

Exact dynamical correlations of the $1/r^2$ model

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We present exact results for the dynamical structure function, i.e., the density-density correlations for the $1/r^2$ system of interacting particles at three special values of the coupling constant. The results are interpreted in terms of exact excitations of the model which are available from Bethe's ansatz, thereby throwing light on the quasiparticle content of the elementary excitations. We also obtain the first moment of the discrete version of the model, which provides a nontrivial check on its structure function. We show that the property of spectrum saturation is a common feature of both versions.

I. INTRODUCTION

The $1/r^2$ system of interacting particles, introduced by Calogero¹ and Sutherland,² and indeed even earlier by Dyson,³ within a relaxational framework in the course of a discussion of the Brownian motion of random matrices, has continued to be of great importance in theoretical physics. In the recent past, it has also generated great interest in the context of a discrete version, i.e., a spin 1/2 model introduced independently by Haldane⁴ and Shastry,⁵ and in terms of its algebraic content;⁶⁻¹² some generalizations have also appeared in the literature.¹³⁻¹⁵ Its interest derives from the combination of beautiful mathematical structure and rich physical phenomena of quantum fluctuations in a low dimensional system (quasi-long-range order, non-Fermi-liquid behavior, etc.), as well as surprising tractability.

In a recent development in an apparently completely different physical system, namely, that of electrons in a random medium, Simons *et al.*^{16,17} have succeeded in computing a certain correlation function depending on two variables, say, space and time, and conjectured that this represents the density-density correlation function of the above $1/r^2$ model at three appropriate values of the coupling constant $\beta = 1, 2,$ and 4 , corresponding to orthogonal, unitary, and symplectic ensembles, respectively. The results are obtained due to a suggested equivalence of the problem to the evolution of energy eigenvalues of a disordered metallic grain subject to an arbitrary perturbation¹⁸ to that of the $1/r^2$ many body problem. The mapping, performed on the level of the two-point, time-dependent, density-density correlation functions, leads to an explicit exact result for the density-density correlation of the $1/r^2$ model for the above values of the coupling constant. The result, astonishingly enough, is valid at *all length and energy scales*, not just in

asymptotic regions. This conjecture has been explicitly confirmed in a recent work by Narayan and Shastry,¹⁹ where they established the correspondence between the evolution of the distribution of eigenvalues of a random matrix subject to a random Gaussian perturbation, and a Fokker-Planck equation which is equivalent to the $1/r^2$ model. At the same time, Simons *et al.*²⁰ have established a direct connection between the $1/r^2$ model and correlations in the spectra of random matrices through a continuous matrix model.

The excitation spectrum of the $1/r^2$ model is available in great detail from the (asymptotic) Bethe's ansatz (ABA) invented by Sutherland.^{2,21} The picture that arises is that of an underlying gas consisting of quasiparticles obeying Fermi statistics, and interacting weakly with each other through a Hartree-Fock interaction leading to a back flow. This picture indeed gives *all* the excited states of the model. The remaining problem, then, is that of an appropriate decomposition of the "bare" particles into "quasiparticles." The explicit knowledge of the correlation functions provides us with an opportunity to describe the intermediate states in $S(q, \omega)$ phenomenologically as combinations of the "quasiparticles," whose energies are available from the ABA. This is analogous to quark spectroscopy in the theory of elementary particles; the ABA quasiparticles and quasiholes are our quarks in the present scheme. We also provide a nontrivial check on the conjecture by comparing the explicitly known functions of the momentum q of the $1/r^2$ model with those obtained by integrating the explicit correlation functions.

The plan of the paper is as follows. In Sec. II, we define the $1/r^2$ model, and summarize the known information about its moments. The dynamical correlations found in Refs. 16, 17 are summarized and their Fourier transforms given. The small q hydrodynamic limit of these correlators is calculated, and the saturation of the

spectrum by a soundlike linear mode (i.e., $\omega = sq$) is demonstrated. Section III contains a summary of the results of the asymptotic Bethe's ansatz for the $1/r^2$ model, where we write down the dispersion of the effective quasiparticles and quasiholes. In Sec. IV, we rework the expressions for the structure function for the three ensembles into forms wherein the energy conserving δ functions are shown to have a natural interpretation in terms of multi-quasiparticle-quasihole pairs. In Sec. V, we discuss the discrete $1/r^2$ model, and display its first three relevant moments explicitly. The similarity to the continuum $1/r^2$ model in terms of the exhaustion of the structure function by "spinons" at low q is pointed out, and this is highlighted to be a unique feature characterizing this family of models. In Sec. VI, we summarize our results.

II. UNIVERSAL CORRELATION FUNCTIONS APPLIED TO THE $1/r^2$ MODEL

The Sutherland-Calogero-Moser system with periodic boundary conditions has a Hamiltonian describing spinless fermions confined to a ring and interacting through a $1/r^2$ pairwise potential:²

$$H = - \sum_{i=1}^N \frac{\partial^2}{\partial r_i^2} + \beta(\beta/2 - 1) \sum_{i>j} \frac{(\pi/L)^2}{\sin^2[\pi(r_i - r_j)/L]} . \quad (1)$$

For simplicity we have chosen the mass to be $1/2$. The ring has length L and the number of particles is N . The statistics of the particles can be chosen arbitrarily since the particles cannot get past each other owing to the singular nature of the interaction at the origin; we declare them to be fermions for convenience (they are indeed so at $\beta = 2$), and one might imagine the system to be that of fermions with either repulsive ($\beta > 2$) or attractive ($\beta < 2$) interactions. Unless explicitly specified, we will assume that the system is in the thermodynamic limit ($L \rightarrow \infty$ and $N \rightarrow \infty$) and has a finite $O(1)$ density ($d = N/L$).

It was first argued in Ref. 16 that the time-dependent correlation functions of this one-dimensional Hamiltonian are equivalent to certain universal correlation functions¹⁸ of the energy spectra of weakly disordered metallic grains when $\beta = 1, 2$, or 4 . For spectra subject to some arbitrary perturbation, X , exact analytical expressions were derived for the two-point density correlation function,

$$k(E, X) = \langle \rho(\bar{E} - E, \bar{X} + X) \rho(\bar{E}, \bar{X}) \rangle - \langle \rho(\bar{E}, \bar{X}) \rangle^2 , \quad (2)$$

where $\rho(\bar{E}, \bar{X}) = \sum_{i=1}^N \delta(\bar{E} - E_i(\bar{X}))$ is the density of states of the system and $\langle \dots \rangle$ denotes a statistical average which can be performed over a range of energy or over X . It was shown that after the following rescaling in which the parameters become dimensionless, $\epsilon_i = E_i/\Delta$ and $x = X \sqrt{[\partial \epsilon_i(X)/\partial X]^2}$, Eq. (2) becomes universal, depending only on the symmetry of the Dyson ensemble.

In fact, the universality is not specific to disordered metals but applies equally to all nonintegrable or quantum chaotic systems.¹⁸

Remarkably, by performing the change of variables

$$x^2 \equiv -2it , \quad \epsilon \equiv r , \quad (3)$$

where t is the time coordinate and r is the spatial coordinate, Eq. (2) becomes equivalent to the two-point particle density correlator of the ground state of the Sutherland model. The coordinate r will be given in units of the mean interparticle distance $1/d$ (although $d \equiv \langle \rho \rangle = 1$, we will continue to display the d dependence in order to retain generality). The resulting correlation function, after the change of variables, is

$$k(r, t) = \langle \rho(\bar{r} - r, \bar{t} + t) \rho(\bar{r}, \bar{t}) \rangle - d^2 , \quad (4)$$

with $\rho(r, t) = \sum_{i=1}^N \delta(r - r_i(t))$. The expectation value $\langle \dots \rangle$ is to be taken on the ground state of H .

Once we have an expression for $k(r, t)$, we can calculate the dynamical structure factor $S(q, \omega)$.²² The explicit connection between $k(r, t)$ and $S(q, \omega)$ is made by taking the space and time Fourier transforms of $k(r, t)$,

$$S(q, \omega) = \frac{1}{2\pi d} \int dr \int dt k(r, t) e^{-i(qr - \omega t)} .$$

$S(q, \omega)$ has a representation in terms of the excited states of the system:

$$S(q, \omega) \equiv \frac{1}{N} \sum_{\nu \neq 0} |\langle \nu | \rho_q | 0 \rangle|^2 \delta(\omega - E_\nu + E_0) , \quad (5)$$

where $H|\nu\rangle = E_\nu|\nu\rangle$, and

$$\rho_q = \int dr \rho(r) e^{-iqr} = \sum_{i=1}^N e^{-iqr_i} . \quad (6)$$

In the following, we will present exact analytical expressions and discuss some of the properties of $S(q, \omega)$ for the three special values of β .

A. Moments of $S(q, \omega)$

We begin by stating some important sum rules²² concerning the function $S(q, \omega)$. Defining the moments of this function as

$$I_n(q) \equiv \int_0^\infty d\omega \omega^n S(q, \omega) , \quad (7)$$

it follows from the velocity independence of the interaction that

$$I_1(q) = q^2 . \quad (8)$$

This is the statement of particle conservation and is the familiar f -sum rule. Another sum rule follows from the compressibility relation²²

$$\lim_{q \rightarrow 0} I_{-1}(q) = \frac{1}{s^2} , \quad (9)$$

where $s^2 = 2(\partial P/\partial d)$ is the square of the sound velocity and P is the pressure. However, since $P = -(\partial E/\partial L)$ and for the Sutherland model the ground state energy is known to be $E_0 = (\pi^2\beta^2/12)(N^3/L^2)$,² it follows that Eq. (9) can also be written as

$$\lim_{q \rightarrow 0} I_{-1}(q) = \frac{1}{\pi^2 d^2 \beta^2}. \quad (10)$$

$$I_0^u(q) = \frac{1}{2k_F} [|q| + (2k_F - |q|)\theta(|q| - 2k_F)] \quad (\beta = 2), \quad (11)$$

$$I_0^o(q) = \begin{cases} \frac{|q|}{k_F} \left[1 - \frac{1}{2} \ln \left(1 + \frac{|q|}{k_F} \right) \right], & |q| < 2k_F \\ 2 - \frac{|q|}{2k_F} \ln \left| \frac{|q| + k_F}{|q| - k_F} \right|, & |q| > 2k_F \end{cases} \quad (\beta = 1), \quad (12)$$

and

$$I_0^s(q) = \begin{cases} \frac{|q|}{4k_F} \left[1 - \frac{1}{2} \ln \left| 1 - \frac{|q|}{2k_F} \right| \right], & |q| < 4k_F \\ 1, & |q| > 4k_F \end{cases} \quad (\beta = 4). \quad (13)$$

For small q we find the limiting behavior

$$I_0(q) \xrightarrow{q \rightarrow 0} \frac{q}{\beta k_F}. \quad (14)$$

At this point we should also mention that Forrester²⁴ has presented a formula for the static $S(r)$ [the inverse Fourier transform of $I_0(q)$] for all even values of β in terms of generalized hypergeometric functions. More recently²⁵ he has shown that one can rewrite his formula in an integral form which allows an asymptotic analysis consisted with Eq. (14).

In terms of the three moments given above, we can calculate two characteristic frequencies whose q dependences give us an idea of the dispersion relations of the excited modes. This approach has been used before by Hohenberg and Brinkman²⁶ in a different context (that of the isotropic Heisenberg spin chains). Here we name the first frequency the ‘‘Feynman spectrum,’’²⁷

$$\omega_{\mathcal{F}}(q) \equiv \frac{I_1(q)}{I_0(q)} = \frac{q^2}{S(q)}, \quad (15)$$

which has the small q behavior,

$$\omega_{\mathcal{F}}(q) \xrightarrow{q \rightarrow 0} \beta q k_F, \quad (16)$$

and the second one we call the ‘‘hydrodynamical spectrum,’’

$$\omega_{\mathcal{H}}(q) \equiv \sqrt{\frac{I_1(q)}{I_{-1}(q)}} \xrightarrow{q \rightarrow 0} \beta q k_F. \quad (17)$$

In Fig. 1 we have plotted these dispersion relations for the three values of β for which we know the moments exactly. For the repulsive and noninteracting cases there is a logarithmic dip at $q = 2k_F$, while for the attractive

Finally, we note that the connection between the $1/r^2$ model in the static limit and the distribution of eigenvalues of a random matrix was established by Sutherland in his early papers.² Consequently, the zeroth moment $I_0(q)$ [also called the static form factor and usually denoted by $S(q)$] had been determined before for all three values of β from random matrix theory (RMT).²³ Therefore, from the results of Ref. 2 we anticipate the following expressions:

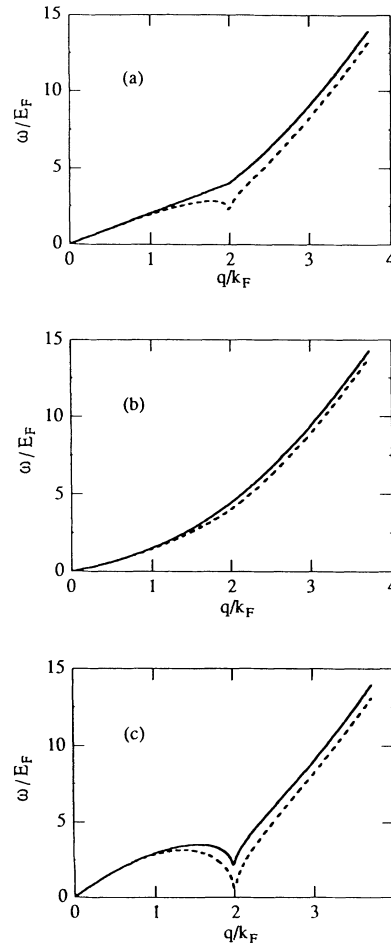


FIG. 1. Characteristic frequencies of $S(q, \omega)$. The solid line corresponds to the Feynman spectrum $\omega_{\mathcal{F}}(q)$ and the dashed line corresponds to the hydrodynamical spectrum $\omega_{\mathcal{H}}(q)$: (a) $\beta = 2$; (b) $\beta = 1$; (c) $\beta = 4$.

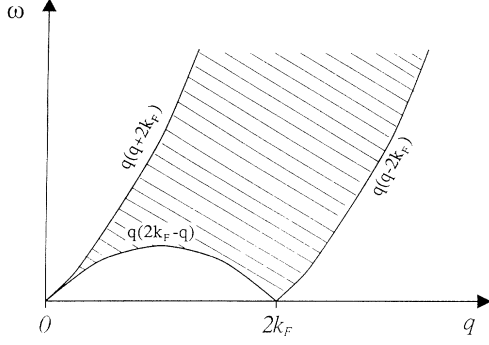


FIG. 2. Region where $S^u(q, \omega) \neq 0$ ($\beta = 2$). The equations for the boundaries are indicated.

case the dispersion grows monotonically. The appearance of a dip can be interpreted as a tendency towards “crystallization” (i.e., the particles tend to arrange themselves in a lattice with spacing $1/d$) as the strength of the repulsive interaction increases.

B. Static correlation functions: Real space

We note that the static density-density correlations are simply related to the Fourier transforms of the moments $I_0(q)$. Writing the density correlation function in the form

$$k(r, 0) = d \delta(r) + d^2 C(r), \quad (18)$$

the dimensionless correlation function $C(r)$ satisfies the relations $\lim_{r \rightarrow 0} C(r) \rightarrow -1$ and $\lim_{r \rightarrow \infty} C(r) \rightarrow 0$. It may be written as

$$C(r) = \frac{1}{d} \int_{-\infty}^{\infty} \frac{dq}{2\pi} \cos(qr) [I_0(q) - 1]. \quad (19)$$

The correlation function, in scaled variables, has the representation

1. Unitary ensemble

We will first examine the simplest case when $\beta = 2$ and the system is noninteracting, which corresponds to the unitary ensemble. It can be readily shown that

$$k^u(r, t) = \frac{d^2}{2} \int_1^{\infty} d\lambda_1 \int_{-1}^1 d\lambda \exp[-ik_F^2 t (\lambda_1^2 - \lambda^2)] \cos[k_F r (\lambda_1 - \lambda)], \quad (21)$$

where $\lambda = k/k_F$, $\lambda_1 = k_1/k_F$, and $k_F = \pi d$ (k_F is the Fermi momentum). Taking the Fourier transform in both space and time we get

$$\begin{aligned} S^u(q, \omega) &= \frac{1}{2k_F^2} \int_1^{\infty} d\lambda_1 \int_{-1}^1 d\lambda \delta(\lambda_1^2 - \lambda^2 - \omega/k_F^2) \delta(\lambda_1 - \lambda - |q|/k_F) \\ &= \frac{1}{4k_F|q|} \theta(\omega + q^2 - 2k_F|q|) \theta(\omega - q^2 + 2k_F|q|) \theta(2k_F|q| + q^2 - \omega). \end{aligned} \quad (22)$$

In Fig. 2 we have plotted the region of support corresponding to Eq. (22), which is nothing but the particle-hole continuum (in this case all excitations are in the form of pairs). The tridimensional plot of $S^u(q, \omega)$ is shown in Fig. 3.

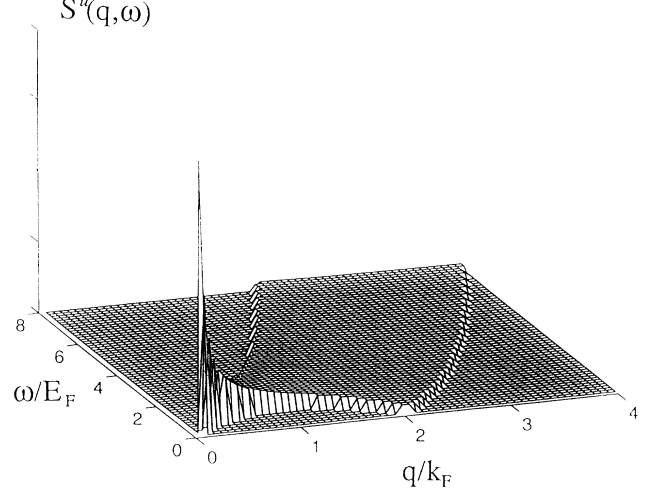


FIG. 3. Tridimensional plot of $S(q, \omega)$ ($\beta = 2$) for the unitary case. The vertical axis has a linear scale.

$$C(\hat{r}/d) = \int_{-\infty}^{\infty} \frac{d\hat{q}}{2} \cos(\pi \hat{q} \hat{r}) [I_0(\hat{q} k_F) - 1]. \quad (20)$$

Explicit expressions are available for the various ensembles from Mehta²⁸ by noting that $C(r = \hat{r}/d)$ is nothing but the two-level cluster function $-Y_2(\hat{r})$, where \hat{r} is the separation in units of the average interparticle spacing $1/d$.

C. Dynamical correlations

We now recapitulate the results from Refs. 16, 17 for the dynamical correlation function and present explicit expressions for $S(q, \omega)$. We note that $S(q, \omega)$ is real and positive; moreover, it vanishes for $\omega < 0$ and it depends only on the absolute value of q .

From Eq. (22) we can of course compute all the moments of $S^u(q, \omega)$ exactly. The three moments I_1^u , I_0^u , and I_{-1}^u are plotted in Fig. 4 and we remark that they are in agreement with the sum rules of Eqs. (8) and (10) and the identity Eq. (11).

2. Orthogonal ensemble

Second, we will consider the orthogonal (attractive) case, when $\beta = 1$. This value of the coupling constant leads to a more complicated expression for the two-point correlation function; after Ref. 16, we have

$$k^\circ(r, t) = d^2 \int_{-1}^1 d\lambda \int_{-1}^\infty d\lambda_1 \int_{-1}^\infty d\lambda_2 \frac{(1 - \lambda^2)(\lambda_1 \lambda_2 - \lambda)^2}{(\lambda_1^2 + \lambda_2^2 + \lambda^2 - 2\lambda\lambda_1\lambda_2 - 1)^2} \\ \times \exp[-ik_F^2 t(2\lambda_1^2 \lambda_2^2 - \lambda_1^2 - \lambda_2^2 - \lambda^2 + 1)/2] \cos[k_F r(\lambda_1 \lambda_2 - \lambda)]. \quad (23)$$

Taking the Fourier transform of $k^\circ(r, t)$ in space and time yields

$$S^\circ(q, \omega) = \frac{2q^2}{k_F^4} \int_{-1}^\infty d\lambda_1 \int_{-1}^\infty d\lambda_2 \frac{[1 - (\lambda_1 \lambda_2 - |q|/k_F)^2]}{(\lambda_1^2 + \lambda_2^2 + q^2/k_F^2 - \lambda_1^2 \lambda_2^2 - 1)^2} \\ \times \delta(\lambda_1^2 + \lambda_2^2 + q^2/k_F^2 - 1 - \lambda_1^2 \lambda_2^2 - 2\lambda_1 \lambda_2 |q|/k_F + 2\omega/k_F^2) \\ \times \theta(\lambda_1 \lambda_2 - |q|/k_F + 1) \theta(1 - \lambda_1 \lambda_2 + |q|/k_F). \quad (24)$$

The region of support of $S^\circ(q, \omega)$ is plotted in Fig. 5 and it shows a continuum similar to the excitation of a single particle-hole pair in the noninteracting case. However, as we will later discuss in detail, the excited states have a more complex structure. The tridimensional plot of $S^\circ(q, \omega)$ obtained by numerically integrating Eq. (24) and using a Gaussian regularization of the δ function is shown in Fig. 6. The ridge along $\omega = q(q + k_F)$ indicates an algebraic (inverse square root) divergence.

The evaluation of I_0° starting from Eq. (24) has been done analytically by Efetov²⁹ and it can be readily generalized for any I_n° , $n > 0$. However, we have only been able to evaluate numerically the moments with $n < 0$. The moments I_1° , I_0° , and I_{-1}° are plotted in Fig. 7 and they obey exactly the sum rules of Eqs. (8) and (10), and the identity Eq. (12).

3. Symplectic ensemble

Finally, we look at the repulsive case, when $\beta = 4$. In the context of RMT this corresponds to the symplectic ensemble. We start with the correlation function originally obtained in Ref. 17,

$$k^s(r, t) = \frac{d^2}{2} \int_{-1}^\infty d\lambda \int_{-1}^1 d\lambda_1 \int_{-1}^1 d\lambda_2 \frac{(\lambda^2 - 1)(\lambda - \lambda_1 \lambda_2)^2}{(\lambda_1^2 + \lambda_2^2 + \lambda^2 - 2\lambda\lambda_1\lambda_2 - 1)^2} \\ \times \exp[-4ik_F^2 t(\lambda_1^2 + \lambda_2^2 + \lambda^2 - 2\lambda_1^2 \lambda_2^2 - 1)] \cos[2k_F r(\lambda - \lambda_1 \lambda_2)]. \quad (25)$$

We take the Fourier transform of $k^s(r, t)$ to get

$$S^s(q, \omega) = \frac{q^2}{64k_F^4} \int_{-1}^1 d\lambda_1 \int_{-1}^1 d\lambda_2 \frac{[(\lambda_1 \lambda_2 + |q|/2k_F)^2 - 1]}{(\lambda_1^2 + \lambda_2^2 + q^2/4k_F^2 - \lambda_1^2 \lambda_2^2 - 1)^2} \\ \times \delta(\lambda_1^2 + \lambda_2^2 + q^2/4k_F^2 - 1 - \lambda_1^2 \lambda_2^2 + \lambda_1 \lambda_2 |q|/k_F - \omega/4k_F^2) \\ \times \theta(\lambda_1 \lambda_2 + |q|/2k_F - 1). \quad (26)$$

The region of support of $S^s(q, \omega)$ is shown in Fig. 8. The continuum reaches $\omega = 0$ not only at $q = 0$ and $2k_F$, but at $q = 4k_F$ as well. The figure is similar to the one obtained when we excite two particle-hole pairs in the noninteracting case; in fact, the real structure of the excited states is more complicated than this simple picture, as we will later demonstrate. In Fig. 9 we show the tridimensional plot of $S^s(q, \omega)$ with a Gaussian regularization and a numerical integration of Eq. (26). The ridges along $\omega = 2q(2k_F - q)$ and $\omega = q^2 - 4k_F^2$ indicate an algebraic (inverse square root) divergence.

As for the orthogonal ensemble, the analytical evaluation of I_n^s can be done for $n > 0$ by generalizing the method of Ref. 29 for $n = 0$. For $n < 0$ we have only been able to proceed with a numerical evaluation. In Fig. 10 we have plotted the moments I_1^s , I_0^s , and I_{-1}^s . It is simple to check that the moments obtained in this way also agree with Eqs. (8), (10), and (13).

D. Spectrum saturation in the hydrodynamic limit

An important property of the Sutherland model is the saturation at $q \rightarrow 0$. As we have pointed out before in Eqs. (16,17), both the Feynman and hydrodynamical spectra tend to the same value in this limit. In fact, this is also

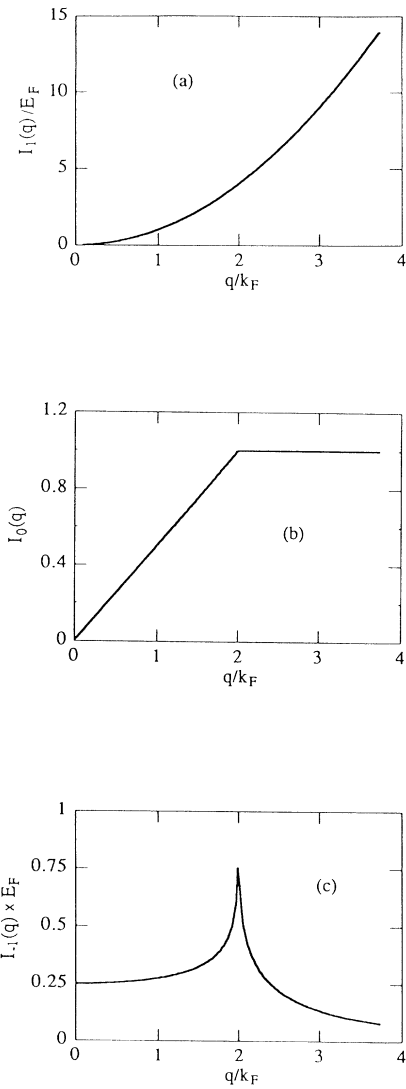


FIG. 4. Moments of $S(q, \omega)$ for $\beta = 2$: (a) $I_1(q)$; (b) $I_0(q)$; (c) $I_{-1}(q)$.

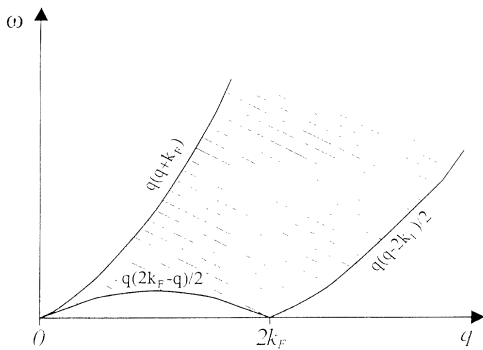


FIG. 5. Region where $S^o(q, \omega) \neq 0$ ($\beta = 1$). The equations for the boundaries are indicated.

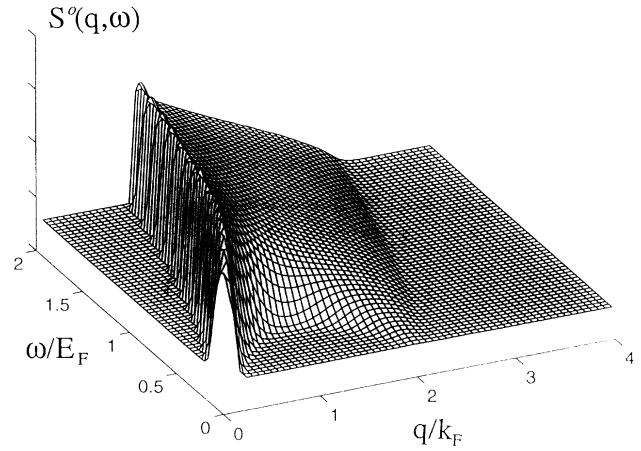


FIG. 6. Tridimensional plot of $S(q, \omega)$ for the orthogonal case ($\beta = 1$). The δ function of Eq. (24) has been regularized by a Gaussian. The vertical axis has a logarithmic scale.

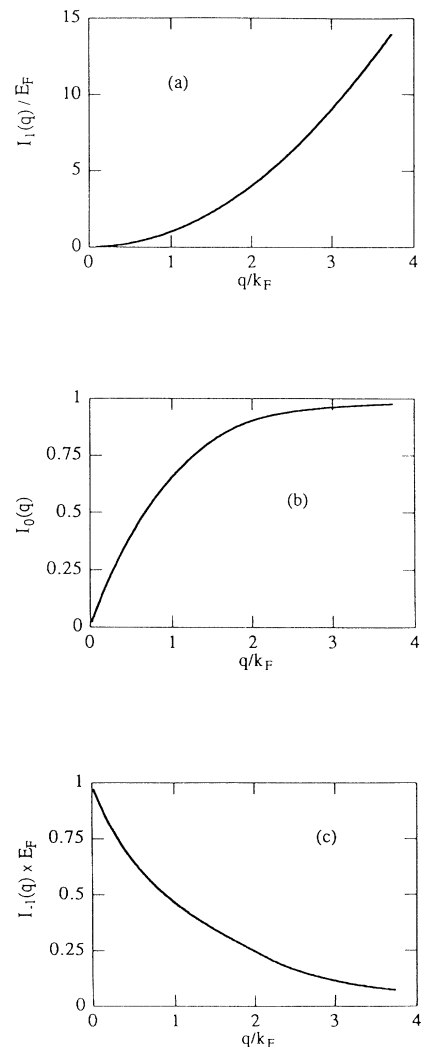


FIG. 7. Moments of $S(q, \omega)$ for $\beta = 1$: (a) $I_1(q)$; (b) $I_0(q)$; (c) $I_{-1}(q)$.

true for any characteristic frequency defined as the ratio of any two moments. We shall prove it for the orthogonal ensemble: In Eq. (24) we change variables to

$$\lambda_{1,2} = 1 + x_{1,2},$$

which yields

$$\begin{aligned} I_n^o(q) &= \frac{2q^2}{k_F^4} \left(\frac{k_F^2}{2}\right)^{n+1} \int_0^\infty dx_1 \int_0^\infty dx_2 \frac{[1 - (1 + x_+ + x_1 x_2 - q/k_F)^2]}{[q^2/k_F^2 - x_1 x_2 (4 + x_1 x_2 + 2x_+)]^2} \\ &\times \left[-\frac{q^2}{k_F^2} + \frac{2q(1 + x_+ + x_1 x_2)}{k_F} + x_1 x_2 (4 + x_1 x_2 + 2x_+) \right]^n \\ &\times \theta(2 + x_+ + x_1 x_2 - q/k_F) \theta(q/k_F - x_+ - x_1 x_2), \end{aligned} \quad (27)$$

where $x_\pm = x_1 \pm x_2$. This last expression can be much simplified in the limit $q \rightarrow 0$; it becomes

$$I_n^o(q) \xrightarrow{q \rightarrow 0} \frac{2(qk_F)^{n+2}}{k_F^4} \int_0^{q/k_F} dx_1 \int_0^{q/k_F - x_1} dx_2 \frac{(q/k_F - x_+)}{(q^2/k_F^2 - x_+^2 + x_-^2)^2}. \quad (28)$$

Changing the integration variables to x_\pm and performing the double integral we obtain

$$I_n^o(q) \xrightarrow{q \rightarrow 0} \frac{1}{k_F^2} (qk_F)^{n+1}. \quad (29)$$

As a result,

$$\left[\frac{I_{n+m}^o(q)}{I_n^o(q)} \right]_{q \rightarrow 0}^{1/m} \rightarrow qk_F. \quad (30)$$

The proof for the symplectic and unitary ensembles is quite analogous; one obtains for the three values of β

$$I_n(q) \xrightarrow{q \rightarrow 0} \frac{1}{\beta^2 k_F^2} (\beta q k_F)^{n+1}. \quad (31)$$

This yields, for any integers n and m ,

$$\left[\frac{I_{n+m}(q)}{I_n(q)} \right]_{q \rightarrow 0}^{1/m} \rightarrow \beta q k_F. \quad (32)$$

The saturation property can also be visualized in Figs. 2, 5, and 8: The fact that the lower and upper branches of parabola have the same linear term as $q \rightarrow 0$

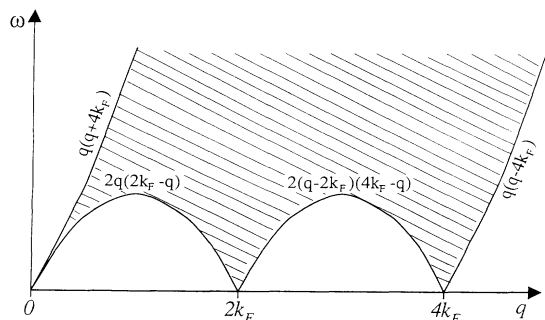


FIG. 8. Region where $S^s(q, \omega) \neq 0$ ($\beta = 4$). The equations for the boundaries are indicated.

implies that all characteristic frequencies must have the same asymptotics.

III. ASYMPTOTIC BETHE'S ANSATZ

We now summarize the results of the asymptotic Bethe's ansatz,²¹ which gives an explicit expression for the “particle-hole”-like excitations underlying the system. The *complete* excitation spectrum for the $1/r^2$ model can be described in remarkably simple terms as follows: The total energy of a state of the system is expressible as

$$E = \sum_n p_n^2, \quad (33)$$

with the “pseudomomenta” p_n satisfying the equation

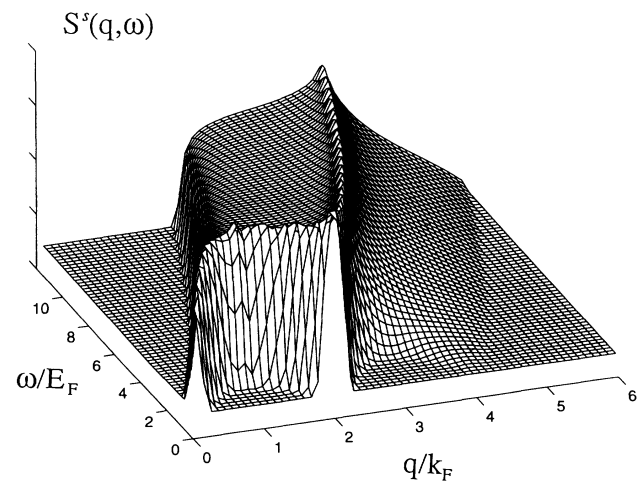


FIG. 9. Tridimensional plot of $S(q, \omega)$ for the symplectic case ($\beta = 4$). The δ function of Eq. (26) has been regularized by a Gaussian. The vertical axis has a logarithmic scale.

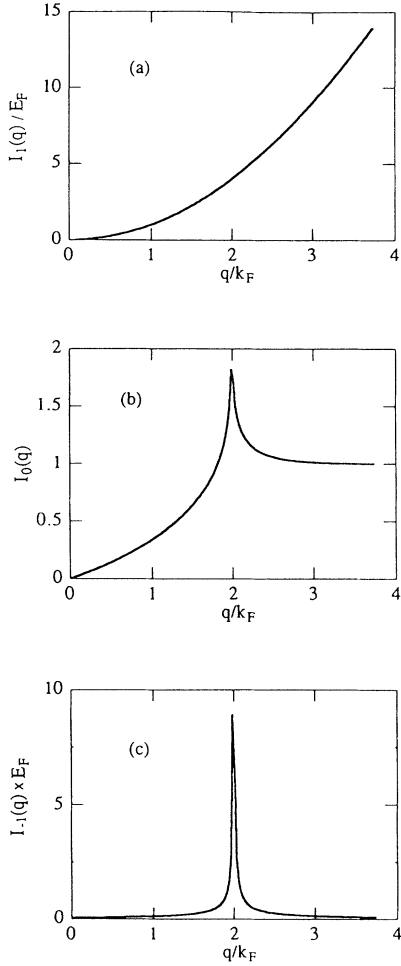


FIG. 10. Moments of $S(q, \omega)$ for $\beta = 4$: (a) $I_1(q)$; (b) $I_0(q)$; (c) $I_{-1}(q)$.

$$p_n = k_n + \frac{\pi(\beta - 2)}{2L} \sum_{n \neq m} \text{sgn}(k_n - k_m). \quad (34)$$

The total momentum of the state is

$$P = \sum_n p_n = \sum_n k_n. \quad (35)$$

The bare momenta are given by $k_n = 2\pi J_n/L$, where the J_n 's are fermionic quantum numbers $J_1 < J_2 < J_3 < \dots < J_N$. Note that at $\beta = 2$ the interaction is turned off and we recover the free-fermion results. The important point is that the totality of states for the N particle sector is obtained by allowing the integers J_n to take on all values consistent with Fermi statistics, not only for $\beta = 2$, but for *all* $\beta \in [1, +\infty]$. The summation in Eq. (34) is trivial to carry out and we find

$$p_n = k_n + \frac{(\beta - 2)\pi}{L} \left(n - \frac{N + 1}{2} \right). \quad (36)$$

We can now select an arbitrary state of the system by

specifying that states $\{k_1, k_2, \dots\}$ are occupied, i.e., by introducing the fermionic occupation numbers $n(k_j) = 0, 1$, such that

$$E = \sum_n \varepsilon(k_n) n(k_n) + \sum_{n \neq m} v(k_n - k_m) n(k_n) n(k_m) + \left[\frac{\pi(\beta - 2)}{2} \right]^2, \quad (37)$$

with $\varepsilon(k) = k^2$ and $v(k) = \pi(\beta - 2)|k|/2L$. For future reference, the ground state is represented by $n_0(k_n) = 1$ for $|k_n| < k_F$ and $n_0(k_n) = 0$ otherwise, where $k_F = \pi d$. We remark that the above expression of the energy takes the form of a renormalized Hartree-Fock theory; a Hartree-Fock energy expectation value of the interacting Hamiltonian in a determinantal state $\prod c_{k_j}^\dagger |0\rangle$ leads to precisely this type of expression. Note that the Fourier transform of the two-body interaction can be deduced from the expansion

$$\left(\frac{\pi}{L} \right)^2 \frac{\beta(\beta - 2)}{\sin^2(\pi r/L)} = -\beta(\beta - 2) \frac{\pi}{L} \sum_q |q| \exp(iqr).$$

Therefore, Eq. (37) states that a Hartree-Fock expression with a renormalization of the coupling constant $\beta(\beta - 2) \rightarrow 2(\beta - 2)$ leads to the *exact spectrum* of the model.

We now consider the excitation spectrum near the ground state, wherein we excite a particle-hole pair in the free Fermi system and ask what the energy of the interacting system is by including the Hartree-Fock back flow term. From this point onwards we measure all momenta in units of k_F and energies in units of the Fermi energy. Let us suppose that one of the particles described by Eq. (37) has initially a momentum k , with $|k| < 1$; we promote it to some state labeled by $k + q$, with $|k + q| > 1$. The energy cost in units of the Fermi energy is equal to

$$\begin{aligned} \Delta(q, k) &= \varepsilon(k + q) - \varepsilon(k) \\ &\quad + 2 \sum_{|k'| < 1} [v(k + q - k') - v(k - k')] \\ &= q^2 + 2kq + \frac{(\beta - 2)}{2} (2|k + q| - k^2 - 1), \end{aligned} \quad (38)$$

and the momentum of this state is simply q . This implies that we can associate a generalized energy corresponding to a particle $\varepsilon_>(k)$ (i.e., $|k| > 1$) and a hole $\varepsilon_<(k)$ (i.e., $|k| < 1$):

$$\begin{aligned} \varepsilon_>(k) &= k^2 + (\beta - 2)|k|, \\ \varepsilon_<(k) &= \frac{\beta}{2} k^2 + \left(\frac{\beta}{2} - 1 \right), \end{aligned} \quad (39)$$

such that

$$\Delta(q, k) = \varepsilon_>(k + q) - \varepsilon_<(k). \quad (40)$$

Note that $\varepsilon_>(k)$ and $\varepsilon_<(k)$ are continuous and have continuous derivatives across the Fermi surface. These expressions for the particle and hole energies look different from the results of Sutherland²¹ but their equivalence may be readily checked. We have chosen to label our

states by the “bare” momenta k_n , while Ref. 21 works with the pseudomomenta p_n ; of course these are in one to one correspondence and so the choice is a matter of convenience.

We will introduce, in the usual way, particle operators $A^\dagger(k)$ and hole operators $B^\dagger(k)$ with the convention that the momenta corresponding to these are constrained by $|k| > k_F$ for particles and $|k| \leq k_F$ for holes, with excitation energies

$$\begin{aligned} E_A(k) &= \varepsilon_>(k) - \mu, \\ E_B(k) &= \mu - \varepsilon_<(-k), \end{aligned} \quad (41)$$

where $\mu \equiv \varepsilon_>(k_F)$ is the “chemical potential.” The quasiparticle-quasihole excitation created by the operator $A^\dagger(k+q)B^\dagger(-k)$ then has energy $E_A(k+q) + E_B(-k)$, which of course is equal to $\Delta(k, q)$. Having introduced the underlying fermionic quasiparticles and quasiholes through Eqs. (39), we would like to see if the excitations generated by the bare density fluctuation operator ρ_q can be expressed in terms of the latter. One of our objectives, then, is to express the excitations of the system probed by the bare density fluctuation operator ρ_q in terms of the quasiparticle-quasihole operators. Recall that in Landau’s Fermi liquid theory³⁰ one expresses the bare particles $c(k)$ in a series involving quasiparticles and quasiholes of the form

$$\begin{aligned} c(k) &= \sqrt{z_k} B^\dagger(-k) \\ &+ \sum_{(p,l)} M[k, p, l] B^\dagger(p) B^\dagger(l) A^\dagger(-k - p - l) + \dots, \end{aligned} \quad (42)$$

where $|k| \leq k_F$, and a similar expansion for particles, where z_k is the quasiparticle residue. The density fluctuation operator $\rho_q = \sum_k c^\dagger(k+q)c(k)$ then has a development in terms of 1, 2, 3, ... pairs of (quasi)particle-(quasi)hole excitations. In one dimension, we expect z_k to vanish for arbitrary nonzero interactions, and hence the particle-hole series is expected to be such that the single pair should not appear. The expansions are somewhat nonunique, in view of the fact that we can add an arbitrary number of “zero-energy” and “zero-momentum” particle-hole excitations to any given scheme.

IV. QUASIPARTICLE CONTENT OF THE STRUCTURE FUNCTION

For the unitary case there is no interaction and consequently quasiparticles and quasiholes are the same as particles and holes: Equation (38) at $\beta = 2$ exactly describes the spectrum of Fig. 2. On the other hand, for the orthogonal and symplectic cases the simple creation of quasiparticle-quasihole pairs cannot account for the whole excitation spectrum. In order to see that in general ($\beta \neq 2$), we begin by considering the spectrum ($q \times \omega$) for a particle-hole excitation [Eq. (38)], which is the familiar pair spectrum renormalized by the interaction. For a fixed q , the maximum value of ω occurs when $k = 1$: $\omega_{\max} = q^2 + \beta|q|$. The minimum value of ω depends on q : For $|q| < 2$ it occurs when $k = 1 - |q|$; for $|q| > 2$ it occurs when $k = -1$. This results in $\omega_{\min} = \beta|q|(2 - |q|)/2$ for $q < 2$, and $\omega_{\min} = (|q| - 2)[|q| + (\beta - 2)]$, for $q > 2$. The curves bound a continuum which does not agree with the hatched regions of either Fig. 5 or 8.

We can also promote two quasiparticles from the Fermi sea, instead of just one. The result is that the upper limit for ω is then given by $|q|(|q| + \beta)$ and the lower limits are $\beta|q|(2 - |q|)/2$, $\beta(|q| - 2)(4 - |q|)/2$, and $(|q| - 4)[|q| + 2(\beta - 2)]/2$ (for the intervals $|q| < 2$, $2 < |q| < 4$, and $|q| > 4$, respectively). Again, we note that the continuum bound by these curves is not equal to the hatched region in Fig. 8.

Below we recast the expressions for the correlation functions in the various ensembles in terms of new variables, in order to reveal their exact quasiparticle content.

A. Orthogonal ensemble: Change of variables

We now turn to the expression Eq. (24) of the structure function. First, we change variables and introduce

$$\begin{aligned} u &= \lambda_1 \lambda_2, \\ z &= \lambda_1 + \lambda_2. \end{aligned} \quad (43)$$

With this change of variables, we find

$$S^o(q, \omega) = 2q^2 \int_{\max\{1, q-1\}}^{(1+q)} du \int_{2\sqrt{u}}^{(1+u)} dz \frac{1}{\sqrt{z^2 - 4u}} [1 - (u - q)^2] \delta(E)/D, \quad (44)$$

where

$$\begin{aligned} \sqrt{D} &= z^2 + q^2 - (1 + u)^2, \\ E &= \sqrt{D} + 2\omega - 2uq. \end{aligned} \quad (45)$$

In the expressions above and hereafter in this section we will set $q > 0$ without loss of generality. We change the integration variable u by defining $k = u - q$, in terms of which the k integration is restricted to $\max\{1 - q, -1\} \leq k \leq 1$, and is immediately recognizable as the momentum of a hole restricted to the Fermi surface with $|k + q|$ restricted to be a particle. The z integration can be conveniently transformed by introducing a “rapidity” variable

$$z = 2\sqrt{(k + q)} \cosh \theta, \quad (46)$$

and recalling that for the orthogonal case $\varepsilon_>(k) = k^2 - |k|$ and $\varepsilon_<(k) = (k^2 - 1)/2$ [see Eq. (39)], so that

$$S^o(q, \omega) = \frac{q^2}{2} \int_{\substack{|k| \leq 1 \\ k+q > 1}} dk \frac{[-\varepsilon_<(k)]}{[\omega - q(k+q)]^2} \int_0^{\ln \sqrt{(k+q)}} d\theta \times \delta(\varepsilon_>(k+q) - \varepsilon_<(k) - 2(k+q) \sinh^2 \theta - \omega). \quad (47)$$

The θ integration can be done simply, and gives the final result

$$S^o(q, \omega) = \frac{q^2}{4} \int_{\substack{|k| \leq 1 \\ k+q > 1}} dk \frac{[-\varepsilon_<(k)]}{[\omega - q(k+q)]^2} \frac{\theta(\omega - \Delta(k, q) + (k+q-1)^2/2) \theta(\Delta(k, q) - \omega)}{\sqrt{\Delta(k, q) - \omega} \sqrt{\Delta(k, q) + 2(k+q) - \omega}}. \quad (48)$$

1. Bethe quasiparticle-quasihole content: Orthogonal ensemble

We can rewrite the energy conserving δ function in Eq. (47) as $\delta(\varepsilon_>^{(o)}(k+q|\theta) - \varepsilon_<(k) - \omega)$, where $\varepsilon_>^{(o)}(k|\theta) = \varepsilon_>(k) - 2k \sinh^2 \theta$. One possible picture suggested then has the excited state particle possessing a “hidden” gauge variable θ , which lies in a limited range, as a hole would, endowed with energy but possessing no “physical momentum.” The particle, say, for $k > 1$ has an energy $\varepsilon_>^{(o)}(k)$ which lies between $\frac{k^2-1}{2}$ and $k^2 - k$. We can also view the excited particle state as a combination of a particle and particle-hole pair, as follows. We write a schematic development for $k \geq 1$,

$$c^\dagger(k) \sim \sum_{\frac{1+k}{2} \leq p \leq k} A^\dagger(p) A^\dagger(k-p+1) B^\dagger(-1). \quad (49)$$

The excitation energy of this complex is readily seen from Eqs. (39) and (41) to be $E_A(p) + E_A(k-p+1) + E_B(-1)$, with $\frac{1+k}{2} \leq p \leq k$. The sum of these three terms reproduces the variation in $\varepsilon_>^{(o)}(k|\theta)$ implied by the rapidity variable. The density fluctuation ρ_q is then seen to be formally a two-quasiparticle-quasihole object: Writing $c(k) \sim B^\dagger(-k)$, we have

$$\begin{aligned} c^\dagger(k) c(k-q) \\ \sim \sum_{\frac{1+k}{2} \leq p \leq k} A^\dagger(p) A^\dagger(k-p+1) B^\dagger(-1) B^\dagger(q-k), \end{aligned} \quad (50)$$

where it is understood here and elsewhere that when two momenta coincide (as they would in, say, $[B^\dagger(-1)]^2$), then these should be separated by the smallest nonzero wave number. The above scheme for the density fluctuation operator $c^\dagger(k)c(k-q)$ is indicated in Fig. 11. We may therefore regard the density fluctuation as being built up from a particular set of (noninteracting) pair states consisting of annihilating two particles at momenta $k-q$ and 1, and creating a pair with total momentum $k+1$, distributed over all possible relative momenta with appropriate form factors.

B. Symplectic ensemble: Change of variables

We recall for the symplectic case, the Bethe energies $\varepsilon_>(p) = p^2 + 2|p|$ and $\varepsilon_<(p) = 2p^2 + 1$ [Eq. (39)]. We now rewrite Eq. [26] using the same variables as in the previous case [Eq. (43)]. We find that the result breaks up naturally into two pieces S_a and S_b , with the second piece S_b only arising for $q > 2$:

$$S^s(q, \omega) = S_a(q, \omega) + \theta(q-2) S_b(q, \omega), \quad (51)$$

with

$$S_a(q, \omega) = \frac{q^2}{16} \int_{\max(0, 1-q/2)}^1 du \int_{2\sqrt{u}}^{(1+u)} \frac{dz}{\sqrt{z^2 - 4u}} \delta(E_a) \frac{N_a}{D_a} \quad (52)$$

and

$$S_b(q, \omega) = \frac{q^2}{16} \int_{\max\{-1, 1-q/2\}}^0 du \int_0^{(1+u)} \frac{dz}{\sqrt{z^2 - 4u}} \delta(E_b) \frac{N_b}{D_b}, \quad (53)$$

where E , N , and D are appropriately defined (see below). We write $u = (1+k)/2$ in S_a and $u = (l-1)/2$ in S_b , in terms of which Eqs. (52) and (53) take a more natural form

$$S_a(q, \omega) = \frac{q^2}{32} \int dk n_0(k) [1 - n_0(k+q)] \int_{\sqrt{2(1+k)}}^{(3+k)/2} \frac{dz}{\sqrt{z^2 - 2(1+k)}} \delta(E_a) \frac{N_a}{D_a} \quad (54)$$

and

$$S_b(q, \omega) = \frac{q^2}{32} \int dl n_0(l) [1 - n_0(l+q-2)] \int_0^{(1+l)/2} \frac{dz}{\sqrt{z^2 + 2(1-l)}} \delta(E_b) \frac{N_b}{D_b}. \quad (55)$$

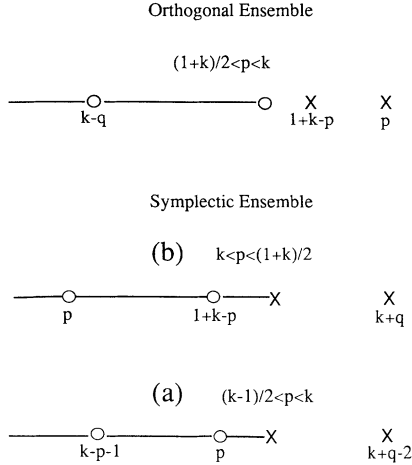


FIG. 11. The two-quasiparticle-quasihole pair scheme for the orthogonal and symplectic cases. \times denotes particles and \circ denotes holes, and the solid line indicates the range $-1 \leq k \leq 1$, i.e., the Fermi sea ($k_F = 1$). The symplectic ensemble has two pieces: case (a) corresponding to S_a and case (b) corresponding to S_b .

In Eq. (54) we further introduce the rapidity variable θ through $z = 8(1+k)\sinh^2\theta$, so as to eliminate the square root in the integrand. The result can be written compactly as

$$S_a(q, \omega) = \frac{q^2}{2} \int_{\substack{|k| \leq 1 \\ k+q > 1}} dk \frac{[\varepsilon_>(k+q) - 3]}{[\omega - 2q(1+k)]^2} \int_0^{\ln \sqrt{2/(1+k)}} d\theta \\ \times \delta(\Delta(k, q) + 8(1+k)\sinh^2\theta - \omega). \quad (56)$$

We can perform explicitly the rapidity integrals and find the final result

$$S_a(q, \omega) = \frac{q^2}{4} \int_{\substack{|k| \leq 1 \\ k+q > 1}} dk \frac{[\varepsilon_>(k+q) - 3]}{[\omega - \Delta(k, q) - (k-1)^2 + q^2]^2} \\ \times \frac{\theta(\omega - \Delta(k, q)) \theta(\Delta(k, q) + (k-1)^2 - \omega)}{\sqrt{\omega - \Delta(k, q)} \sqrt{\omega - \Delta(k, q) + 8(1+k)}}. \quad (57)$$

We next turn to the other piece for $q > 2$. With $\alpha \equiv (q-2)$ and introducing the rapidity variable ϕ through $z = 8(1-l)\sinh^2(\phi)$, Eq. (55) is expressible in the form

$$S_b(q, \omega) = \frac{q^2}{2} \int_{\substack{|l| \leq 1 \\ l+\alpha > 1}} dl \frac{[\varepsilon_>(l+\alpha) - 3]}{[\omega - 2q(l-1)]^2} \int_0^{\ln \sqrt{2/(1-l)}} d\phi \\ \times \delta(\Delta(l, \alpha) + 8(1-l)\sinh^2\phi - \omega). \quad (58)$$

Performing the rapidity integration we find

$$S_b(q, \omega) = \frac{q^2}{4} \int_{\substack{|l| \leq 1 \\ l+\alpha > 1}} dl \frac{[\varepsilon_>(l+\alpha) - 3]}{[\omega - \Delta(l, \alpha) - (l+1)^2 + q^2]^2} \\ \times \frac{\theta(\omega - \Delta(l, \alpha)) \theta(\Delta(l, \alpha) + (l+1)^2 - \omega)}{\sqrt{\omega - \Delta(l, \alpha)} \sqrt{\omega - \Delta(l, \alpha) + 8(1-l)}}. \quad (59)$$

1. Bethe quasiparticle-quasihole content: Symplectic ensemble

The energy conserving δ function in Eq. (56) can be rewritten using an effective energy variable $\varepsilon_<^a(k|\theta) = \varepsilon_<(k) - 8(1+k)\sinh^2\theta$, which implies $\omega = \varepsilon_>(k+q) - \varepsilon_<^a(k|\theta)$. The energy $\varepsilon_<^a(k|\theta)$ varies between the limits $2k^2 + 1$ [i.e., $\varepsilon_<(k)$] and $k^2 + 2k$ [i.e., $\varepsilon_<(k) - (k-1)^2$]. It is thus evident that we may interpret $\varepsilon_<^a(k|\theta)$ as an effective hole, i.e., a composite object. One possible way to decompose it is to write schematically for the bare annihilation operator a representation as a quasihole plus a quasiparticle-quasihole pair:

$$c(k) \sim \sum_{k \leq p \leq \frac{k+1}{2}} B^\dagger(p-k-1)B^\dagger(-p)A^\dagger(1). \quad (60)$$

The restriction on the range of p is such that we avoid double counting the pair and have a natural ordering of the two quasiholes. The energy of the effective hole is then $E_B(-p) + E_B(p-k-1) + E_A(1)$, with the constraint $k \leq p \leq \frac{k+1}{2}$, which from Eqs. (39) and (41) reproduces the range required by the rapidity variation. Therefore, the operator ρ_q is seen to be formally a two-quasiparticle-quasihole object,

$$c^\dagger(k+q)c(k) \\ \sim \sum_{k \leq p \leq \frac{k+1}{2}} B^\dagger(p-k-1)B^\dagger(-p)A^\dagger(1)A^\dagger(k+q). \quad (61)$$

This scheme for the density fluctuation is illustrated in Fig. 11.

In the second piece of S^s [Eq. (58)], once again the energy conserving δ function can be rewritten introducing an effective energy variable for the hole $\varepsilon_<^b(l|\phi) = \varepsilon_<(l) - 8(1-l)\sinh^2\phi$, which implies $\omega = \varepsilon_>(l+\alpha) - \varepsilon_<^b(l|\phi)$. The effective hole energy $\varepsilon_<^b(l|\phi)$ varies between $\varepsilon_<(l)$ and $\varepsilon_<(l) - (1+l)^2$, corresponding to a predominantly left moving object, and may be decomposed again into a hole and a particle-hole pair. Schematically we have

$$c(l) \sim \sum_{\frac{l-1}{2} \leq p \leq l} B^\dagger(-p)A^\dagger(-1)B^\dagger(1+p-l).$$

Using the quasienergies Eqs. (39) and (41) this complex has energy $E_B(p) + E_B(1+p-l) + E_A(-1)$, with the physical constraint $\frac{l-1}{2} \leq p \leq l$, which reproduces the range implied by the variation of the rapidity. Owing to momentum conservation, we must regard the creation operator $c^\dagger(l+q)$ as $A^\dagger(l+q-2)$ times a particle-hole pair with energy zero and momentum 2, i.e., $A^\dagger(1)B^\dagger(1)$. We may eliminate a “zero pair” $A^\dagger(-1)B^\dagger(1)$ and thus obtain the scheme for the density fluctuation operator ρ_q ,

$$c^\dagger(l+q)c(l) \\ \sim \sum_{\frac{l-1}{2} \leq p \leq l} A^\dagger(l+q-2)A^\dagger(1)B^\dagger(-p)B^\dagger(1+p-l). \quad (62)$$

The term S_b then evidently may be regarded as a two-quasiparticle-quasihole object, and is illustrated in Fig. 11.

Summarizing, in process (a), we may regard the density fluctuation as being built up from a particular set of (noninteracting) pair states consisting of creating two particles at momenta 1 and $k + q$, and destroying a pair with total momentum $k + 1$, distributed over all possible relative momenta with appropriate form factors. Likewise, in process (b), we may regard the density fluctuation as being built up from a particular set of (noninteracting) pair states consisting of creating two particles at momenta 1 and $l + q - 2$, and destroying a pair with total momentum $l - 1$, distributed over all possible relative momenta with appropriate form factors.

V. SPIN 1/2 HEISENBERG SYSTEMS

In this section we will study the moments I_n of the $1/r^2$ spin system. For comparison purposes, we will also briefly review some well-known results for another standard 1/2 Heisenberg antiferromagnetic system, namely, the Bethe chain.³¹ The Heisenberg spin chain model with a $1/r^2$ interaction was introduced by Haldane⁴ and Shastry.⁵ It is defined by the Hamiltonian

$$H = J\phi^2 \sum_{i < j} \frac{\mathbf{S}_i \cdot \mathbf{S}_j}{\sin^2[\phi(r_i - r_j)]}, \quad (63)$$

where $\phi = \frac{\pi}{L}$, $r_i = 0, 1, \dots, L-1$ (L integer), and the spins are 1/2. In this case the natural operators one can use to introduce a dynamical structure function $S^d(Q, \omega)$ are the \hat{S}_i^z : We define a ‘‘charge operator’’ $\hat{\rho}_i = (\hat{S}_i^z + 1/2)$ and use (5) to write

$$S^d(Q, \omega) \equiv \frac{1}{N} \sum_{\nu \neq 0} |\langle \nu | \hat{S}_Q | 0 \rangle|^2 \delta(\omega - E_\nu + E_0), \quad (64)$$

with E_ν as the energy eigenvalues of the Hamiltonian and the lattice Fourier transform

$$D(r) = - \left[\frac{\sin(2\pi r \hat{d})}{2\pi r \hat{d}} \right]^2 + \left(\int_0^{2\pi r \hat{d}} dt \frac{\sin t}{t} \right) \frac{(2\pi r \hat{d}) \cos(2\pi r \hat{d}) - \sin(2\pi r \hat{d})}{(2\pi r \hat{d})^2}. \quad (69)$$

Using the relation $\hat{S}_i^z = \hat{\rho}_i - \frac{1}{2}$, we find

$$\langle \hat{S}_0^z \hat{S}_r^z \rangle = \frac{1}{4} + \hat{d}(1 - \hat{d})(\delta_{r,0} - 1) + (1 - \delta_{r,0}) \hat{d}^2 D(r). \quad (70)$$

Inverting the Fourier series, we have

$$I_0^d(Q) = 1 + \hat{d} \sum_r \exp(iQr) D(r). \quad (71)$$

We may convert the sum over r to an integral, remembering that Q and $Q + 2\pi \times \text{integer}$ are equivalent. We

$$\hat{S}_Q^z = \sum_{j=1}^L \hat{S}_j^z e^{-iQr_j}, \quad (65)$$

where Q is the lattice momentum $Q = (2\pi/L) \times \text{integer}$. N is the number of spin deviations or the number of hard-core bosons. We set $N = \hat{d}L$, so that $\hat{d} = 1/2$ for half filling. The Fermi momentum is then $k_F = \pi \hat{d}$, and we will scale $Q = \pi \hat{d} \hat{Q}$ in order to compare with the continuum model results.

We cannot calculate $S^d(Q, \omega)$ directly, but we know some moments of this distribution, just as we did for the continuum model. As in the continuum, we restrict ourselves to $n = 0, 1$, and -1 .

A. Static correlation functions and the zeroth moment

We begin with the zeroth moment, or the static structure factor

$$I_0^d(Q) = \int_0^\infty S^d(Q, \omega) d\omega. \quad (66)$$

There is a remarkable theorem by Mehta and Mehta³² stating that the static correlation function is identical to that of the repulsive ($\beta = 4$) continuum model in real space. This object was also calculated independently for half filling in Ref. 33, and the result (in Q space) is for the half filled case:

$$I_0^d(Q) = -\frac{1}{2} \ln \left(1 - \frac{|Q|}{\pi} \right), \quad (67)$$

in the scheme where we restrict $|Q| \leq \pi$.

The correlation function I_0 is also available from Mehta and Mehta,³² for *arbitrary* densities $\hat{d} \leq \frac{1}{2}$. The density-density correlator can be written as

$$\langle \hat{\rho}_0 \hat{\rho}_r \rangle = \hat{d} \delta_{r,0} + (1 - \delta_{r,0}) \hat{d}^2 [1 + D(r)], \quad (68)$$

in a manner similar to Eq. (18). The function D is given³² explicitly in the thermodynamic limit as

will work in the reduced zone scheme $|Q| \leq \pi$, for which two cases may be distinguished, case (A) $\hat{d} \leq \frac{1}{4}$ and case (B) $\frac{1}{4} < \hat{d} \leq \frac{1}{2}$. The correlations are given for $Q \geq 0$, and may be obtained for negative Q by using the evenness in Q of I_0 . For case (A) $\hat{d} \leq \frac{1}{4}$ we find

$$I_0^d(Q) = 1 + \theta(4\pi \hat{d} - Q) A(Q), \quad (72)$$

$$A(Q) = \frac{Q}{4\pi \hat{d}} - 1 - \frac{Q}{8\pi \hat{d}} \ln \left| 1 - \frac{Q}{2\pi \hat{d}} \right|, \quad (73)$$

and for case (B) $\frac{1}{4} < \hat{d} \leq \frac{1}{2}$ we find

$$I_0^d(Q) = 1 + A(Q) + \theta(Q - 2\pi + 4\pi\hat{d}) A(2\pi - Q). \quad (74)$$

It can be checked for $\hat{d} = 1/2$ that Eq. (74) is identical with Eq. (67). For $\hat{d} \leq 1/4$, the expression in Eq. (73) is identical to the symplectic case Eq. (13), apart from a scale factor of \hat{d} . Equation (74) is in fact nothing but the umklapp reduction of the continuation of Eq. (73); i.e., Q is allowed to extend up to $4\hat{d}\pi$, and the part beyond π is declared to belong to $Q - 2\pi$, after subtracting unity from the structure function.

The correlation function, in scaled variables, is given by [compare Eq. (20)]

$$D(\hat{r}/\hat{d}) = \int_{-1/\hat{d}}^{1/\hat{d}} \frac{d\hat{Q}}{2} \exp(i\hat{Q}\pi\hat{r}) \left[I_0^d(\hat{Q}\pi\hat{d}) - 1 \right]. \quad (75)$$

The theorem of Mehta and Mehta asserts the equality of the scaled correlation functions for $\beta = 2, 4$ for all integer r , i.e., for $\hat{r} = \hat{d} \times \text{integer}$,

$$D(r \rightarrow \hat{r}/\hat{d}) = C(r \rightarrow \hat{r}/d). \quad (76)$$

$$I_1^d(Q) = \frac{\pi J}{2L} \sum_{|k| \leq \pi} \ln(1 - |k|/\pi) \sum_{r \neq 0} \left[\frac{\phi^2}{\sin^2(\phi r)} \right] [1 - \cos(kr)] \cos(kr). \quad (79)$$

Using the fact that, for $|k| \leq \pi$,

$$\begin{aligned} \sum_{r \neq 0} \left[\frac{\phi^2}{\sin^2(\phi r)} \right] \cos(kr) \\ = \frac{\pi^2}{3} (1 - 1/L^2) - \pi |k| \left(1 - \frac{|k|}{2\pi} \right), \end{aligned} \quad (80)$$

we find with $[k] \equiv (k - 2\pi m)$ (m integer $|| [k] | \leq \pi$)

$$\begin{aligned} I_1^d(Q) = \frac{\pi J}{4L} \sum_{|k| \leq \pi} \ln \left(1 - \frac{|k|}{\pi} \right) \left[|Q + k| + |Q - k| \right. \\ \left. - 2|k| - \frac{1}{2\pi} \{ [Q + k]^2 + [Q - k]^2 - 2k^2 \} \right]. \end{aligned} \quad (81)$$

After turning the sum to an integral, we can integrate the expression and find, for $|Q| < \pi$,

$$\begin{aligned} I_1^d(Q) = \frac{J\pi^2}{8} \left[1 - \left(1 - \frac{|Q|}{\pi} \right)^2 \right. \\ \left. + 2 \left(1 - \frac{|Q|}{\pi} \right)^2 \ln \left(1 - \frac{|Q|}{\pi} \right) \right]. \end{aligned} \quad (82)$$

In the limit $Q \rightarrow 0$, we see that $I_1^d(Q) \rightarrow JQ^2/4$ but for larger Q there is substantial departure from the pure quadratic behavior of the continuum models Eq. (8).

In the hydrodynamic limit the moment $I_{-1}(Q)$ can be obtained from the spin susceptibility χ_{spin} , which is known explicitly:⁹

B. Other moments

As opposed to the case when particles are in the continuum, the moment $I_1(Q)$ is not interaction independent for a system where the particles sit on a lattice: This is a well-known effect of the lattice systems with interaction.⁶ However, we can work out an expression for the $1/r^2$ spin chain, using the usual definition as a double commutator. In the remaining part we will assume that $\hat{d} = 1/2$ and write

$$I_1(Q) = \langle [[\hat{S}_Q^z, H], \hat{S}_{-Q}^z] \rangle. \quad (77)$$

Calculating the commutator, we find

$$\begin{aligned} I_1^d(Q) &= \frac{2}{L} \sum_{i,j} (-J_{i,j}) [1 - \cos\{Q(r_i - r_j)\}] \langle \hat{S}_i^z \hat{S}_j^z \rangle \\ &= -2J\phi^2 \sum_{r \neq 0} \frac{1}{\sin^2(\phi r)} [1 - \cos(Qr)] \langle \hat{S}_0^z \hat{S}_r^z \rangle. \end{aligned} \quad (78)$$

Using Eq. (67), we rewrite this as

$$\begin{aligned} I_{-1}^d(Q) \xrightarrow{Q \rightarrow 0} \left[L \left(\frac{\partial^2 E_0}{\partial \mathcal{M}^2} \right)_{\mathcal{M}=0} \right]^{-1} &= \frac{\chi_{\text{spin}}(Q \rightarrow 0)}{L} \\ &= \frac{1}{\pi^2 J}, \end{aligned} \quad (83)$$

where $\mathcal{M} = (\hat{N}_\uparrow - \hat{N}_\downarrow)/L$ is the magnetization and the operators $\hat{N}_{\uparrow\downarrow}$ count the number of up or down spins in the chain (notice that we have $\hat{N}_\uparrow + \hat{N}_\downarrow = L/2$).

With these moments at hand, we can define characteristic frequencies for the excitation spectrum, in the same way as we did before. The important point is that all those frequencies will have the same value as $Q \rightarrow 0$:

$$\frac{I_1^d(Q)}{I_0^d(Q)} \xrightarrow{Q \rightarrow 0} \frac{I_0^d(Q)}{I_{-1}^d(Q)} \xrightarrow{Q \rightarrow 0} \sqrt{\frac{I_1^d(Q)}{I_{-1}^d(Q)}} \xrightarrow{Q \rightarrow 0} J\pi \frac{|Q|}{2}. \quad (84)$$

This means that in the hydrodynamic limit the spectrum is exhausted by the excitations with dispersion relation $\omega = J\pi|Q|/2$.^{4,5} Although this might be taken as a common feature for continuum systems, this is not so in general, as we shall see below.

We see that, by choosing appropriate energy scale ($J \rightarrow \frac{16}{\pi^2}$), the $1/r^2$ discrete spin model moments map exactly onto the continuum symplectic ones for $Q \rightarrow 0$. The dimensionless moments are identical in this limit,

$$I_n^d(Q = \pi\bar{q}/2) \xrightarrow{Q \rightarrow 0} \frac{1}{k_F^n} I_n^s(q = k_F\bar{q}), \quad (85)$$

where the factor of $1/2$ arises from the different normalization in Eqs. (5) and (64). It would be very interesting to pursue the calculation of more moments at $q > 0$ for

both models to check whether the discrete one shares the same unusual characteristics of the continuum ($\beta = 4$) model excitation spectrum.

We now turn to the Bethe chain, which is defined by the Hamiltonian $H = J_B \sum_i \mathbf{S}_i \cdot \mathbf{S}_{i+1}$. The literature about this model is very extensive; here, for the sake of brevity, we only refer to contributions related to our discussion. For instance, Hohenberg and Brinkman²⁶ have studied this model and found that it shows no saturation at $Q \rightarrow 0$. Let us describe a simple argument favoring this conclusion. The moments I_1 and I_{-1} for this model are known,²⁶ but not I_0 :

$$I_1^B(Q) = \frac{J_B}{4} (2 \ln 2 - 1/2)(1 - \cos Q) \quad (86)$$

and

$$I_{-1}^B(Q) \xrightarrow{Q \rightarrow 0} \frac{1}{\pi^2 J_B}. \quad (87)$$

On the other hand, the spinon spectrum of the Bethe chain is known from Faddeev and Takhtajan's work:³⁴

$$\omega_{\text{sp}}(Q) = (\pi J_B / 2) \sin |Q| \quad (88)$$

(the same dispersion relation had been obtained long before by des Cloizeaux and Pearson,³⁵ but they concluded that the excitation had a threefold degeneracy, rather than fourfold, as shown in Ref. 34). Therefore, we could look for saturation by the spinon, and hence form the ratio

$$\lim_{Q \rightarrow 0} \left[\frac{1}{\omega_{\text{sp}}(Q)} \sqrt{\frac{I_1^B(Q)}{I_{-1}^B(Q)}} \right] \approx 1.08706. \quad (89)$$

Consequently, it is clear that the small Q behavior of $S(Q, \omega)$ is not exhausted by the spinons in the Bethe chain, unlike in the $1/r^2$ model, where it is.

VI. CONCLUSIONS

We have seen that the results of the calculation of Simons *et al.*¹⁶⁻¹⁸ for the dynamical structure function have a representation in terms of the Bethe quasiparticle-

quasihole energies. The representation obtained in this work has the character of two particle-hole pairs representing the bare density fluctuation. We should note, however, that this representation is far from unique, and for one thing one may add an arbitrary number of "zero pairs." Also, for example, we could decompose our two holes and a particle as three holes and two particles, by, e.g., forcing the momenta of two holes to coincide, and by suitably restricting the momenta. However, it is not possible to decompose the results into those of a single particle-hole pair; our representation in terms of two pairs appears to be minimal in some sense. The striking feature which underlines the dynamical structure factors presented in this paper is the truncation at very low orders of series like Eq. (42). Therefore, the excited states for systems with $\beta = 1, 2$, and 4 will always involve a small number of quasiparticle-quasihole pairs. This may not be true for other (arbitrary) values of the coupling constant.

We have also shown that the discrete model shares the property of saturation of the Feynman sum rule by the lowest mode as $q \rightarrow 0$. The static structure function obtained by a direct calculation³³ is shown to be consistent with the older calculation of Mehta's³² provided one interprets the weight outside the Brillouin zone appropriately by unklapping it. The first moment of the discrete model is obtained using the known result for the two-point static correlator, and shows interesting structure and departure from the first moment of the continuum model, and should provide a nontrivial check on the structure function of the discrete model.

We stress that the saturation of the structure function by the sound modes at small q is a general property characterizing this class of models.

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