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Generalized Toeplitz–Hankel matrices and their application to a layered electron gas

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Abstract

We extend the standard result for the eigenspectrum of the Toeplitz matrix $C_{ij} = \mathrm{e}^{-\kappa|i-j|}$ with $0 \leqslant i, j \leqslant N$ and $\kappa \in \mathbb{C}$ to a combination of a Toeplitz matrix and a Hankel matrix. We apply this result to find the plasma modes of a layered assembly of a two-dimensional electron gas. We find a sum rule relating the geometric mean of the frequencies of the plasma modes to the determinant of this Toeplitz matrix, for which an analytical expression is obtained. We apply the same technique to the generalized case when the layers are not evenly spaced, where the corresponding matrix is not a Toeplitz–Hankel combination. Despite this fact, it is possible to find properties of the eigenspectrum, and the eigenmodes are localized to a few layers instead of extending across the system.

Keywords: Toeplitz, 2d electron gas, plasma modes

(Some figures may appear in colour only in the online journal)

1. Introduction

In the course of our study of layered electronic systems initiated in reference [1], we came across an interesting Toeplitz matrix

$$C_{ij} = \mathrm{e}^{-\kappa|i-j|}, \quad 0 \leqslant i, j \leqslant N. \tag{1}$$

Eigenfunctions and eigenvalues of Toeplitz matrices are usually found by the Wiener–Hopf technique [2, 3]. The special case of equation (1) is called a Kac–Murdock–Szegö matrix [4], which can be solved [5–8] by noting that its inverse is a simple tridiagonal matrix, whose eigenfunctions and eigenvalues can easily be found. The method of finding the eigenfunctions and eigenvalues of the matrix C_{ij} by constructing its inverse can be generalized to a matrix that

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is the combination of a Hankel matrix and a Toeplitz matrix:

$$M_{ij} = a\mathrm{e}^{-\kappa|i-j|} + b\mathrm{e}^{\kappa|i-j|} + c\left[\mathrm{e}^{-\kappa(i+j)} + \mathrm{e}^{-\kappa(2\ N-i-j)}\right] \tag{2}$$

with arbitrary $a, b, c \in \mathbb{C}$ with $a \neq b$. It can also be generalized to the case when the matrix no longer has the Toeplitz form, and is

$$D = \begin{pmatrix} 1 & \mathrm{e}^{-\kappa_{01}} & \mathrm{e}^{-(\kappa_{01}+\kappa_{12})} & \mathrm{e}^{-(\kappa_{01}+\kappa_{12}+\kappa_{23})} & \dots \\ \mathrm{e}^{-\kappa_{01}} & 1 & \mathrm{e}^{-\kappa_{12}} & \mathrm{e}^{-(\kappa_{12}+\kappa_{23})} & \dots \\ \mathrm{e}^{-(\kappa_{01}+\kappa_{12})} & \mathrm{e}^{-\kappa_{12}} & 1 & \mathrm{e}^{-\kappa_{23}} & \dots \\ \mathrm{e}^{-(\kappa_{01}+\kappa_{12}+\kappa_{23})} & \mathrm{e}^{-(\kappa_{12}+\kappa_{23})} & \mathrm{e}^{-\kappa_{23}} & 1 & \dots \\ \vdots & & \vdots & & \ddots \end{pmatrix} \tag{3}$$

which, as we shall discuss in this paper, is relevant for disordered systems. (Here $\kappa_{i,i+1}$ for $i = 0, 1, \dots, N-1$ are complex random variables.) These generalizations are the subject of this paper.

The two generalizations given above are relevant for solving the plasma modes of a layered assembly of two-dimensional electron gas, when the layers are equally spaced and randomly spaced respectively. For the ordered case, the plasma mode frequencies have been found numerically by a method different from ours [9–11], and have in fact been studied experimentally using Raman scattering [12]. Applying our method to this problem yields a simple expression for the eigenfunctions associated with these plasma modes, as well as a sum rule relating the frequencies of the $(N+1)$ branches of the plasma frequency as functions of the parallel component of the photon wave vector. The density of states of eigenvalues is also of interest experimentally [14] and evaluated analytically here.

In the rest of this paper, we first summarize the calculation of the inverse of C and its eigenspectrum. We then generalize the method to calculate the eigenspectrum of the matrix M in equation (2) and the matrix D in equation (3). We apply the results to the layered electron gas.

2. Properties of C_{ij}

2.1. Eigenvalues and eigenvectors

In this subsection, we review previous results [5–8] for the spectra of Toeplitz matrices. It is easy to verify that the tridiagonal matrix A with

$$A_{ii} = \coth\ \kappa, \qquad A_{i,i+1} = A_{i,i-1} = -\frac{1}{2\ \sinh\ \kappa} \tag{4}$$

for $N > i > 0$,

$$A_{00} = A_{NN} = \frac{\mathrm{e}^{\kappa}}{2\ \sinh\ \kappa}, \qquad A_{01} = A_{N-1,N} = -\frac{1}{2\ \sinh\ \kappa}, \tag{5}$$

then $A = C^{-1}$.

It is easier to diagonalize C^{-1} than to diagonalize C. For $0 \leqslant j \leqslant N$ we denote the basis column vector $\hat{e}_j$ (with 1 at the jth row and 0 elsewhere) as $|j\rangle$, and write an operator $\hat{C}$ such that $\hat{C}|j\rangle = \sum_{l=0}^{N} C_{lj}|l\rangle$. To find the eigenvectors of $\hat{C}$ and $\hat{C}^{-1}$, we try states of the form

$$|\Psi(q)\rangle = \sum_{j=0}^{N} \cos(qj - \Phi(q))|j\rangle \tag{6}$$

and seek to satisfy the condition

$$\hat{C}^{-1}|\Psi(q)\rangle = \Lambda^{-1}|\Psi(q)\rangle \tag{7}$$

which implies $\hat{C}|\Psi(q)\rangle = \Lambda|\Psi(q)\rangle$. Here q and $\Phi(q)$ as well as the eigenvalue Λ are to be determined. The interior terms $1 \leqslant j \leqslant N-1$ are satisfied by this wavefunction provided

$$\Lambda^{-1} = \coth\,\kappa - \frac{\cos\,q}{\sinh\,\kappa}. \tag{8}$$

The amplitude at $j = 0$ requires the condition

$$\left(\Lambda^{-1} - \frac{\mathrm{e}^{\kappa}}{2\,\sinh\,\kappa}\right)\cos\,\Phi = -\frac{1}{2\,\sinh\,\kappa}\,\cos(q-\Phi). \tag{9}$$

Simplifying further, we find the phase shift determined by

$$\Phi(q) = \mathrm{arccot}\left\{\frac{\sin\,q}{\cos\,q - \mathrm{e}^{-\kappa}}\right\}. \tag{10}$$

The phase shift $\Phi(q)$ varies continuously with q in the interval $0 \leqslant q \leqslant \pi$, decreasing monotonically from $\pi/2$ to $-\pi/2$. It is thus a convenient parameterization for finding all the eigenvalues. The amplitude at $j = N$ is satisfied if

$$\left(\Lambda^{-1} - \frac{\mathrm{e}^{\kappa}}{2\,\sinh\,\kappa}\right)\cos(qN-\Phi(q)) = -\frac{1}{2\,\sinh\,\kappa}\,\cos(q(N-1)-\Phi(q)). \tag{11}$$

Simplifying further,

$$\sin(qN - 2\Phi(q)) = 0. \tag{12}$$

Alternatively, we can observe that the eigenfunctions must be odd or even functions of the index j measured from the midpoint of $j = N/2$ (this is true even if N is odd), so that either $\sin(qN/2-\Phi(q))$ or $\cos(qN/2-\Phi(q))$ is zero for each eigenfunction. The product of the two expressions, and therefore $\sin(qN-2\Phi(q))$, must be zero for every eigenfunction.

It is straightforward to verify that the N values $\nu = 0, 1, \ldots, N$ yield the $N+1$ distinct eigenvalues

$$\Lambda(q_\nu, \kappa) = \frac{\sinh\,\kappa}{\cosh\,\kappa - \cos\,q_\nu} \tag{13}$$

with

$$q_\nu N = \nu\pi + 2\Phi(q_\nu). \tag{14}$$

We will usually denote $\Lambda(q_\nu, \kappa)$ as $\Lambda(q_\nu)$. At finite N the values $q = 0$ and $q = \pi$ are excluded since for these the wavefunction $|\Psi(q)\rangle$ vanishes identically, formally these correspond to $\nu = -1$ and $\nu = N+1$ respectively. Also we note that in the limit $\kappa \to +\infty$, the phase shift $\Phi(q) = \pi/2 - q$ and hence $q_\nu = \frac{\nu+1}{N+2}\pi$.

2.2. Density of states

For large N it is useful to employ the density of states of the exact eigenvalues, these can be found straightforwardly. We note the identity

$$\frac{\mathrm{d}\Phi(q)}{\mathrm{d}q} = -\frac{1}{2}(1+\Lambda(q)), \tag{15}$$

so that we can write the difference in successive solutions from equation (14) in the form

$$\pi\Delta\nu = N\Delta q_\nu - 2\Delta\Phi(q_\nu) = \Delta q_\nu(N+1+\Lambda(q_\nu)). \tag{16}$$

So that any function of the eigenvalues that is summed over all eigenvalues can be converted to an integral in the large N limit:

$$\sum_{\nu=0}^{N} f(\Lambda_\nu) \to \int_{q_0}^{q_N} \mathrm{d}q \frac{\Delta\nu}{\Delta q} f(\Lambda(q,\kappa)) = \int_{q_0}^{q_N} \frac{\mathrm{d}q}{\pi} f(\Lambda(q,\kappa))\left\{N+1+\Lambda(q)\right\}. \tag{17}$$

From equation (14),

$$\frac{\mathrm{d}q}{\mathrm{d}\Lambda} = -\frac{\sinh\,\kappa}{\Lambda^2\left[1-\left(\cosh\,\kappa-\frac{\sinh\,\kappa}{\Lambda}\right)^2\right]^{1/2}} \tag{18}$$

and hence we can convert a sum over solutions to an integral over eigenvalues with a density of states

$$\frac{1}{N}\sum_{\nu} f(\Lambda_\nu) \to \frac{1}{\pi N}\int_{\Lambda_<}^{\Lambda_>} \frac{\mathrm{d}\Lambda}{\Lambda^2} \frac{\{N+1+\Lambda\}\sinh\,\kappa}{\left[1-\left(\cosh\,\kappa-\frac{\sinh\,\kappa}{\Lambda}\right)^2\right]^{1/2}} f(\Lambda) + \mathcal{O}\left(\frac{1}{N^2}\right), \tag{19}$$

where

$$\Lambda_< = \frac{\sinh\,\kappa}{\cosh\,\kappa+1}, \qquad \Lambda_> = \frac{\sinh\,\kappa}{\cosh\,\kappa-1}. \tag{20}$$

Here we assume that $f(\Lambda)$ is a function that is well-defined in the large-N limit. Thus the integral in equation (19) gives the leading term as well as the first correction term at large N with further corrections relegated to $\mathcal{O}(\frac{1}{N^2})$. We checked that using this density of states to compute the sum $\sum_\nu \log\Lambda(q_\nu,\kappa)$, yields the log-determinant of C correct to the terms quoted in equation (27).

2.3. Szegö's theorem for the determinant of C_{ij}

The determinant of the matrix C is well known, and is given here because it is relevant to the experiments discussed later in this paper. It is elementary to calculate the determinant exactly by using Gauss's method of triangulation, leading to

$$\det(C) = (1-\mathrm{e}^{-2\kappa})^N. \tag{21}$$

An alternative approach exploits the tridiagonal nature of C^{-1}. If one defines A_j to be the $j \times j$ submatrix of $(2\sinh\kappa)C^{-1}$ that ends at the bottom right corner of C^{-1}, it is easy to verify that $\det(A_{j+1}) = V_j \det(A_j) - \det(A_{j-1})$ for $j = 1, 2 \ldots, N$ with $V_j = \mathrm{e}^{\kappa} - \delta_{j,N}\mathrm{e}^{-\kappa}$. The boundary condition is $\det(A_0) = 1$ and $\det(A_1) = \mathrm{e}^{\kappa}$. The solution to the recurrence relation is $\det(A_j) = \mathrm{e}^{j\kappa}$ for $0 \leqslant j \leqslant N$, and so $\det(A_{N+1}) = \mathrm{e}^{(N+1)\kappa}(1 - \mathrm{e}^{-2\kappa})$. Therefore $\det(C) = (2\ \sinh\ \kappa)^{N+1}/[\mathrm{e}^{(N+1)\kappa}(1 - \mathrm{e}^{-2\kappa})] = (1 - \mathrm{e}^{-2\kappa})^N$.

We can also calculate the determinant from the strong theorem of Szegö [15], assuming for this step a real and positive κ. This theorem is guaranteed to give the two leading terms in the limit of large N. Specifically the theorem says that when the $(N + 1) \times (N + 1)$ Toeplitz matrix C is generated by a density $\varphi(\mathrm{e}^{\mathrm{i}\theta})$ through a Fourier series, i.e.

$$C_{ij} = \int_{-\pi}^{\pi} \frac{\mathrm{d}\theta}{2\pi} \mathrm{e}^{-\mathrm{i}\theta(i-j)} \varphi(\mathrm{e}^{\mathrm{i}\theta}) \tag{22}$$

and further if

$$\log\ \varphi(\mathrm{e}^{\mathrm{i}\theta}) = \sum_{l=-\infty}^{\infty} \mathrm{e}^{\mathrm{i}l\theta} \nu_l \tag{23}$$

then the determinant for large N is given by

$$\log\ \det(C) = \left\{ (N+1)\nu_0 + \sum_{l=1}^{\infty} l|\nu_l|^2 + o(N) \right\}. \tag{24}$$

In the present case of equation (1) it is readily seen that

$$\varphi(\mathrm{e}^{\mathrm{i}\theta}) = \frac{\sinh\ \kappa}{\cosh\ \kappa - \cos(\theta)}, \tag{25}$$

and

$$\nu_l = \delta_{l,o}\,(\log\ 2\ \sinh\ \kappa - \kappa) + (1 - \delta_{l,o}) \frac{\mathrm{e}^{-\kappa|l|}}{|l|}. \tag{26}$$

Substituting into equation (24) and carrying out the summation over l we find

$$\begin{aligned} \log\ \det(C) &= (N+1)\left[\log(1 - \mathrm{e}^{-2\kappa})\right] - \left[\log(1 - \mathrm{e}^{-2\kappa})\right] + o(N) \\ &= N\left[\log(1 - \mathrm{e}^{-2\kappa})\right] + o(N). \end{aligned} \tag{27}$$

Comparing with equation (21) we see that the above expression is *exact if we drop the* $o(N)$ *correction terms altogether*. The rather unexpected vanishing of the $o(N)$ correction term, as explained to us by Prof. Ehrhardt, is the consequence of the following general result [16]. With $z = \mathrm{e}^{\mathrm{i}\theta}$, the density $\varphi(z)$ in equation (22) clearly has an infinite Laurent series expansion in z. However in special cases such as the one considered here, the *inverse* of the density i.e. $\varphi^{-1}(z)$ the Laurent series is truncated to a finite number of terms. In such cases the coefficients of terms z^k vanish, both for $k > m$ and for $k < -m$. Under these conditions, $\det(C_k)$ the determinant in k dimensions (i.e. with $N + 1 = k$) satisfies the following condition: for $k \geqslant m$ we have a purely exponential growth $\det(C_{k+1})/\det(C_k) = G$, where G is independent of k. Therefore $\det(C_M) = G^{M-m} \det(C_m)$. For our matrix equation (25), $\varphi(z) = \coth\ \kappa - \frac{1}{2}\,\mathrm{csch}\,\kappa(z + z^{-1})$, and hence $m = 1$, and therefore $\det(C_1), \det(C_2), \det(C_3) \ldots$ form a geometrical series. We note that, while Szegö's theorem is limited to real and positive κ, our results apply to general complex κ.

3. Generalization to combined Toeplitz Hankel matrices

Toeplitz matrices are closely related to Hankel matrices: the elements H_{ij} of a Hankel matrix H only depend on $i + j$. It is clear that any Hankel matrix is related to some Toeplitz matrix through reflection about the midpoint: $i \to N - i$ or $j \to N - j$. In particular, the matrix

$$H_{ij} = \mathrm{e}^{-\kappa|i+j-N|}, \quad 0 \leqslant i, j \leqslant N \tag{28}$$

is a reflection of the Toeplitz matrix C_{ij} which we have analyzed. Since $\hat{H} = \hat{R}\hat{C}$, where $\hat{R}$ is the reflection operator, any eigenvector of $\hat{C}$ satisfies $\hat{H}|\Psi\rangle = \hat{R}\hat{C}|\Psi\rangle = \Lambda\hat{R}|\Psi\rangle$. Since, as we have remarked earlier, the eigenvectors of $\hat{C}$ are even or odd under reflection about the midpoint, $\hat{H}|\Psi\rangle = (-1)^P\Lambda|\Psi\rangle$, where P is the parity of the eigenvector.

A related Hankel matrix, $H'_{ij} = \exp[-\kappa(i + j)]$, for which there is no cusp on the diagonal, is even simpler to solve. Define the vector $|\Theta\rangle$ by $\langle j|\Theta\rangle = \exp[-\kappa j]$. Then $\hat{H}' = |\Theta\rangle\langle\Theta|$. A vector $|\Psi\rangle$ is a null vector of $\hat{H}'$ if $|\Theta\rangle\langle\Theta|\Psi\rangle = 0$, i.e. $\langle\Theta|\Psi\rangle = 0$. Thus the null space of $\hat{H}'$ is N-dimensional. The $N + 1$'th eigenvector of $\hat{H}'$ is the vector that is orthogonal to this null space, $|\Theta\rangle$. The eigenvalue is obtained from the equation $\hat{H}'|\Theta\rangle = |\Theta\rangle\langle\Theta|\Theta\rangle$, i.e. the eigenvalue is $\langle\Theta|\Theta\rangle = \sum_{j=0}^{N} \exp[-2\kappa j]$.

We now consider the problem of finding the eigenvalues of a combination of Hankel and Toeplitz matrices:

$$\begin{aligned} M_{ij} = {} & a\, \exp\left[-\kappa|i-j|\right] + b\, \exp\left[\kappa|i-j|\right] \\ & + c\left\{\exp\left[-\kappa(i+j)\right] + \exp\left[-\kappa(2N-i-j)\right]\right\} \end{aligned} \tag{29}$$

with the restriction $a \neq b$. This can be expressed as $M = aC + b\tilde{C} + cH$, where C and $\tilde{C}$ are Toeplitz matrices and H is a Hankel matrix. Considering the three parts of M separately, we have $C^{-1}C = I$, and

$$C^{-1}\tilde{C} = -I + \begin{pmatrix} 1 & \mathrm{e}^{\kappa} & \mathrm{e}^{2\kappa}\ldots \\ 0 & 0 & 0\ldots \\ \vdots & & \ddots \\ \mathrm{e}^{N\kappa} & \mathrm{e}^{(N-1)\kappa} & \ldots \end{pmatrix} \tag{30}$$

while

$$C^{-1}H = \begin{pmatrix} 1 & \mathrm{e}^{-\kappa} & \mathrm{e}^{-2\kappa}\ldots \\ 0 & 0 & 0\ldots \\ \vdots & & \ddots \\ \mathrm{e}^{-N\kappa} & \mathrm{e}^{-(N-1)\kappa} & \ldots \end{pmatrix}. \tag{31}$$

Putting all these together, with

$$T = \frac{1}{a-b}C^{-1} = \frac{1}{2(a-b)\sinh\,\kappa}\begin{pmatrix} \mathrm{e}^{\kappa} & -1 & 0\ldots \\ -1 & 2\,\cosh\,\kappa & -1\ldots \\ \vdots & & \ddots \end{pmatrix} \tag{32}$$

we obtain

$$TM = I + \begin{pmatrix} \alpha_0 & \alpha_1 & \dots \\ 0 & 0 & \dots \\ \vdots & \vdots & \ddots \\ \alpha_N & \alpha_{N-1} & \dots \end{pmatrix} \tag{33}$$

with the elements of additional matrix on the right-hand side equal to

$$\alpha_i = \frac{1}{a-b}(b\ \exp[\kappa \mathrm{i}] + c\ \exp[-\kappa \mathrm{i}]). \tag{34}$$

From equation (33),

$$\begin{aligned} \det(M) &= [(1+\alpha_0)^2 - \alpha_N^2]/\det(T) \\ &= (a-b)^{N-1}(1-\mathrm{e}^{-2\kappa})^N[(a+c)^2 - (c\mathrm{e}^{-N\kappa} + b\mathrm{e}^{N\kappa})^2]. \end{aligned} \tag{35}$$

We assume that this is non-zero, so that the matrix M is invertible. The inverse of M is then

$$M^{-1} = T + \begin{pmatrix} x_0 & x_1 & \dots \\ 0 & 0 & \dots \\ \vdots & \vdots & \ddots \\ x_N & x_{N-1} & \dots \end{pmatrix}, \tag{36}$$

where

$$TM + \begin{pmatrix} x_0 & x_1 & \dots \\ 0 & 0 & \dots \\ \vdots & \vdots & \ddots \\ x_N & x_{N-1} & \dots \end{pmatrix} M = I + \begin{pmatrix} \alpha_0 & \alpha_1 & \dots \\ 0 & 0 & \dots \\ \vdots & \vdots & \ddots \\ \alpha_N & \alpha_{N-1} & \dots \end{pmatrix} + \begin{pmatrix} x_0 & x_1 & \dots \\ 0 & 0 & \dots \\ \vdots & \vdots & \ddots \\ x_N & x_{N-1} & \dots \end{pmatrix} M = I. \tag{37}$$

Since $M^{\mathrm{T}} = M$, the condition to be satisfied by the x_i's is

$$\begin{pmatrix} x_0 \\ x_1 \\ \vdots \\ x_N \end{pmatrix} = -M^{-1} \begin{pmatrix} \alpha_0 \\ \alpha_1 \\ \vdots \\ \alpha_N \end{pmatrix} = -T \begin{pmatrix} \alpha_0 \\ \alpha_1 \\ \vdots \\ \alpha_N \end{pmatrix} - \begin{pmatrix} \sum \alpha_i x_i \\ 0 \\ \vdots \\ \sum \alpha_{N-i} x_i \end{pmatrix}. \tag{38}$$

Substituting equations (32) and (34) in the first term on the right-hand side, all the elements of $T \cdot \alpha$ except the first and last ones are zero. Therefore, $x_1, x_2, \dots, x_{N-1} = 0$ and we are left with the coupled equations

$$\begin{aligned} \begin{pmatrix} x_0 \\ x_N \end{pmatrix} &= -\begin{pmatrix} T_{00}\alpha_0 + T_{01}\alpha_1 \\ T_{N,N-1}\alpha_{N-1} + T_{NN}\alpha_N \end{pmatrix} - \begin{pmatrix} x_0 & x_N \\ x_N & x_0 \end{pmatrix} \begin{pmatrix} \alpha_0 \\ \alpha_N \end{pmatrix} \\ &= -\frac{1}{(a-b)^2} \begin{pmatrix} c \\ b\mathrm{e}^{N\kappa} \end{pmatrix} - \frac{1}{a-b} \begin{pmatrix} x_0 & x_N \\ x_N & x_0 \end{pmatrix} \begin{pmatrix} b+c \\ b\mathrm{e}^{N\kappa} + c\mathrm{e}^{-N\kappa} \end{pmatrix} \end{aligned} \tag{39}$$

which has the solution

$$x_0 = \frac{b^2 e^{2\kappa N} - ac + bc - c^2}{(a-b)[(a+c)^2 - (ce^{-N\kappa} + be^{N\kappa})^2]}$$
$$x_N = \frac{c^2 e^{-N\kappa} - abe^{N\kappa}}{(a-b)[(a+c)^2 - (ce^{-N\kappa} + be^{N\kappa})^2]}. \tag{40}$$

From equation (36), we see that M^{-1} is a tridiagonal matrix with an extra element on the top right and bottom left. As before with the Toeplitz matrix C, it is easy to see that the eigenvectors can be written with elements $\psi_q(j) = \cos[q(j - N/2)]$ or $\psi_q(j) = \sin[q(j - N/2)]$ with the eigenvalues related to q through

$$\Lambda^{-1} = \frac{1}{a-b}\left[\coth\,\kappa - \frac{\cos\,q}{\sinh\,\kappa}\right]. \tag{41}$$

The boundary conditions for the even and odd eigenvectors

$$(x_0 + x_N)\cos\,qN/2 = \frac{1}{2(a-b)\sinh\,\kappa}[e^{-\kappa}\cos\,qN/2 - \cos\,q(N/2+1)]$$
$$(x_0 - x_N)\sin\,qN/2 = \frac{1}{2(a-b)\sinh\,\kappa}[e^{-\kappa}\sin\,qN/2 - \sin\,q(N/2+1)] \tag{42}$$

respectively determine the allowed values of q.

When $c = 0$, the eigenvalues and eigenvectors obtained in this section are given in reference [7]. The $N \to \infty$ limit discussed in that paper assumes that $a - b = 1$, in which case there are two eigenvalues that tend to $\pm\infty$, leading to the divergence in $\det(M)$. As seen in the next section, the physically appropriate $N \to \infty$ limit has $b/a \sim \exp[-2N\kappa]$ and c/a being independent of N. With this choice, for $N \to \infty$, $x_N = 0$ and the boundary conditions for even and odd eigenvectors reduce to

$$-\frac{c}{a+c}\cos\,qN/2 = \frac{1}{e^{\kappa} - e^{-\kappa}}[e^{-\kappa}\cos\,qN/2 - \cos\,q(N/2+1)] = 0$$
$$-\frac{c}{a+c}\sin\,qN/2 = \frac{1}{e^{\kappa} - e^{-\kappa}}[e^{-\kappa}\sin\,qN/2 - \sin\,q(N/2+1)] = 0 \tag{43}$$

respectively.

Unlike the case for a single Toeplitz matrix, discussed in section 2, all the eigenvectors of the Toeplitz–Hankel matrix need not be oscillatory. We can seek eigenvectors of the form

$$|\Psi_{\pm}(z)\rangle = \sum_j \left(z^{j-N/2} \pm z^{N/2-j}\right)|j\rangle, \tag{44}$$

where the $\pm$ characterizes eigenvectors that are even or odd about the midpoint of the system. Then $\cos q$ is replaced by $(z + 1/z)/2$ in equation (41), and since the eigenvalues must be real, z either has unit magnitude or is real. The second case corresponds to solutions that have components that grow or decay exponentially as j is varied. We do not have a general expression

for the number of such solutions in terms of a, b, c, κ and N. However, if the $N \to \infty$ limit is taken as discussed above, equation (43) reduces to

$$\frac{c}{a+c}\mathrm{e}^{\kappa} + \frac{a}{a+c}\mathrm{e}^{-\kappa} = \lim_{N\to\infty} \frac{z^{N/2+1} \pm z^{-N/2-1}}{z^{N/2} \pm z^{-N/2}}. \tag{45}$$

If z is real, choosing $|z| > 1$ without loss of generality, the right-hand side reduces to z. Thus for $N \to \infty$, if $(c\mathrm{e}^{\kappa} + a\mathrm{e}^{-\kappa})/(a+c) > 1$, there is exactly one odd and one even exponentially varying eigenvector, with the same z and therefore the same Λ^{-1} for both. Equivalently, there is exactly one eigenvector that decays exponentially as one moves away from $j = 0$ boundary, and similarly for the $j = N$ boundary. All other solutions are of the form $\psi_q(j) = \cos[q(j - N/2)]$ or $\psi_q(j) = \sin[q(j - N/2)]$. If $(c\mathrm{e}^{\kappa} + a\mathrm{e}^{-\kappa})/(a+c) < 1$, all the eigenmodes are oscillatory for $N \to \infty$.

4. 2D plasmon spectrum

As mentioned in the introduction, the Toeplitz matrix C_{ij} arises in the context of plasmons in multilayer systems, a system that has been studied extensively earlier. The original systems studied in the work of Olego, Pinczuk, Gossard and Wiegmann [12, 13] consists of alternating layers of insulating GaAs and conducting $(\mathrm{Al}_x\mathrm{Ga}_{1-x})\mathrm{As}$. Here the conducting planes are coupled by the Coulomb interaction only, i.e. one ignores the direct hopping of electrons between layers [1]. Recent advances in materials allows a vast range of composite materials, generalizing this initial system [17–20]. To understand plasmons in these systems, one needs to understand the dielectric function of layered systems [9–11], where the plasmon is a pole of a charge response function, probed by either a charged particle surface scattering, or as in the case of [9, 12, 13] by photons using Raman scattering. Within the widely used random phase approximation for these systems, the plasmon is found as the eigen-solution of a homogeneous Fredholm equation [9] satisfied by $\delta\rho(l)$, the induced charge density on layer l due to a small excess external charge:

$$\delta\rho(l) = D_0(k_\parallel, \omega)V(k_\parallel)\sum_{m=0}^{N} \mathrm{e}^{-k_\parallel d|l-m|}\delta\rho(m), \tag{46}$$

where $k_\parallel$ is the magnitude of the component of the photon parallel to the 2d layer, d the separation between the $N+1$ layers, $V(k_\parallel) = \frac{2\pi\mathrm{e}^2}{k_\parallel \varepsilon_\mathrm{M}}$ and ε_M is the material dielectric constant. Here $D_0(k_\parallel, \omega)$ is the 'bubble' polarization in 2d; it is approximated well in terms of the 2d density n and effective mass m^* by

$$D_0 \sim \frac{nk_\parallel^2}{m^*\omega^2}.$$

When the dielectric constants in the different layers are different, one must also add image charges to equation (46) as explained in [9], who provide a complete numerical solution for all cases.

When image charges can be neglected, equation (46) corresponds to the Kac–Murdock–Szegö matrix reviewed in section 2. Comparing equation (46) with equation (1) we see that the plasmon frequencies for the $N+1$ layer problem are obtained from Λ_ν in

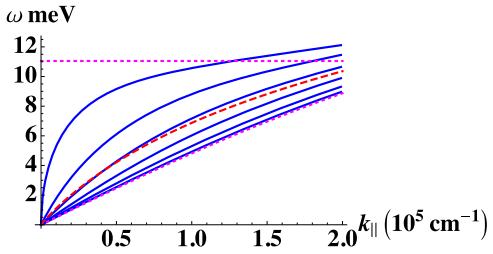

Figure 1. The six plasmon branches for a six layer system in blue solid curves, the geometric mean frequency from equation (49) in red dashed curve, and the 3d bulk and 2d bulk plasmon in magenta dotted curves. The parameters used are similar to those of sample 1 in [12], we used $d = 900A^0$, $n = 7.3 \times 10^{11}$ cm^{-2}, $m^* = 0.07m_e$, $\varepsilon = 13.1$.

equation (13)

$$\omega_\nu(k_\parallel) = \sqrt{\frac{2\pi n e^2}{\varepsilon_M m^*}}\sqrt{k_\parallel \Lambda(q_\nu, \kappa)} \tag{47}$$

by identifying

$$\kappa = k_\parallel d. \tag{48}$$

The allowed q_ν's are given by equation (14), and are not evenly spaced. The exact determination of the Toeplitz determinant implies that the geometric mean of the plasmon frequencies is

$$\begin{aligned}\langle\omega(k_\parallel)\rangle_{gm} &\equiv \left[\prod_{\nu=0}^{N}\omega_\nu(k_\parallel)\right]^{\frac{1}{N+1}} \\ &= \sqrt{\frac{2\pi n e^2}{\varepsilon_M m^*}}\sqrt{k_\parallel}\left[(1-e^{-2k_\parallel d})\right]^{\frac{N}{2(N+1)}}.\end{aligned} \tag{49}$$

Equivalently, this is a sum rule for the logarithms of the plasmon frequencies, $\log(\omega_\nu(k_\parallel))$. It is worth mentioning that the various plasma branches can be measured experimentally for a fixed $k_\parallel$, and hence this sum rule has the potential for experimental verification. In figure 1, we illustrate the plasmon solutions for the case of 6 layers using parameters close to those in [12], and also display the geometric mean.

It is useful to note that in general layered systems, the background dielectric function varies between layers, often described as a ϵ_0–ϵ–ϵ_0 configuration of the layers [9]. The pure Toeplitz spectrum is obtained when $\epsilon_0 = \epsilon$. In fact the experiment in [13] corresponds to such a case, with a vanishing dielectric contrast.

In the generic ϵ_0–ϵ–ϵ_0 configuration of the layers [9], the problem corresponds to the more complicated Toeplitz–Hankel combination discussed in the previous section, with (in the notation of reference [9], with $N \to N+1$)

$$\begin{aligned}c/a &= \frac{\epsilon-\epsilon_0}{\epsilon+\epsilon_0} \\ b/a &= \left(\frac{\epsilon-\epsilon_0}{\epsilon+\epsilon_0}\right)^2 \exp[-2\kappa N].\end{aligned} \tag{50}$$

(Here we have assumed that the width L of the region of the system with dielectric constant ϵ is equal to Nd, where d is the separation between the layers. The case when $L > Nd$ is a straightforward extension of the analysis given here, but with more complicated expressions.) Equation (49) is then modified to

$$\langle\omega(k_\parallel)\rangle_{gm} = \sqrt{\frac{2\pi n e^2}{\varepsilon_M m^*}}\sqrt{k_\parallel}\left[(1-e^{-2k_\parallel d})\right]^{\frac{N}{2(N+1)}} \times \left(\frac{(a+c)^2-(ce^{-N\kappa}+be^{N\kappa})^2}{(a-b)^2}\right)^{\frac{1}{2(N+1)}}. \tag{51}$$

In the $N\to\infty$ limit, $(b/a)\exp[N\kappa]\to 0$, and this reduces to $(1+c/a)^{1/(N+1)}$. The nature of the eigenvectors, and the possibility of zero or one eigenmodes that decay exponentially away from the boundary of the system for $N\to\infty$, was discussed at the end of section 3.

5. Disordered case

So far, we have assumed that the two-dimensional electron gas layers are equally spaced. With this assumption, we have found that all the eigenfunctions have an oscillatory behavior as one moves from one layer to the next, and extend across the entire system. However, it is not difficult to relax this assumption and construct a system where the spacings between the layers are irregular. In this case, the coupling between the i'th layer and j'th layer is still an exponentially decaying function of the distance between the i'th and j'th layers, but this distance is the sum of the random distances between the successive layers in the interval from i to j. Modifying equation (48) appropriately, we have to find the spectrum of the matrix (limiting ourselves to the ϵ–ϵ–ϵ case, where the background dielectric medium is the same in the layered region as everywhere else, and there are therefore no image charges)

$$D = \begin{pmatrix} 1 & e^{-\kappa_{01}} & e^{-(\kappa_{01}+\kappa_{12})} & e^{-(\kappa_{01}+\kappa_{12}+\kappa_{23})} & \dots \\ e^{-\kappa_{01}} & 1 & e^{-\kappa_{12}} & e^{-(\kappa_{12}+\kappa_{23})} & \dots \\ e^{-(\kappa_{01}+\kappa_{12})} & e^{-\kappa_{12}} & 1 & e^{-\kappa_{23}} & \dots \\ e^{-(\kappa_{01}+\kappa_{12}+\kappa_{23})} & e^{-(\kappa_{12}+\kappa_{23})} & e^{-\kappa_{23}} & 1 & \dots \\ \vdots & & \vdots & & \ddots \end{pmatrix}. \tag{52}$$

We define the tridiagonal matrix T with the elements

$$T_{i,i-1} = -\frac{1}{2\sinh\kappa_{i-1,i}}$$
$$T_{i,i} = \frac{1}{2}(\coth\kappa_{i-1,i}+\coth\kappa_{i,i+1})$$
$$T_{i,i+1} = -\frac{1}{2\sinh\kappa_{i,i+1}}. \tag{53}$$

for $0 < i < N$ and

$$T_{00} = \frac{e^{\kappa_{01}}}{2\sinh\kappa_{01}}$$
$$T_{01} = -\frac{1}{2\sinh\kappa_{01}}$$

$$T_{NN} = \frac{e^{\kappa_{N-1,N}}}{2 \sinh \kappa_{N-1,N}}$$
$$T_{N,N-1} = -\frac{1}{2 \sinh \kappa_{N-1,N}} \tag{54}$$

for the first and last rows. Then it is possible to verify that $TD = I$, i.e. T is the inverse of D. The crucial observation here is that, when the i'th row of any tridiagonal matrix T acts on the j'th column of D, all the columns for $j > i$ yield expressions that are multiples of each other, and all the columns for $j < i$ yield expressions that are multiples of each other. Therefore, it is sufficient to choose the elements of T in a manner such that $(TD)_{i-1,i} = (TD)_{i+1,i} = 0$ and $(TD)_{ii} = 1$, which is possible since there are three elements in each row of T. (This argument is slightly modified for the first and last rows.)

The eigenfunctions of T are the same as the eigenfunctions of D, and the eigenvalues of the two matrices are the reciprocals of each other. The matrix T is the Hamiltonian of a tight-binding model for a chain with disordered on-site potentials and nearest neighbor hoppings. The hopping elements are all random and uncorrelated with each other, but the on-site potentials can be obtained from the adjacent hopping elements.

One dimensional disordered chains generically have a spectrum that consists entirely of localized modes. Because the correlation in the randomness is short-range, one would not expect this result to be affected. To confirm this, we use the standard procedure of converting the system with the Hamiltonian T into an open system, with transfer matrix

$$\begin{pmatrix} \psi_{i+1} \\ \psi_i \end{pmatrix} = \mathcal{T}_i \begin{pmatrix} \psi_i \\ \psi_{i-1} \end{pmatrix} \tag{55}$$

with

$$\mathcal{T}_i = \begin{pmatrix} \sinh(\kappa_{i-1,i} + \kappa_{i,i+1}) / \sinh \kappa_{i-1,i} - 2E \sinh \kappa_{i,i+1} & -\sinh \kappa_{i,i+1} / \sinh \kappa_{i-1,i} \\ 1 & 0 \end{pmatrix} \tag{56}$$

at an energy E. We can rewrite the product of transfer matrices as

$$\ldots \mathcal{T}_{i+1} \mathcal{T}_i \mathcal{T}_{i-1} \ldots = \ldots (O^{-1}_{i+2,i+1} \mathcal{T}_{i+1} O_{i+1,i})(O^{-1}_{i+1,i} \mathcal{T}_i O_{i,i-1}) \\ (O^{-1}_{i,i-1} \mathcal{T}_{i-1} O_{i-1,i-2}) \ldots \tag{57}$$

where the $O_{j+1,j}$'s are multiples of the identity matrix, with diagonal elements $(\sinh \kappa_{j,j+1})^{1/2}$. Since

$$O^{-1}_{i+1,i} \mathcal{T}_i O_{i,i-1} = \begin{pmatrix} \sinh(\kappa_{i-1,i} + \kappa_{i,i+1})/\alpha_i - 2E\alpha_i & \mu_i \\ -1/\mu_i & 0 \end{pmatrix}, \tag{58}$$

where $\alpha_i = (\sinh \kappa_{i,i+1} \sinh \kappa_{i-1,i})^{1/2}$ and $\mu_i = -(\sinh \kappa_{i,i+1} / \sinh \kappa_{i-1,i})^{1/2}$, this is a product of matrices, each of which belongs to the group SL(2, R). The product of the two eigenvalues of each such matrix is unity, and the sum is equal to the top left element of the matrix. Since the $\kappa_{j,j+1}$'s are chosen randomly, in the matrix product in equation (57) we will generically find successive matrices of the form in equation (58) with unequal top left elements. This means that they have no common eigenvalues or eigenvectors. Invoking the results proved by Furstenberg and Ishii [21, 22], the matrix product has a diverging norm as the number of matrices is increased. Thus we conclude that *all states* of the matrix D are localized. This is somewhat unexpected since D has long distance coupling, where it is not usually possible to make such a strong statement.

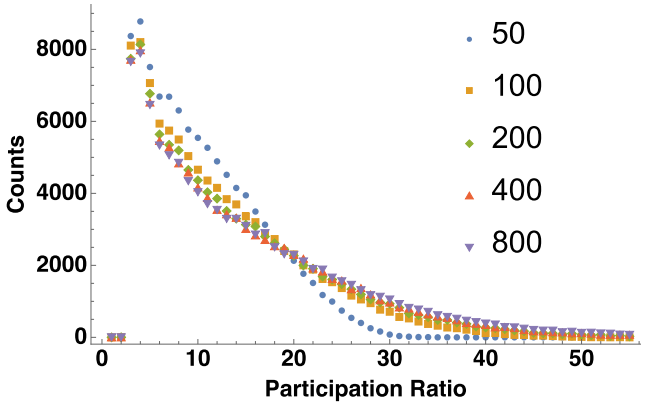

Figure 2. The histogram of the participation ratio $\sum_j |\psi_j|^4/(\sum_j |\psi_j|^2)^2$ for $N+1 = 50, 100, 200, 400$ and 800 is shown. For each N, the number of random systems that were randomly generated was sufficient to yield 10^5 eigenstates. The counts in each bin are shown; the bins each have width 1. The histograms approach a limiting form as N is increased, as one expects from localized states. (Note that the participation ratio, not the inverse participation ratio, is plotted here.)

For a 1000×1000 system, with $\kappa_{j,j+1}$ as a uniform random variable between 0.5 and 1.5, we have verified numerically that the inverse participation ratio $\sum_j |\psi_j|^4/(\sum_j |\psi_j|^2)^2$ is of the order of 0.1, which is much greater than $\sim 1/N$. Visually, many of the eigenfunctions are seen to be significantly different from zero over approximately 10 to 20 layers, and the distribution of participation ratios is seen in figure 2 to approach a limiting form as N increases. (The numerical results are intended to estimate how localized the eigenfunctions are, rather than as a proof of localization which we have given analytically.) The localized states mean that each plasmon mode is restricted to a small range of layers instead of extending over the entire system. Thus the two-dimensional plasmon spectrum of multilayer systems with random spacing can potentially be used to test one of the canonical doctrines of Anderson localization: that in the presence of disorder, all states are localized in one dimension.

6. Conclusion

In this paper, we have extended standard techniques for Toeplitz matrices to a combination of Toeplitz and Hankel matrices, and obtained the eigenmodes of two-dimensional electron gas multilayers. Apart from reproducing earlier results [9], we have obtained a sum rule for the eigenvalues, and a simple form for the eigenfunctions. We have further extended this approach to the case when the spacings between layers are random, where the matrix in question is no longer Toeplitz–Hankel, and shown that, in such a system, every eigenmode will be localized to a few layers.

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Data availability statement

All data that support the findings of this study are included within the article (and any supplementary files).

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