_r] t_{r,\mathbf{l}} \mathbf{g}_{\mathbf{l},i}[\tau_\mathbf{l}, \tau_m],
\end{aligned} \tag{32}$$

and

$$\begin{aligned}
\left(\frac{\delta}{\delta \mathcal{V}_r} \mathbf{g}_{i,i}[\tau_i, \tau_m] \right)^{(l)} &= \mathbf{g}_{i,r}[\tau_i, \tau_\mathbf{k}]\, A_r^{(l)}[\tau_\mathbf{k}, \tau_\mathbf{l}; \tau_r] \mathbf{g}_{r,i}[\tau_\mathbf{l}, \tau_m] \\
&\quad + \mathbf{g}_{i,r}[\tau_i, \tau_\mathbf{k}]\, B_r^{(l)}[\tau_\mathbf{k}, \tau_\mathbf{l}; \tau_r] t_{r,\mathbf{l}} \mathbf{g}_{\mathbf{l},i}[\tau_\mathbf{l}, \tau_m].
\end{aligned} \tag{33}$$

In the limit of large dimensions, $r \to i$. We can therefore (using the inductive hypothesis) write the RHS of Eqs. (32) and (33) as $\lambda^l (R_i)^m$. Applying Eq. (28) (which has been shown to hold for $\chi_i^{(n)}$, $\Psi_i^{(n)}$, and $\gamma^{(n)}[i]$) to Eq. (31), we may write

$$\begin{aligned}
A_i^{(n+1)} &= \sum_{r=0}^{n} \lambda^{r+1} \left(\frac{\delta}{\delta \mathcal{V}_i} (R_i)^m \right)^{(n-r)}, \\
B_i^{(n+1)} &= \sum_{r=0}^{n} \lambda^{r+1} \left(\frac{\delta}{\delta \mathcal{V}_i} (R_i)^m \right)^{(n-r)}.
\end{aligned} \tag{34}$$

Eq. (34), in conjunction with Eqs. (32) and (33), shows that $A_i^{(n+1)}$ and $B_i^{(n+1)}$ have the required form. This completes the proof.

Since $t_{i,j}$ is independent of the source, the substitution $t_{i,j} \to t_{i,j} + \frac{u_0}{2}\delta_{i,j}$ can be made directly into all of the above equations. The only problem that could potentially arise involves Eqs. (26) and (27), where the large d simplifications are actually used. However, one can check that this substitution does not affect the simplifications. Therefore, this substitution merely adds the term $\lambda \frac{u_0}{2} \delta_{i,m} \Psi_i[\tau_i, \tau_m] - \lambda \frac{u_0}{2} \delta[i, m] \gamma[i]$ to $\mathbf{g}^{-1}[i, m]$, and everywhere replaces the local function $t_{i,\mathbf{j}} \mathbf{g}_{\mathbf{j},i}[\tau_n, \tau_m]$ with the local function $t_{i,\mathbf{j}} \mathbf{g}_{\mathbf{j},i}[\tau_n, \tau_m] + \frac{u_0}{2} \mathbf{g}_{i,i}[\tau_n, \tau_m]$. This can be seen explicitly in the $O(\lambda^2)$ equations in Section 3.4, and does not change the general structure of the solution.

3.2. The zero source limit

Setting the sources to zero, the system becomes translationally invariant so that all objects can be written in momentum space. Additionally, $\gamma[i] \to \frac{n}{2}$. Then, the above results can be summed up in the following formulae (in which we set $\lambda = 1$):

$$\mathbf{g}^{-1}(k) = i\omega_k + \boldsymbol{\mu} - \varepsilon_k\left(1 - \frac{n}{2}\right) - \chi(i\omega_k) - \varepsilon_k\Psi(i\omega_k),$$
$$\mu(k) = 1 - \frac{n}{2} + \Psi(i\omega_k), \tag{35}$$

where $\Psi(i\omega_k)$ and $\chi(i\omega_k)$ are the two momentum independent self-energies of the ECFL in infinite dimensions. In terms of these self-energies, the physical Green's function is written as

$$\mathcal{G}(k) = \frac{1 - \frac{n}{2} + \Psi(i\omega_k)}{i\omega_k + \boldsymbol{\mu} - \varepsilon_k\left(1 - \frac{n}{2}\right) - \chi(i\omega_k) - \varepsilon_k\Psi(i\omega_k)}. \tag{36}$$

Comparing with the standard form of the Green's function in terms of the Dyson–Mori self energy

$$\mathcal{G}(k) = \frac{1 - \frac{n}{2}}{i\omega_k + \boldsymbol{\mu} - \epsilon_k\left(1 - \frac{n}{2}\right) - \Sigma_{DM}(k)}, \tag{37}$$

we see the momentum independence of the Dyson–Mori self energy $\Sigma_{DM}(k) = \Sigma_{DM}(i\omega_k)$, and

$$\Sigma_{DM}(i\omega_k) = \frac{(i\omega_k + \boldsymbol{\mu})\Psi(i\omega_k) + \left(1 - \frac{n}{2}\right)\chi(i\omega_k)}{1 - \frac{n}{2} + \Psi(i\omega_k)}. \tag{38}$$

3.3. Conductivity in the limit of large dimensions

It is well known that for the finite-U Hubbard model in the limit of large dimensions, for zero wave vector, vertex corrections can be neglected in the current–current correlation function [10,9]. This simple observation allows one to express the optical conductivity in terms of the single particle Green's function as in Eq. (50). We show that this is also the case for the infinite dimensional t–J model. Moreover, a question of practical importance for the purpose of calculating the optical conductivity within the framework of ECFL, is whether or not Eq. (50) can be applied at each order in the λ expansion (as is done in Ref. [23]). We show that it can be applied and is the correct procedure. First, we define the relevant objects.

The Schrödinger picture current operator for site j in the direction α is defined as follows:

$$J_j^\alpha = i\sum_{k\sigma} v_{k,j}^\alpha X_k^{\sigma 0} X_j^{0\sigma}; \quad v_{k,j}^\alpha = t_{k,j}(\vec{R}_k - \vec{R}_j)_\alpha, \tag{39}$$

so that v is a velocity. Using the notation $J^\alpha[i] = J_i^\alpha(\tau_i)$; $\widetilde{J}^\alpha[i] = J^\alpha[i] - \langle J^\alpha[i]\rangle$, we define the correlation function $\Pi_{\alpha\beta}[i, f]$ and its Fourier transform as

$$\Pi_{\alpha\beta}[i, f] = \langle T_\tau \widetilde{J}^\alpha[i]\widetilde{J}^\beta[f]\rangle;$$
$$\Pi_{\alpha\beta}(\vec{q}, i\Omega_n) = \int_0^\beta d(\tau_i - \tau_f)\, e^{i\Omega_n(\tau_i - \tau_f)} \sum_{i-f} e^{-i\vec{q}\cdot(\vec{R}_i - \vec{R}_f)}\, \Pi_{\alpha\beta}[i, f]. \tag{40}$$

The optical conductivity can be given in terms of this object as

$$\sigma^{\alpha\beta}(\omega) = \frac{1}{i\omega - \eta}\left[\Pi_{\alpha\beta}(\vec{0}, \omega + i\eta) - \Pi_{\alpha\beta}(\vec{0}, i\eta)\right], \tag{41}$$

where $\eta = 0^+$. We would like to express the object $\Pi_{\alpha\beta}[i,f]$ as a functional derivative of the Green's function. To this end, we add a source which couples to the current operator

$$\mathcal{A} \to \mathcal{A} + \sum_{j\alpha} \int_0^\beta d\tau \kappa_j^\alpha(\tau) J_j^\alpha(\tau). \tag{42}$$

In terms of the κ source derivative of the Green's function, and using the definitions $v^\alpha[i,j] = v^\alpha_{i,j}\delta(\tau_i - \tau_j)$; $\kappa_i^\alpha = \kappa_i^\alpha(\tau_i)$, $\Pi_{\alpha\beta}[i,f]$ is given as

$$\Pi_{\alpha\beta}[i,f] = -i \,\mathrm{Tr}\left(\frac{\delta}{\delta \kappa_i^\alpha} \mathcal{G}[f,\mathbf{j}]\; v^\beta[\mathbf{j}, f^+] \right)_{\mathcal{A}\to 0}, \tag{43}$$

where the trace is over the spin degrees of freedom only. We expand the RHS of this equation using Eq. (17) (which holds equally well for the κ source derivative), finally obtaining an expression for $\Pi_{\alpha\beta}[i,f]$ in terms of the κ source derivatives of $\mathbf{g}^{-1}$ and μ.

$$\begin{aligned} \Pi_{\alpha\beta}[i,f] =\; & i\,\mathrm{Tr}\left(\mathbf{g}[f,\mathbf{x}]\, \frac{\delta}{\delta\kappa_i^\alpha} \mathbf{g}^{-1}[\mathbf{x},\mathbf{y}]\, \mathbf{g}[\mathbf{y},\mathbf{k}] \mu[\mathbf{k},\mathbf{j}]\; v^\beta[\mathbf{j}, f^+] \right)_{\mathcal{A}\to 0} \\ & - i\,\mathrm{Tr}\left(\mathbf{g}[f,\mathbf{k}] \frac{\delta}{\delta\kappa_i^\alpha} \mu[\mathbf{k},\mathbf{j}]\; v^\beta[\mathbf{j}, f^+] \right)_{\mathcal{A}\to 0}. \end{aligned} \tag{44}$$

We now consider how the additional source Eq. (42) affects the ECFL equations of motion (Eqs. (15) and (16)). The source enters into the equations of motion in the same way as the Hamiltonian does, via its commutator with the destruction operator, $X_i^{0\sigma}$. Moreover, the source has the same form as the Hamiltonian, with the hopping in the kinetic energy replaced by the velocity in the current operator. Therefore, the additional source affects the equations of motion only through the substitution

$$t[i,f] \to t[i,f] - i \sum_\alpha \kappa_f^\alpha\; v^\alpha[i,f]. \tag{45}$$

Thus, the new equations of motion can be read off from Eq. (15) as

$$\begin{aligned} \mathbf{g}^{-1}[i,m] \;=\; & (\boldsymbol{\mu} - \partial_{\tau_i} - \mathcal{V}_i)\,\delta[i,m] + \left(t[i,m] - i \sum_\alpha \kappa_m^\alpha\; v^\alpha[i,m] \right) \\ & \times (1 - \lambda\gamma[i]) + \lambda \left(t[i,\mathbf{j}] - i \sum_\alpha \kappa_{\mathbf{j}}^\alpha\; v^\alpha[i,\mathbf{j}] \right) \xi^* \cdot \mathbf{g}[\mathbf{j},\mathbf{n}] \cdot \Lambda_*[\mathbf{n}, m; i], \\ \mu[i,m] = \; & (1-\lambda\gamma[i])\delta[i,m] - \lambda \left(t[i,\mathbf{j}] - i \sum_\alpha \kappa_{\mathbf{j}}^\alpha\; v^\alpha[i,\mathbf{j}] \right) \xi^* \cdot \mathbf{g}[\mathbf{j},\mathbf{n}] \cdot \mathcal{U}_*[\mathbf{n}, m; i]. \end{aligned} \tag{46}$$

Since there is no source derivative with respect to κ in the equations of motion and $v^\alpha[i,f]$ is of the same order in $\frac{1}{\sqrt{d}}$ as $t[i,f]$, all of the results derived in Section 3.1 continue to hold after making the substitution in Eq. (45). In particular, we showed that $\mathbf{g}^{-1}[i,m]$ and $\mu[i,m]$ have the following form (Eq. (22)):

$$\begin{aligned} \mathbf{g}^{-1}[i,m] \;=\; & (\boldsymbol{\mu} - \partial_{\tau_i} - \mathcal{V}_i)\,\delta[i,m] - \lambda\,\delta_{i,m}\,\chi_i[\tau_i,\tau_m] + \left(t[i,m] - i \sum_\alpha \kappa_m^\alpha\; v^\alpha[i,m] \right) \\ & \times (1-\lambda\gamma[i]) + \lambda \left(t_{i,m} - i \sum_\alpha \kappa_m^\alpha\; v_{i,m}^\alpha \right) \Psi_i[\tau_i,\tau_m], \\ \mu[i,m] = \; & \delta[i,m](1-\lambda\gamma[i]) + \lambda\,\delta_{i,m}\,\Psi_i[\tau_i,\tau_m], \end{aligned} \tag{47}$$

where χ_i, Ψ_i, and $\gamma[i]$ have properties (a)–(c) of class L functions (Section 2.4), and are defined by Eqs. (22) through (25). We shall now further assume that they also satisfy property (d) (Eq. (21))

and show that this assumption is consistent with their definitions. This, in turn, will allow us to demonstrate the validity of Eq. (50).

Our task is then to show that χ_i, Ψ_i, and $\gamma[i]$, as defined in the last line of Eqs. (23) and (25), satisfy Eq. (21). By Eq. (24), A_i and B_i satisfy Eq. (21) since they inherit this property from χ_i, Ψ_i, and $\gamma[i]$. It remains to show that $\mathbf{g}_{x,x}[\tau_n,\tau_m]$ and $(t_{x,\mathbf{j}}-i\sum_\alpha \kappa^\alpha_{\mathbf{j}}(\tau_n)v^\alpha_{x,\mathbf{j}})\,\mathbf{g}_{\mathbf{j},x}[\tau_n,\tau_m]$ (the time indices are arbitrary) satisfy this equation.

Defining the notation $w_{i,f}(\tau_i)\equiv t_{i,f}-i\sum_\alpha \kappa^\alpha_f(\tau_i)\,v^\alpha_{i,f}$, and using (the κ source derivative version of) Eq. (17) as well as Eq. (47), we find that

$$\begin{aligned}\left(\frac{\delta}{\delta\kappa^\alpha_i}w_{x,\mathbf{j}}(\tau_n)\,\mathbf{g}_{\mathbf{j},x}[\tau_n,\tau_m]\right)_{\mathcal{A}\to0} &= -i\delta[\tau_i,\tau_n]v^\alpha_{x,i}\,\mathbf{g}_{i,x}[\tau_i,\tau_m]+it_{x,\mathbf{j}}\mathbf{g}_{\mathbf{j},\mathbf{a}}[\tau_n,\tau_{\mathbf{a}}]\\ &\quad\times(1-\lambda\gamma[\mathbf{a}]\delta[\tau_{\mathbf{a}},\tau_i]+\lambda\Psi_{\mathbf{a}}[\tau_{\mathbf{a}},\tau_i])v^\alpha_{\mathbf{a},i}g_{i,x}[\tau_i,\tau_m]\\ &\quad+\lambda\,t_{x,\mathbf{j}}\mathbf{g}_{\mathbf{j},\mathbf{a}}[\tau_n,\tau_{\mathbf{a}}]\frac{\delta}{\delta\kappa^\alpha_i}(\gamma[\mathbf{a}]\delta[\tau_{\mathbf{a}},\tau_{\mathbf{b}}]-\Psi_{\mathbf{a}}[\tau_{\mathbf{a}},\tau_{\mathbf{b}}])\\ &\quad\times t_{\mathbf{a},\mathbf{b}}g_{\mathbf{b},x}[\tau_{\mathbf{b}},\tau_m]+\lambda\,t_{x,\mathbf{j}}\mathbf{g}_{\mathbf{j},\mathbf{a}}[\tau_n,\tau_{\mathbf{a}}]\\ &\quad\times\frac{\delta}{\delta\kappa^\alpha_i}(\chi_{\mathbf{a}}[\tau_{\mathbf{a}},\tau_{\mathbf{b}}])\,g_{\mathbf{a},x}[\tau_{\mathbf{b}},\tau_m],\end{aligned}\tag{48}$$

where the RHS is also evaluated in the $\mathcal{A}\to0$ limit. We now substitute this into Eq. (21) (with $s=0$). The last two terms must vanish by assumption (where $\mathbf{a}$ has taken the place of $\mathbf{x}$). The first term contains two paths from i to f, both via $\mathbf{x}$. Hence, this term must vanish in the large d limit unless $\mathbf{x}=i$ or $\mathbf{x}=f$. The former also vanishes since $v^\alpha_{i,i}=0$ while the latter must vanish due to the sum over $i-f$ and the odd parity of $v^\alpha_{i,f}$. The same reasoning applies to the second term except that in this term the $\mathbf{x}=i$ case vanishes by the odd parity of $v^\alpha_{i,f}$. Hence, we have shown that $(t_{x,\mathbf{j}}-i\sum_\alpha\kappa^\alpha_{\mathbf{j}}(\tau_n)v^\alpha_{x,\mathbf{j}})\,\mathbf{g}_{\mathbf{j},x}[\tau_n,\tau_m]$ satisfies Eq. (21) with $s=0$. A completely analogous argument shows that this is also the case for $\mathbf{g}_{x,x}[\tau_n,\tau_m]$. Using Eqs. (26) and (27) (in particular the fact that the RHS is made up of the same objects as the LHS), the above argument can be used to show that the result holds for any value of s. Thus, we have demonstrated the self-consistency of our ansatz (Eq. (21)).

Substituting Eq. (47) into Eq. (44), and using Eq. (21), we find that

$$\sum_{i-f}\Pi_{\alpha\beta}[i,f]=\sum_{i-f}\mathrm{Tr}\left(\mathcal{G}[f,\mathbf{k}]v^\alpha[\mathbf{k},i]\mathcal{G}[i,\mathbf{j}]\,v^\beta[\mathbf{j},f^+]\right)_{\mathcal{A}\to0}.\tag{49}$$

Substituting this equation into Eq. (41), the optical conductivity may be expressed as

$$\sigma^{\alpha\beta}(\omega)=\frac{2}{i\omega}\sum_{\vec{p},i\omega_p}\mathcal{G}(\vec{p},i\omega_p)v^\alpha_{\vec{p}}v^\beta_{\vec{p}}[\mathcal{G}(\vec{p},\omega+i\eta+i\omega_p)-\mathcal{G}(\vec{p},i\eta+i\omega_p)].\tag{50}$$

We now want to prove that this result holds to each order in λ. We do this via an inductive argument, in which we assume that through nth order in λ, $\left(\frac{\delta}{\delta\kappa^\alpha_i}L_x\right)^{(n)}_{\mathcal{A}\to0}$ (where L_i can be Ψ_i, χ_i, or $\gamma[i]$) satisfies a certain explicit form (Eq. (51)), and then show that this form holds for $n+1$st order. We then plug Eq. (47) into $\sum_{i-f}\Pi_{\alpha\beta}[i,f]$ (Eq. (44)), and use the explicit form of $\left(\frac{\delta}{\delta\kappa^\alpha_i}L_x\right)^{(n)}_{\mathcal{A}\to0}$ to simplify the resulting expressions, thereby proving Eqs. (49) and (50) to each order in λ.

For the reason given below (Eq. (46)), we are free to use any of the results from Section 3.1, after making the substitution in Eq. (45). *We define* X_i to be a product of local functions of the type in Eq. (28) (i.e. $X_i=(R_i)^m$) *and* $Y_{i,f}$ to be a *proxy* for either $\mathbf{g}_{i,f}[\tau_n,\tau_m]$ or $t_{i,\mathbf{j}}\mathbf{g}_{\mathbf{j},f}[\tau_n,\tau_m]$ where the time indices are again arbitrary. *Inductive hypothesis*: through nth order in λ, the κ source derivative of the objects Ψ_i, χ_i, and $\gamma[i]$ (denoted below by the generic symbol L_i) can be written as

$$\begin{aligned}\left(\frac{\delta}{\delta\kappa^\alpha_i}L_x\right)^{(n)}_{\mathcal{A}\to0} &= \lambda^n X_xY_{x,\mathbf{x}_1}X_{\mathbf{x}_1}Y_{\mathbf{x}_1,\mathbf{x}_2}X_{\mathbf{x}_2}\ldots X_{\mathbf{x}_{m-1}}Y_{\mathbf{x}_{m-1},\mathbf{x}_m}X_{\mathbf{x}_m}\\ &\quad\times v^\alpha_{\mathbf{x}_m,i}\,Y_{i,\mathbf{x}_{m-1}}X_{\mathbf{x}_{m-1}}\ldots X_{\mathbf{x}_1}Y_{\mathbf{x}_1,x}X_x,\end{aligned}\tag{51}$$

where the number m is arbitrary. In the base case of zeroth order, the objects Ψ_i, χ_i, and $\gamma[i]$ are

$$\Psi_i^{(0)}[\tau_i, \tau_m] = 0; \qquad \gamma^{(0)}[i] = \mathbf{g}^{(k)}[i, i];$$
$$\chi_i^{(0)}[\tau_i, \tau_m] = -\left(t_{i,\mathbf{j}} - i\sum_\alpha \kappa_{\mathbf{j}}^\alpha(\tau_i) v_{\mathbf{ij}}^\alpha\right) \xi^* \cdot \mathbf{g}_{\mathbf{j},i}[\tau_i, \tau_i]\delta[\tau_i, \tau_m]. \tag{52}$$

We note that $\left(\frac{\delta}{\delta\kappa_i^\alpha} w_{x,\mathbf{j}}(\tau_n)\, \mathbf{g}_{\mathbf{j},x}[\tau_n, \tau_m]\right)^{(l)}_{\mathcal{A}\to 0}$ is given by Eq. (48) with the appropriate objects on the RHS evaluated to the appropriate order in λ. An analogous formula holds for $\left(\frac{\delta}{\delta\kappa_i^\alpha} \mathbf{g}_{x,x}[\tau_n, \tau_m]\right)^{(l)}_{\mathcal{A}\to 0}$. Using these formulas with $l = 0$ shows that the hypothesis is satisfied for the base case.

We now prove the inductive step. Eq. (28) continues to hold with $t_{i,\mathbf{j}} \to w_{i,\mathbf{j}}(\tau_n)$ (the time index is again arbitrary). Therefore, using the notation $\widetilde{R}_i = [R_i]_{t_{i,\mathbf{j}}\to w_{i,\mathbf{j}}(\tau_n)}$, we may write

$$\left(\frac{\delta}{\delta\kappa_i^\alpha} L_x\right)^{(n+1)}_{\mathcal{A}\to 0} = \sum_{r=0}^{n+1} \lambda^r \left(\frac{\delta}{\delta\kappa_i^\alpha} (\widetilde{R}_x)^m\right)^{(n+1-r)}_{\mathcal{A}\to 0}. \tag{53}$$

Substituting the formulas for $\left(\frac{\delta}{\delta\kappa_i^\alpha} w_{x,\mathbf{j}}(\tau_n)\, \mathbf{g}_{\mathbf{j},x}[\tau_n, \tau_m]\right)^{(l)}_{\mathcal{A}\to 0}$ and $\left(\frac{\delta}{\delta\kappa_i^\alpha} \mathbf{g}_{x,x}[\tau_n, \tau_m]\right)^{(l)}_{\mathcal{A}\to 0}$ (Eq. (48)) for $l \le n + 1$ into Eq. (53), and using the inductive hypothesis, shows that $\left(\frac{\delta}{\delta\kappa_i^\alpha} \Psi_x\right)^{(n+1)}_{\mathcal{A}\to 0}$, $\left(\frac{\delta}{\delta\kappa_i^\alpha} \chi_x\right)^{(n+1)}_{\mathcal{A}\to 0}$, and $\left(\frac{\delta}{\delta\kappa_i^\alpha} \gamma[x]\right)^{(n+1)}_{\mathcal{A}\to 0}$ all have the desired form (Eq. (51)). Thus, Eq. (51) holds to all orders in λ.

Substituting Eq. (47) into $\sum_{i-f} \Pi_{\alpha\beta}[i, f]$ (Eq. (44)), and using Eq. (51), the only non vanishing terms are those which involve a derivative of the explicit factor $(t_{\mathbf{x},\mathbf{y}} - i\sum_\alpha \kappa_{\mathbf{y}}^\alpha\, v_{\mathbf{x},\mathbf{y}}^\alpha)$ from Eq. (47). The other terms vanish due to the following reasoning. Upon substituting Eq. (51), in each of these terms there are two paths from i to f, both of which pass through the point $\mathbf{x}$ as well as the points $\mathbf{x}_1 \ldots \mathbf{x}_{m-1}$ in Eq. (51). Hence, in the large d limit, all of these points must be chosen to be either i or f for these terms to be non vanishing. Then, if we choose $\mathbf{x}_{m-1} = i$, the term vanishes due to parity, while if we choose $\mathbf{x}_{m-1} = f$, the term vanishes due to parity combined with the sum $\sum_{i-f}$. Therefore, after making these simplifications, we find that Eq. (49) and consequently Eq. (50) hold to each order in λ.

3.4. $O(\lambda^2)$ theory in the limit of large dimensions

To obtain self-consistent integral equations to any order in λ for the objects $\mathbf{g}^{-1}[i, f]$ and $\mu[i, f]$, we expand Eqs. (22) through (25) iteratively in λ, and set the sources to zero. Once the sources are set to zero, the system becomes translationally invariant in both space and time and we may express the equations in momentum/frequency space. Using the definitions

$$\mathbf{g}_{\text{loc},m}(i\omega_k) \equiv \sum_{\vec{k}} \mathbf{g}(k)\epsilon_{\vec{k}}^m, \tag{54}$$

$$I_{m_1m_2m_3}(i\omega_k) \equiv -\sum_{\omega_p,\omega_q} \mathbf{g}_{\text{loc},m_1}(i\omega_q)\mathbf{g}_{\text{loc},m_2}(i\omega_p)\mathbf{g}_{\text{loc},m_3}(i\omega_q + i\omega_p - i\omega_k), \tag{55}$$

the resulting equations to $O(\lambda^2)$ are

$$a_{\mathcal{G}} \equiv 1 - \lambda\frac{n}{2} + \lambda^2\frac{n^2}{4}, \tag{56}$$

$$\mathbf{g}^{-1}(k) = i\omega_k + \boldsymbol{\mu}' - a_{\mathcal{G}}\left(\varepsilon_k - \frac{u_0}{2}\right) - \lambda\left(\epsilon_{\vec{k}} - \frac{u_0}{2}\right)\Psi(i\omega_k) - \lambda\chi(i\omega_k), \tag{57}$$

$$\mu(i\omega_k) = a_{\mathcal{G}} + \lambda\Psi(i\omega_k), \tag{58}$$

$$\boldsymbol{\mu}' = \boldsymbol{\mu} - u_0\left(\lambda\frac{n}{2} - \lambda^2\frac{n^2}{8}\right) + \lambda\sum_p \varepsilon_p \mathbf{g}(p) - a_{\mathcal{G}}\frac{u_0}{2}, \tag{59}$$

$$\Psi(i\omega_k) = -\lambda u_0 I_{000}(i\omega_k) + 2\lambda I_{010}(i\omega_k), \tag{60}$$

$$\chi(i\omega_k) = -\frac{u_0}{2}\Psi(i\omega_k) - u_0\lambda I_{001}(i\omega_k) + 2\lambda I_{011}(i\omega_k). \tag{61}$$

Before solving the equations, one must set $\lambda = 1$. The two Lagrange multipliers $\boldsymbol{\mu}$ and u_0 are determined by the two sum rules:

$$\sum_k \mathbf{g}(k) = \frac{n}{2}; \qquad \sum_k \mathcal{G}(k) = \frac{n}{2}. \tag{62}$$

The objects $\mathbf{g}_{\text{loc},m}(i\omega_k)$ are given by an appropriate integral over the non-interacting density of states of a function composed of the two self energies $\chi(i\omega_k)$ and $\Psi(i\omega_k)$ and the energy ϵ (Eq. (57)). Therefore, these constitute a self-consistent set of equations for the two self energies. These equations have been solved numerically and compared to DMFT calculations in Ref. [23].

4. Anderson model

A word is needed at this point on the notation used, since similar looking symbols represent quite different objects in the t–J model and the AIM. We use the functions $\mathcal{G}(\{\tau_j\})$, $\mathbf{g}(\{\tau_j\})$, $\mu(\{\tau_j\})$ or $\mathcal{G}(\{i\omega_j\})$, $\mathbf{g}(\{i\omega_j\})$, $\mu(\{i\omega_j\})$ and the related vertex functions for the impurity site of the AIM as well, but distinguish them from the t–J model variables by dropping the spatial or momentum labels. Therefore in an equation such as Eq. (88), the object on the left (right) hand side corresponds to the t–J model (AIM).

4.1. Equations of motion for the Anderson model

In DMFT [8,9], the local Green's function of the infinite-dimensional finite-U Hubbard model is mapped onto the impurity Green's function of the finite-U AIM, with a self-consistently determined set of parameters. Using the ECFL equations of motion for both models, we show that the same mapping can be made between the infinite-dimensional t–J model and the infinite-U AIM. Further, we show that this mapping also extends to the auxiliary Green's function $\mathbf{g}$, and the caparison factor μ individually. In this section, we briefly review the ECFL theory of the AIM [22], and we establish the mapping in the following section.

Consider the AIM in the limit $U \to \infty$ which has the following Hamiltonian:

$$H = \sum_\sigma \epsilon_d X^{\sigma\sigma} + \sum_{k\sigma}\widetilde{\epsilon}_k n_{k\sigma} + \sum_{k\sigma}(V_k X^{\sigma 0} c_{k\sigma} + V_k^* c_{k\sigma}^\dagger X^{0\sigma}), \tag{63}$$

where we have set the Fermi energy of the conduction electrons to be zero. The impurity Green's function is given by the following expression:

$$\mathcal{G}_{\sigma_i\sigma_f}[\tau_i, \tau_f] = -\langle\langle X^{0\sigma_i}(\tau_i)\, X^{\sigma_f 0}(\tau_f)\rangle\rangle. \tag{64}$$

The ECFL solution of the Anderson model is presented in Ref. [22]. The impurity Green's function is factored into the auxiliary Green's function and the caparison factor:

$$\mathcal{G}[\tau_i, \tau_f] = \mathbf{g}[\tau_i, \tau_{\mathbf{j}}] \cdot \mu[\tau_{\mathbf{j}}, \tau_f]. \tag{65}$$

The equations of motion for $\mathbf{g}$ and μ can be written as

$$\begin{aligned}(\partial_{\tau_i} + \epsilon_d + \mathcal{V}(\tau_i))\mathbf{g}[\tau_i, \tau_f] \;=\; & -\delta(\tau_i - \tau_f) - (1 - \lambda\gamma[\tau_i]) \cdot \varDelta[\tau_i, \tau_{\mathbf{j}}] \cdot \mathbf{g}[\tau_{\mathbf{j}}, \tau_f] \\ & - \lambda\, \xi^* \varDelta[\tau_i, \tau_{\mathbf{j}}] \cdot \mathbf{g}[\tau_{\mathbf{j}}, \tau_{\mathbf{x}}] \cdot \varLambda_*[\tau_{\mathbf{x}}, \tau_{\mathbf{y}}; \tau_i] \cdot \mathbf{g}[\tau_{\mathbf{y}}, \tau_f],\end{aligned} \tag{66}$$

$$\mu[\tau_i, \tau_f] = \delta(\tau_i - \tau_f)(\mathbb{1} - \lambda\gamma[\tau_i]) + \lambda\, \xi^* \cdot \varDelta[\tau_i, \tau_{\mathbf{j}}] \cdot \mathbf{g}[\tau_{\mathbf{j}}, \tau_{\mathbf{x}}] \cdot \mathcal{U}_*[\tau_{\mathbf{x}}, \tau_f; \tau_i], \tag{67}$$

where the conduction band enters through the ($\mathcal{V}$ independent) function

$$\Delta[\tau_i, \tau_f] = -\mathbb{1} \sum_k |V_k|^2 (\partial_{\tau_i} + \widetilde{\epsilon}_k)^{-1} \delta(\tau_i - \tau_f). \tag{68}$$

We have also made use of the following definitions:

$$\begin{aligned} &\Lambda[\tau_n, \tau_m; \tau_i] = -\frac{\delta}{\delta \mathcal{V}(\tau_i)} \mathbf{g}^{-1}[\tau_n, \tau_m]; \qquad \mathcal{U}[\tau_n, \tau_m; \tau_i] = \frac{\delta}{\delta \mathcal{V}(\tau_i)} \mu[\tau_n, \tau_m]; \\ &\gamma[\tau_i] = \mu^{(k)}[\tau_\mathbf{n}, \tau_i^+] \cdot \mathbf{g}^{(k)}[\tau_i, \tau_\mathbf{n}]. \end{aligned} \tag{69}$$

4.2. Mapping of the t–J model onto the Anderson model in infinite dimensions

Now let us consider the t–J model in the limit of infinite dimensions. Inverting Eq. (15), the equations of motion for $\mathbf{g}_{i,i}[\tau_i, \tau_f]$ and $\mu_{i,i}[\tau_i, \tau_f]$ are

$$\begin{aligned} (\partial_{\tau_i} - \mu + \mathcal{V}_i(\tau_i))\, \mathbf{g}_{i,i}[\tau_i, \tau_f] \;=\; & -\delta(\tau_i - \tau_f) + (1 - \lambda\gamma[i]) \cdot t_{i,\mathbf{j}}\, \mathbf{g}_{\mathbf{j},i}[\tau_i, \tau_f] \\ & + \lambda\, t_{i,\mathbf{j}}\, \xi^* \cdot \mathbf{g}_{\mathbf{j},i}[\tau_i, \tau_\mathbf{x}] \cdot A_{i,*}[\tau_\mathbf{x}, \tau_\mathbf{y}; \tau_i] \cdot \mathbf{g}_{i,i}[\tau_\mathbf{y}, \tau_f] \\ & + \lambda\, t_{i,\mathbf{j}}\, \xi^* \cdot \mathbf{g}_{\mathbf{j},i}[\tau_i, \tau_\mathbf{x}] \cdot B_{i,*}[\tau_\mathbf{x}, \tau_\mathbf{y}; \tau_i] \cdot t_{i,\mathbf{y}}\, \mathbf{g}_{\mathbf{y},i}[\tau_\mathbf{y}, \tau_f], \end{aligned} \tag{70}$$

$$\mu_{i,i}[\tau_i, \tau_f] = (1 - \lambda\gamma[i])\delta(\tau_i - \tau_f) + \lambda\, t_{i,\mathbf{j}}\, \xi^* \cdot \mathbf{g}_{\mathbf{j},i}[\tau_i, \tau_\mathbf{x}] \cdot B_{i,*}[\tau_\mathbf{x}, \tau_f; \tau_i]. \tag{71}$$

By mapping $\mathbf{g}_{i,i}[\tau_i, \tau_f]$ and $\mu_{i,i}[\tau_i, \tau_f]$ onto $\mathbf{g}[\tau_i, \tau_f]$ and $\mu[\tau_i, \tau_f]$ of the AIM, we would like to show that the equations of motion of the AIM (Eqs. (66) and (67)) and those of the infinite dimensional t–J model (Eqs. (70) and (71)) map onto each other. To do this, we need the analog of the object $\mathbf{g}^{-1}[\tau_i, \tau_f]$ of the AIM in the t–J model. We denote this new object by $\mathbf{g}^{-1}_{\mathrm{loc},i}[\tau_i, \tau_f]$ and define it to be the temporal inverse of the local auxiliary Green's function:

$$\mathbf{g}_{i,i}[\tau_i, \tau_\mathbf{j}] \cdot \mathbf{g}^{-1}_{\mathrm{loc},i}[\tau_\mathbf{j}, \tau_f] = \delta(\tau_i - \tau_f). \tag{72}$$

Note that $\mathbf{g}^{-1}_{\mathrm{loc},i}[\tau_i, \tau_f] \neq \mathbf{g}^{-1}_{i,i}[\tau_i, \tau_f]$. We also define the corresponding vertex:

$$\Lambda_{\mathrm{loc},i}[\tau_n, \tau_m; \tau_i] = -\frac{\delta}{\delta \mathcal{V}_i(\tau_i)} \mathbf{g}^{-1}_{\mathrm{loc},i}[\tau_n, \tau_m]. \tag{73}$$

We now make use of the following identity:

$$\Lambda_{\mathrm{loc},i}[\tau_x, \tau_\mathbf{y}; \tau_i] \cdot \mathbf{g}_{i,i}[\tau_\mathbf{y}, \tau_f] = A_i[\tau_x, \tau_\mathbf{y}; \tau_i] \cdot \mathbf{g}_{i,i}[\tau_\mathbf{y}, \tau_f] + B_i[\tau_x, \tau_\mathbf{y}; \tau_i] \cdot t_{i,\mathbf{y}}\, \mathbf{g}_{\mathbf{y},i}[\tau_\mathbf{y}, \tau_f]. \tag{74}$$

This identity is easily proven by considering $\frac{\delta}{\delta \mathcal{V}_i(\tau_i)} \mathbf{g}_{i,i}[\tau_x, \tau_f]$:

$$\frac{\delta}{\delta \mathcal{V}_i(\tau_i)} \mathbf{g}_{i,i}[\tau_x, \tau_f] = \mathbf{g}_{i,i}[\tau_x, \tau_\mathbf{j}] \Lambda_{\mathrm{loc},i}[\tau_\mathbf{j}, \tau_\mathbf{y}; \tau_i] \mathbf{g}_{i,i}[\tau_\mathbf{y}, \tau_f]. \tag{75}$$

The LHS can also be expressed as

$$\begin{aligned} \frac{\delta}{\delta \mathcal{V}_i(\tau_i)} \mathbf{g}_{i,i}[\tau_x, \tau_f] = \mathbf{g}_{i,i}[\tau_x, \tau_\mathbf{j}] \cdot (& A_i[\tau_\mathbf{j}, \tau_\mathbf{y}; \tau_i] \cdot \mathbf{g}_{i,i}[\tau_\mathbf{y}, \tau_f] \\ & + B_i[\tau_\mathbf{j}, \tau_\mathbf{y}; \tau_i] \cdot t_{i,\mathbf{y}}\, \mathbf{g}_{\mathbf{y},i}[\tau_\mathbf{y}, \tau_f]). \end{aligned} \tag{76}$$

Left multiplying the above 2 equations by $\mathbf{g}^{-1}_{\mathrm{loc},i}$, we recover the identity Eq. (74). Substituting this identity into Eq. (70), we obtain

$$\begin{aligned} (\partial_{\tau_i} - \mu + \mathcal{V}_i(\tau_i))\mathbf{g}_{i,i}[\tau_i, \tau_f] = & -\delta(\tau_i - \tau_f) + (1 - \lambda\gamma[i]) \cdot t_{i,\mathbf{j}} \mathbf{g}_{\mathbf{j},i}[\tau_i, \tau_f] \\ & + \lambda\, t_{i,\mathbf{j}}\, \xi^* \cdot \mathbf{g}_{\mathbf{j},i}[\tau_i, \tau_\mathbf{x}] \cdot \Lambda_{\mathrm{loc},i*}[\tau_\mathbf{x}, \tau_\mathbf{y}; \tau_i] \cdot \mathbf{g}_{i,i}[\tau_\mathbf{y}, \tau_f]. \end{aligned} \tag{77}$$

We are now ready to map the t–J model onto the Anderson model. To do this, we map the local objects $\mathbf{g}_{i,i}[\tau_i, \tau_f]$ and $\mu_{i,i}[\tau_i, \tau_f]$ of the t–J model to the objects $\mathbf{g}[\tau_i, \tau_f]$ and $\mu[\tau_i, \tau_f]$ of the Anderson model. We also map μ to $-\epsilon_d$. The following mappings also follow as a consequence of these.

$$\begin{aligned} &\gamma[i] \to \gamma[\tau_i]; \qquad \Lambda_{\text{loc},i}[\tau_n, \tau_m; \tau_i] \to \Lambda[\tau_n, \tau_m; \tau_i]; \\ &B_i[\tau_n, \tau_m; \tau_i] \to -\mathcal{U}[\tau_n, \tau_m; \tau_i]. \end{aligned} \tag{78}$$

Comparing Eq. (77) with Eq. (66) and Eq. (71) with Eq. (67), we see that the equations of motion map onto each other if the following constraint is satisfied:

$$t_{i,\mathbf{j}}\, \mathbf{g}_{\mathbf{j},i}[\tau_i, \tau_f] = -\Delta[\tau_i, \tau_{\mathbf{j}}] \cdot \mathbf{g}[\tau_{\mathbf{j}}, \tau_f]. \tag{79}$$

4.3. *Mapping to each order in λ*

The $O(\lambda^2)$ equations for the infinite-dimensional t–J model and infinite-U AIM are solved numerically in Ref. [23] and Ref. [22] respectively. This can in principle be done to higher orders in λ as well, and it is therefore interesting to know if the mapping from the previous section holds to each order in λ. We show that it does, and give a simple prescription for obtaining the ECFL integral equations for one model from those of the other one (Eq. (83)).

We review the λ expansion for the Anderson model from Ref. [22]. There, Eqs. (66) and (67) are written as

$$\begin{aligned} \mathbf{g}^{-1}[\tau_i, \tau_f] \;=\; &-(\partial_{\tau_i} + \epsilon_d + \mathcal{V}(\tau_i))\delta(\tau_i - \tau_f) - (1 - \lambda\gamma[\tau_i]) \cdot \Delta[\tau_i, \tau_f] \\ &- \lambda\xi^* \Delta[\tau_i, \tau_{\mathbf{j}}] \cdot \mathbf{g}[\tau_{\mathbf{j}}, \tau_{\mathbf{x}}] \cdot \Lambda_*[\tau_{\mathbf{x}}, \tau_f; \tau_i], \end{aligned} \tag{80}$$

$$\mu[\tau_i, \tau_f] = \delta(\tau_i - \tau_f)(\mathbb{1} - \lambda\gamma[\tau_i]) + \lambda\xi^* \cdot \Delta[\tau_i, \tau_{\mathbf{j}}] \cdot \mathbf{g}[\tau_{\mathbf{j}}, \tau_{\mathbf{x}}] \cdot \mathcal{U}_*[\tau_{\mathbf{x}}, \tau_f; \tau_i]. \tag{81}$$

The λ expansion is obtained in the same way as for the t–J model, by iterating the equations in $\mathbf{g}^{-1}$ and μ and keeping track of explicit powers of λ. The details to $O(\lambda^2)$ can be found in Ref. [22]. To relate this to the λ expansion for the infinite-dimensional t–J model, recall from Eq. (28) that to each order in λ, Ψ_i, χ_i, $\gamma[i]$, A_i, and B_i can be written as a product of the functions $\mathbf{g}_{i,i}[\tau_n, \tau_m]$ and $t_{i,\mathbf{j}}\mathbf{g}_{\mathbf{j},i}[\tau_n, \tau_m]$. We can now state our *inductive hypothesis*: through nth order in λ, the λ expansion for the Anderson model has the form

$$\begin{aligned} \mathbf{g}^{-1}[\tau_i, \tau_m] \;=\; &-(\partial_{\tau_i} + \epsilon_d + \mathcal{V}(\tau_i))\, \delta[\tau_i, \tau_m] - \lambda\, \chi[\tau_i, \tau_m] \\ &- (1 - \lambda\gamma[\tau_i])\Delta[\tau_i, \tau_m] - \lambda\, \Psi[\tau_i, \tau_{\mathbf{j}}]\Delta[\tau_{\mathbf{j}}, \tau_m], \\ \mu[\tau_i, \tau_m] = &\;\delta[\tau_i, \tau_m](1 - \lambda\gamma[\tau_i]) + \lambda\Psi[\tau_i, \tau_m], \\ \Lambda[\tau_n, \tau_m; \tau_i] = &\; A[\tau_n, \tau_m; \tau_i] - B[\tau_n, \tau_{\mathbf{j}}; \tau_i]\Delta[\tau_{\mathbf{j}}, \tau_m], \\ \mathcal{U}[\tau_n, \tau_m; \tau_i] = &-B[\tau_n, \tau_m; \tau_i], \end{aligned} \tag{82}$$

where through nth order in λ, the objects $A[\tau_n, \tau_m; \tau_i]$ and $B[\tau_n, \tau_m; \tau_i]$, and through $n-1$st order in λ, the objects $\gamma[\tau_i]$, $\chi[\tau_i, \tau_m]$, and $\Psi[\tau_i, \tau_m]$, can be obtained from their infinite dimensional t–J model counterparts via the substitution

$$\mathbf{g}_{i,i}[\tau_n, \tau_m] \to \mathbf{g}[\tau_n, \tau_m]; \qquad \boldsymbol{\mu} \to -\epsilon_d; \qquad t_{i,\mathbf{j}}\mathbf{g}_{\mathbf{j}i}[\tau_n, \tau_m] \to -\Delta[\tau_n, \tau_{\mathbf{j}}] \cdot \mathbf{g}[\tau_{\mathbf{j}}, \tau_m]. \tag{83}$$

We first examine the base case of zeroth order:

$$A^{(0)}[\tau_n, \tau_m; \tau_i] = \delta[\tau_i, \tau_n]\delta[\tau_i, \tau_m]; \qquad B^{(0)}[\tau_n, \tau_m; \tau_i] = 0. \tag{84}$$

Comparing with Eq. (29), the hypothesis clearly holds. We now prove the inductive step. Eq. (69) together with Eqs. (80) through (82) implies the following:

$$\begin{aligned} &\chi^{(n)}[\tau_n, \tau_m] = \xi^* \Delta[\tau_n, \tau_{\mathbf{j}}] \cdot \mathbf{g}[\tau_{\mathbf{j}}, \tau_{\mathbf{x}}] \cdot A_*^{(n)}[\tau_{\mathbf{x}}, \tau_m; \tau_n], \\ &\Psi^{(n)}[\tau_n, \tau_m] = -\xi^* \Delta[\tau_n, \tau_{\mathbf{j}}] \cdot \mathbf{g}[\tau_{\mathbf{j}}, \tau_{\mathbf{x}}] \cdot B_*^{(n)}[\tau_{\mathbf{x}}, \tau_m; \tau_n], \\ &A^{(n+1)}[\tau_n, \tau_m; \tau_i] = \lambda \left(\frac{\delta}{\delta \mathcal{V}(\tau_i)} \chi[\tau_n, \tau_m] \right)^{(n)}, \end{aligned}$$

$$B^{(n+1)}[\tau_n, \tau_m; \tau_i] = \lambda\, \delta[\tau_n, \tau_m] \left(\frac{\delta}{\delta \mathcal{V}(\tau_i)} \gamma[\tau_n] \right)^{(n)} - \lambda \left(\frac{\delta}{\delta \mathcal{V}(\tau_i)} \Psi[\tau_n, \tau_m] \right)^{(n)},$$
$$\gamma^{(n)}[\tau_i] = -\lambda\, \gamma^{(k)(n-1)}[\tau_i] \mathbf{g}^{(k)}[\tau_i, \tau_i] + \lambda\, \Psi^{(k)(n-1)}[\tau_{\mathbf{j}}, \tau_i] \mathbf{g}^{(k)}[\tau_i, \tau_{\mathbf{j}}]. \tag{85}$$

Comparing with Eq. (30), we see that $\chi^{(n)}[\tau_n, \tau_m]$, $\Psi^{(n)}[\tau_n, \tau_m]$, and $\gamma^{(n)}[\tau_i]$ have the desired form. We also note that

$$\left(\frac{\delta}{\delta \mathcal{V}(\tau_r)} \mathbf{g}[\tau_i, \tau_m] \right)^{(l)} = \mathbf{g}[\tau_i, \tau_{\mathbf{x}}](A^{(l)}[\tau_{\mathbf{x}}, \tau_{\mathbf{y}}; \tau_r] - B^{(l)}[\tau_{\mathbf{x}}, \tau_{\mathbf{j}}; \tau_r] \Delta[\tau_{\mathbf{j}}, \tau_{\mathbf{y}}]) \mathbf{g}[\tau_{\mathbf{y}}, \tau_m]. \tag{86}$$

Comparing this with Eqs. (26) and (27), we see that by the inductive hypothesis, the mapping Eq. (83) continues to hold through order $l \leq n$ even after both sides have been acted on with a functional derivative. Furthermore, in evaluating $A^{(n+1)}[\tau_n, \tau_m; \tau_i]$ and $B^{(n+1)}[\tau_n, \tau_m; \tau_i]$ using Eq. (85), we will at most need to set $l = n$ in Eq. (86). Finally, comparing Eq. (85) with Eq. (31), we see that $A^{(n+1)}[\tau_n, \tau_m; \tau_i]$ and $B^{(n+1)}[\tau_n, \tau_m; \tau_i]$ have the desired form. Thus, we have proven our inductive hypothesis.

Setting the sources to zero, and Fourier transforming Eq. (82), we may write ($\lambda \to 1, \gamma[\tau_i] \to \frac{n_d}{2} \equiv \frac{n}{2}$)

$$\mathbf{g}^{-1}(i\omega_k) = i\omega_k - \epsilon_d - \left(1 - \frac{n}{2}\right) \Delta(i\omega_k) - \chi(i\omega_k) - \Delta(i\omega_k)\Psi(i\omega_k),$$
$$\mu(i\omega_k) = 1 - \frac{n}{2} + \Psi(i\omega_k). \tag{87}$$

Comparing with Eq. (35), it immediately follows that under the mapping Eq. (83), $\mu_{i,i}(i\omega_k) \to \mu(i\omega_k)$. Furthermore, multiplying both sides of the equation for $\mathbf{g}^{-1}(k)$ by $\mathbf{g}(k)$, summing over $\vec{k}$, and using the mapping Eq. (83), it follows that $\mathbf{g}_{i,i}(i\omega_k) \to \mathbf{g}(i\omega_k)$. Therefore, the ECFL solution of the infinite dimensional t–J model maps onto the ECFL solution of the AIM to each order in λ as long as the following self-consistency condition is satisfied:

$$\sum_{\vec{k}} \epsilon_{\vec{k}} \mathbf{g}(k) = \sum_{\vec{k}} \frac{|V_{\vec{k}}|^2}{i\omega_n - \widetilde{\epsilon}_{\vec{k}}} \mathbf{g}(i\omega_k). \tag{88}$$

This mapping and self-consistency condition can be understood by referring back to DMFT. In DMFT [9], the physical Green's function $\mathcal{G}_{i,f}(i\omega_k)$ is determined for any separation of i and f by the local Green's function $G_{i,i}(i\omega_k)$ or equivalently the local self energy $\Sigma(i\omega_k)$. The impurity Green's function of the Anderson model $\mathcal{G}(i\omega_k)$ can be set equal to $G_{i,i}(i\omega_k)$ as long as $\widetilde{\varepsilon_k}$ and V_k satisfy a self-consistency condition relating them to $\mathcal{G}(i\omega_k)$ (see Eqs. (13) and (15) of Ref. [9]). In the ECFL mapping, the auxiliary Green's function $\mathbf{g}_{i,f}(i\omega_k)$ is determined for any separation of i and f by the local auxiliary Green's function $\mathbf{g}_{i,i}(i\omega_k)$ and by the local caparison factor $\mu_{i,i}(i\omega_k)$, or equivalently by the two local self energies $\Psi(i\omega_k)$ and $\chi(i\omega_k)$. $\mu_{i,f}(i\omega_k)$ is itself local and related simply to $\Psi(i\omega_k)$. The impurity auxiliary Green's function of the Anderson model $\mathbf{g}(i\omega_k)$ can be set equal to $\mathbf{g}_{i,i}(i\omega_k)$ and the caparison factor of the Anderson model $\mu(i\omega_k)$ set equal to $\mu_{i,i}(i\omega_k)$ as long as $\widetilde{\varepsilon_k}$ and V_k satisfy the self-consistency condition (Eq. (88)). We now show that Eq. (88) can be put into the form of Eqs. (13) and (15) of Ref. [9]. Using Eq. (35) the LHS can be written as

$$\sum_{\vec{k}} \epsilon_{\vec{k}} \mathbf{g}(k) = \frac{-1}{1 - \frac{n}{2} + \Psi(i\omega_k)} [1 - (i\omega_k + \boldsymbol{\mu} - \chi(i\omega_k))\mathbf{g}(i\omega_k)]. \tag{89}$$

Using Eqs. (2), (5), (35), (38) and the relation $\mathcal{G}(i\omega_k) = \mathbf{g}(i\omega_k) \cdot \mu(i\omega_k)$, the above equation becomes

$$\Sigma_D(i\omega_k) + \frac{1}{\mathcal{G}(i\omega_k)} - (i\omega_k + \boldsymbol{\mu}) = -\sum_{\vec{k}} \epsilon_{\vec{k}} \mathbf{g}(k) \frac{1}{\mathbf{g}(i\omega_k)}. \tag{90}$$

Substituting Eq. (88) into the RHS of the above equation, we recover Eqs. (13) and (15) from Ref. [9].

5. Conclusion

In this work we provide a detailed analysis of the simplifications arising from the large dimensionality limit of the t–J model, and have given the first few terms in the λ series that leads to practically usable results. It is clear that the formal result of a local Dysonian self energy is already implied by the large d results for the Hubbard model reviewed in Ref. [9], if we take the limit of infinite U; that is indeed another description of the model studied here. However it must be kept in mind that the present calculation starts with the infinite U limit already taken, and thus provides a non trivial check on the uniqueness of the limit of $U \to \infty$ and $d \to \infty$, i.e. its independence on the order of these two limits. Also the present work uses the novel ECFL methodology that rests on a different set of tools from the ones usually used to study the Hubbard model and its large dimensional limit. We use the Schwinger equations of motion, as opposed to the usual Feynman–Wick theory, and we have obtained analytical results that do not rely on Wick's theorem.

Summarizing, we have considered the ECFL theory for the t–J model ($J = 0$) by establishing the simplifications that arise in the equations of motion in the limit of large dimensions. The auxiliary Green's function $\mathbf{g}(k)$ and the caparison factor $\mu(k)$ can be written in terms of two local self energies $\Psi(i\omega_k)$ and $\chi(i\omega_k)$ as in Eq. (35). This insight into the structural form of the physical Green's function $\mathcal{G}(k)$ has been used in a concurrent publication (Ref. [23]), to benchmark and compare the ECFL and DMFT calculations. The ECFL integral equations in the large d limit, derived here to $O(\lambda^2)$, have been solved numerically in Ref. [23], and their solution compares favorably with DMFT results. It can be seen explicitly from these equations that Eq. (35) holds to second order in λ, with $\Psi(i\omega_k)$ and $\chi(i\omega_k)$ written as a product of the functions $\mathbf{g}_{\text{loc},m}(i\omega_k)$ (Eq. (54)) with $m = 0$ or $m = 1$. This continues to hold to each order in λ. We have analyzed the optical conductivity and have shown that it is given by Eq. (50) in general and to each order in λ. We have separately also studied the ECFL theory of the infinite-U AIM[22], and have shown that there is a mapping between the ECFL of the infinite dimensional t–J model and the ECFL of the AIM with a self-consistently determined set of parameters (Eq. (88)). This mapping holds to each order in λ and there is a simple prescription for obtaining the ECFL integral equations for one model from those of the other (Eq. (83)).

In conclusion this work provides a solid foundation for the study of the t–J model, and in particular for the ECFL formalism, in the limit of infinite dimensions, by providing exact statements about the k dependence of self energies, the absence of vertex corrections in computing the conductivity and finally in yielding a systematic expansion in the parameter λ that enables a quantitative comparison with other methods as in Ref. [23].

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