Persistent currents in a one-dimensional ring for a disordered Hubbard model

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We consider a one-dimensional Hubbard model in the presence of disorder. We compute the charge stiffness for a mesoscopic ring as a function of the size L, which is a measure of the persistent currents. We find that for finite disorder the persistent currents of the system with repulsive interactions are larger than those of the system with attractive interactions. This counterintuitive result is due to the fact that local-density fluctuations are reduced in the presence of repulsive interactions.

I. INTRODUCTION

A hallmark of mesoscopic systems is the presence of persistent current upon application of an external flux.[1] Although a noninteracting electron theory is quite successful in describing qualitatively the features of such an effect,[2-5] it strongly underestimates the magnitude of the observed persistent currents compared to the observed experimental values.[6,7] One possible way out of this discrepancy would be to include the electron-electron interactions. In general the combined study of disorder and interactions proves difficult to tackle, so that one has to resort to various approximations. In three dimensions, Hartree-Fock-like calculations[8] or numerical perturbative calculations in interactions[9] tend to suggest that persistent currents are enhanced by the interactions. In one dimension, on the other hand, numerical and analytical techniques alike allow us to treat the interactions exactly enough so that, in principle, firm results can be obtained. Arguments based on level statistics[10,11] and on perturbative calculations[9] reach the same conclusions, namely, an increase of the persistent current with repulsive interactions. Surprisingly, however, various independent calculations performed on a spinless Hubbard model with nearest-neighbor interactions[12,13] or long-range Coulomb interactions[14] have reached the opposite conclusion, namely, that repulsive interactions are detrimental to persistent currents, which, therefore, casts doubts on the validity of the approximate calculations in higher dimensions.

Here, we consider the full problem of interacting electrons with spin, in the presence of disorder, and examine the persistent currents in such a system. We will mainly concentrate for the sake of simplicity on a purely local interaction (Hubbard model), but will also consider briefly finite range interactions. The interactions are treated exactly and we use a renormalization-group (RG) calculation[15,16] to take care of the disorder. We show analytically that the persistent currents are *enhanced* by the presence of repulsive interactions, and that the result of the

spinless model, although correct, was an artifact. As a by-product of the study, we also give a very simple formulation of the Bethe-ansatz equations for the pure attractive Hubbard model.

The plan of the paper is as follows. Section II relates the so-called charge stiffness to the persistent currents, and discusses various peculiarities that can occur for finite temperature. Section III introduces the model and treats the interactions using the bosonization procedure. This section is merely to fix the notations. Section IV computes the stiffness, using a renormalization-group calculation. Section V focuses on the case of attractive interactions, both with a large $-U$ expansion and using Bethe ansatz, and Sec. VI examines an extended (nearest-neighbors) Hubbard model. Finally, the conclusions of our study can be found in Sec. VII.

II. STIFFNESS VERSUS MEISSNER

Instead of computing the persistent currents, it is simpler on the so-called conductivity stiffness[17-19] D, which measures the strength of the Drude peak in a macroscopic system $\sigma(\omega) = D\delta(\omega) + \sigma_{\text{reg}}$. The stiffness D can be related to the change of the energy of the ground state of the system in presence of an external flux by

$$D = \frac{L}{2} \frac{d^2 E_0}{d\phi^2} \bigg|_{\phi=0}, \tag{1}$$

E_0 being the ground-state energy of a ring in a field. ϕ denotes the boundary angle $\phi = 2\pi f / f_0$, where f is the flux threading the ring and $f_0 = hc/e$ is the flux quantum. On the other hand, for a mesoscopic system, the persistent current measures the response to a finite flux by

$$J = L \frac{dE_0}{d\phi} \bigg|_{\phi}. \tag{2}$$

Therefore, the stiffness D provides a measure of the persistent currents for small (or close to a multiple of 2π)

flux since $J = 2D\phi$. Although the complete calculation of the persistent currents at finite flux is also possible for a one-dimensional interacting system, the calculation is more complicated in the presence of disorder, and the stiffness carries enough information for our present purposes.

In order to compute the persistent currents, one should, in principle, distinguish between an odd and even number of electrons. For an odd number of electrons the energy is minimum in zero external flux, and the number of right and left moving electrons is equal. For an even number of electrons, due to the $k = 0$ state, the number of right and left moving electrons cannot be equal, and the energy is not minimum at zero flux. If one has nondisordered noninteracting electrons, it is obvious that in that case the minimum of the energy corresponds to half a flux quantum. An external flux can be incorporated in the wave function by making the usual gauge transformation,

$$c_i \rightarrow e^{iAr_i} c_i , \tag{3}$$

where A is the vector potential. This transformation modifies the kinetic-energy term and absorbs an external flux at the cost of a twist in boundary conditions. Since this transformation is purely local and the interactions and disorder do depend on density only, it will not affect the interaction and disorder terms. Therefore, the persistent currents for a system with an even number of electrons, even in the presence of disorder and interactions, will be identical to those of a system with an odd number of electrons, with a shift corresponding to half a flux quantum. In the calculation of the stiffness, one implicitly assumes that the equilibrium state is a minimum of energy with zero current (equal number of right and left carriers). Therefore, the stiffness for the case of an even number of electrons, measures the persistent current produced as a function of $(\phi - \phi_0/2)$.

The above connection between the persistent currents and the stiffness is valid only at zero temperature. At finite temperature the persistent current is given by the derivative of the free energy. One could think that the stiffness would be given by the second derivative of the free energy. But such a quantity is the so-called Meissner fraction,

$$\rho_s = \frac{L}{2} \frac{d^2F}{d\phi^2} . \tag{4}$$

Although both of ρ_s and D are related to the current correlation function, they correspond to different limits. ρ_s being a thermodynamic quantity corresponds to the limit $\omega \rightarrow 0$ first and then to $q \rightarrow 0$, whereas D, which is related to transport, corresponds to the limits taken in the reverse order. For finite T the two quantities are distinct. For a macroscopic system ρ_s measures the superfluid density and will be zero for a nonsuperfluid system, whereas D can be nonzero if the system is perfectly conducting but not superconducting (e.g., free electrons in the absence of impurities). If one has a finite system, then ρ_s need not be zero, even if the system is not superconducting. In general, the Meissner fraction is

given by

$$\rho_s = \frac{L}{2} \frac{d^2F}{d\phi^2}\bigg|_{\phi=0} = \sum_n \left\langle n \left| \frac{\partial^2 H}{\partial \phi^2} \right| n \right\rangle e^{-\beta E_n}$$

$$+ \sum_{\alpha,\gamma,E_\alpha \neq E_\gamma} \frac{e^{-\beta E_\alpha} - e^{-\beta E_\gamma}}{E_\alpha - E_\gamma} \left| \left\langle \alpha \left| \frac{\partial H}{\partial \phi} \right| \gamma \right\rangle \right|^2$$

$$- \beta \sum_{\alpha,\gamma,E_\alpha = E_\gamma} \left| \left\langle \alpha \left| \frac{\partial H}{\partial \phi} \right| \gamma \right\rangle \right|^2 e^{-\beta E_\alpha} , \tag{5}$$

where n denotes a state of the system, H is the Hamiltonian, and E_n the energy of the eigenstates. We have separated out the contribution from the manifold of degenerate states, which are of special importance for charge transport. In (5) the limits $L \rightarrow \infty$ and $T \rightarrow 0$ do not commute in general. If one takes the limit $L \rightarrow \infty$ first, then ρ_s tends to zero unless the system is really superconducting, as can be checked by computing explicitly ρ_s for, e.g., free electrons. For a mesoscopic system, where L is finite, ρ_s will be finite and gives the slope of the persistent currents with the flux $J \sim \rho_s \phi$. One now has to distinguish whether we have an odd or even number of electrons, or more generally whether the ground state is not degenerate (odd number of electrons) or whether it has a twofold degeneracy (even number of electrons). In the first case, for small enough temperatures, the sums in (5) will be dominated by the ground state, other terms are suppressed by factors like $e^{-\beta\Delta}$, where Δ would be the gap between the ground state and the first excited states. Such a gap remains finite for a system of finite size. If the ground state is not degenerate, then the average value of the current in zero external flux is zero in the ground state. One has

$$\left\langle \phi \left| \frac{\partial H}{\partial \phi} \right| \phi \right\rangle = \langle \phi | J | \phi \rangle = 0 . \tag{6}$$

The last term in (5) vanishes and one finds a positive ρ_s, roughly temperature independent [dominated by the first term in (5)]. There is, therefore, a paramagnetic persistent current for small flux.

On the other hand, if the ground state has a twofold degeneracy, which occurs, e.g., for the case of an even number of electrons, then in each of the degenerate ground states ϕ_v, the current can (and will in general) be nonzero,

$$\left\langle \phi_v \left| \frac{\partial H}{\partial \phi} \right| \phi_v \right\rangle = \langle \phi_v | J | \phi_v \rangle \neq 0 . \tag{7}$$

In that case (5) is dominated by the last term, which gives rise to a Curie type behavior, $\rho_s \sim -1/T$. There will, therefore, be a diamagnetic persistent current, with a slope proportional to $1/T$.

All these properties are well known for free electrons (see, e.g., Ref. 3 and references therein), but the arguments presented here show that they are much more general and rest only on the degeneracy of the ground state and are valid for interacting electrons as well.

III. ONE-DIMENSIONAL HUBBARD MODEL: NOTATIONS

Only a short derivation will be given here in order to fix the notations. More details can be found in Refs. 20 and 21.

Let us consider, for example, the discrete 1D Hubbard model with L sites,

$$H = -t \sum_{\langle i,j \rangle, \sigma} c_{i,\sigma}^{\dagger} c_{j,\sigma} + U \sum_{i} n_{i,\uparrow} n_{i,\downarrow} , \qquad (8)$$

where $\langle \ \rangle$ stands for nearest neighbors. Using the well-known boson representation of fermion operators in one dimension the complete Hamiltonian (8) becomes, away from half filling[20,21]

$$H = H_{\rho} + H_{\sigma} + \frac{2g_{11}}{(2\pi\alpha)^2} \int dx \cos(\sqrt{8}\phi_{\sigma}(x)) , \qquad (9)$$

where H_{ρ} and h_{σ} are defined by

$$H_{\nu} = \frac{1}{2\pi} \int dx \left[(u_{\nu}K_{\nu})(\pi\Pi_{\nu})^2 + \left[\frac{u_{\nu}}{K_{\nu}} \right] (\partial_x \phi_{\nu})^2 \right] . \qquad (10)$$

Π and ϕ are canonically conjugate variables and α is a short-distance cutoff that can be identified with the lattice spacing. The ρ and σ parts of the Hamiltonian (9), respectively, describe the charge and spin degrees of freedom of the system. The g_{11} term is the scattering between electrons of opposite spins with an exchange of momentum of $2k_F$. The Hamiltonian (9) describes, in fact, the most general one-dimensional Hamiltonian with spin conserving interactions, provided that the proper values of the K and u parameters are used. K_{ρ} controls the charge excitations. $K_{\rho} > 1$ corresponds to dominant superconducting fluctuations, whereas $K_{\rho} < 1$ corresponds to charge- or spin-density wave (depending on the spin part of the Hamiltonian) dominant fluctuations.

For the Hubbard Hamiltonian (8) the various coefficients in (9) and (10) are given, if U is small compared to v_F, by

$$u_{\rho}K_{\rho} = u_{\sigma}K_{\sigma} = v_F , \quad u_{\rho}/K_{\rho} = v_F + U/\pi , \qquad (11)$$

$$u_{\sigma}/K_{\sigma} = v_F - U/\pi , \quad g_{11} = U . \qquad (12)$$

For a macroscopic system, g_{11} renormalizes to zero and K_{σ} to one for repulsive interactions. In the asymptotic limit u_{ρ}, u_{σ}, K_{ρ} are the only parameters needed to describe the long-range properties and have been computed exactly using Bethe ansatz.[22] For attractive interactions there is no gap in the spin sector, and only u_{ρ} and K_{ρ} are needed to describe the low-energy properties of the model. They can also be computed from Bethe ansatz[23] as will be seen in more details in Sec. V. The charge stiffness D can be obtained[22,24] from the bosonized Hamiltonian (9), and is simply given by $D = 2u_{\rho}K_{\rho}$.

IV. EFFECT OF DISORDER ON $D(L)$

Disorder can be added to (9) by

$$H_{\text{dis}} = \int dx \, W(x)\rho(x) , \qquad (13)$$

where W is a random potential. As is well known, the forward scattering due to the potential does not affect the conductivity in one dimension (see, e.g., Ref. 18) and one can retain only the $2k_F$ Fourier components of the random potential. Higher Fourier components are less effective since they do not scatter electrons on the Fermi surface, and do not correspond to the low-energy processes. A notable exception are $4k_F$ components that will be discussed later. When expressed in terms of the boson variables, (13) becomes

$$H_{\text{dis}} - \int dx \, \zeta(x) e^{i\sqrt{2}\phi_{\rho}(x)} \cos(\sqrt{2}\phi_{\sigma}(x)) + \text{H.c.} , \qquad (14)$$

where $\xi(x)$ is a complex random potential corresponding to the part of the random potential having Fourier components close to $2k_F$. For simplicity, we will take ζ Gaussian,

$$\langle \zeta(x)\zeta^*(x') \rangle = W_{\zeta}\delta(x - x') . \qquad (15)$$

The effect of (14) can be computed using a renormalization-group calculation,[15,16] where one changes the cutoff (lattice spacing) α into αe^l. We will just quote the results here,

$$\frac{dK_{\rho}(l)}{dl} = -\frac{1}{2} \left[\frac{K_{\rho}^2 u_{\rho}}{u_{\sigma}} \right] \Delta(l) , \qquad (16)$$

$$\frac{dK_{\sigma}(l)}{dl} = -\frac{1}{2}[\Delta(l) + y(l)^2]K_{\sigma}^2 , \qquad (17)$$

$$\frac{du_{\rho}(l)}{dl} = -\frac{u_{\rho}^2 K_{\rho}}{2u_{\sigma}}\Delta(l) , \qquad (18)$$

$$\frac{du_{\sigma}(l)}{dl} = -\frac{u_{\sigma}K_{\sigma}}{2}\Delta(l) , \qquad (19)$$

$$\frac{dy(l)}{dl} = [2 - 2K_{\sigma}(l)]y(l) - \Delta(l) , \qquad (20)$$

$$\frac{d\Delta(l)}{dl} = [3 - K_{\rho}(l) - K_{\sigma}(l) - y(l)]\Delta(l) \qquad (21)$$

with the dimensionless quantities

$$\Delta = \frac{2W_{\zeta}\alpha}{\pi u_{\sigma}^2} \left[\frac{u_{\sigma}}{u_{\rho}} \right]^{K_{\rho}} , \quad y = g_{11}/(\pi u_{\sigma}) . \qquad (22)$$

The renormalizations of K_{ν} and u_{ν} are of first order in Δ and y^2, and, consequently, can be neglected on the right sides of the first three equations. Contrary to the pure case, charge and spin degrees of freedom are now no longer decoupled.

For a macroscopic system, the physics implied by the equations (16)–(21) has been studied at length.[15,16] As can be seen from (21), there are two regimes depending on the initial values of K_{ρ}, K_{σ}, y. Δ can scale to zero, the system is in that case delocalized, and as shown in Refs. 15 and 16 is dominated by divergent superconducting fluctuations. In the other regime Δ scales to large values, and the corresponding phase can be identified with a localized phase. In that case, the RG equations break down below a certain length scale that can be identified with the localization length. In the limit where $\Delta \rightarrow 0$, as can be seen from (21), the localized-delocalized transition

occurs when $2-K_\rho=0$, if $y>0$ (since y renormalizes to zero and K_σ to one) or $3-K_\rho=0$ if $y<0$. For a mesoscopic system of size L, one can expect the size to play the role of an infrared cutoff in the RG equations. When the renormalized cutoff $\alpha e^{l^*}\sim L$, i.e., $l^*=\ln(L/\alpha)$, one can treat the disorder term in perturbation, and the stiffness is, therefore, simply given by the quadratic part of the Hamiltonian, with the renormalized coefficients $D=2u(l^*)K(l^*)$. Provided the size L is smaller than the length at which the equations cease to be valid, one can use them to compute the various values of K, u, y as function of the size L of the system, and from that to obtain the stiffness D. Such a calculation is similar to the one performed for a macroscopic system to get the exponent K_ρ (Ref. 16) at finite temperatures. In that case, the cutoff is provided by the dephasing length v_F/T.

The full dependence of the stiffness on the size of the sample needs a numerical integration of the RG equations, but the qualitative features can be understood by looking at the very small disorder limit. In that case, one can neglect the renormalization of u, K, y in the equation for Δ, which gives

$$\Delta=\Delta_0 e^{(3-K_\rho-K_\sigma-g_{1\perp})\ln(L/a)} , \qquad (23)$$

where Δ_0 is the initial value of the disorder. Here, we focus on the case of the Hubbard model. For small U one can use the values of the parameters (11), and one gets

$$\Delta=\Delta_0(L/a)^{1-U/(\pi v_F)} . \qquad (24)$$

Therefore, the disorder grows more slowly for the repulsive Hubbard model than in the attractive one. In the same limit of a very small initial Δ, the stiffness is roughly given by

$$D[l^*=\ln(L/\alpha)]=D(l=0)-\text{const}\times\int_{l=0}^{l=l^*}\Delta(l) . \qquad (25)$$

The bare stiffness $D(l=0)$ can be considered roughly independent of the interactions if the system is far enough from half filling as can be seen from (11). The dependence in U comes only from lattice effects[18,22,25] that break Galilean invariance and are sensitive, for repulsive interactions, only for fillings close to a commensurate filling (mainly half filling where the pure system would be a Mott insulator). For attractive interactions, the renormalization of the stiffness of the pure system due to interactions will become much more important and will be discussed in Sec. V. We will ignore, in the following, the change of the bare stiffness due to the interactions and will only focus on the effects due to the disorder. As can be seen from (25), the disorder term tends to drastically reduce the stiffness D, and this effect will be smaller for the repulsive model than the attractive one and the stiffness (the persistent currents) will be *enhanced* by repulsive interactions for a given size and a given disorder. Note that in our calculation the interactions re treated exactly and all the effects of density and superconducting fluctuations are included in (23).

For finite disorder one has to numerically integrate the RG equations. The result is shown in Fig. 1 and is in agreement with the simplified analysis above. This rather
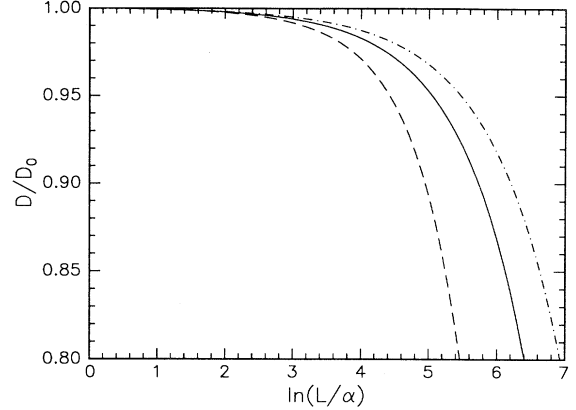

FIG. 1. Normalized stiffness D/D_0 as a function of the size of the system (in units of the lattice spacing α), obtained by numerically integrating the RG equations (16)–(21). D_0 is the stiffness in the absence of disorder. All energies are in units of the original Fermi velocity v_F. The disorder W_ξ/v_F is fixed to $W/v_F=5\times10^{-4}$. The full line is $U/v_F=0$, the dotted line $U/v_F=-0.5$, and the dash-dotted line $U/v_F=0.5$. For a given size L, systems with repulsive interactions have a larger stiffness than those with attractive ones.

counterintuitive result can be simply understood with the following argument: with the repulsive Hubbard model, the ground state is almost a spin-density wave (with a power-law decay of the correlation functions) whose density is uniform. Such a ground state couples very weakly to nonmagnetic impurities as is obvious from (13). To couple to disorder, one has to distort the spin-density wave and make a fluctuation of the density, a process that will cost an energy increasing with U. The disorder effect is, therefore, very weak, at least if the size of the system is not too big. On the other hand, the attractive Hubbard model has a ground state that contains charge-density wave fluctuations (although superconductive fluctuations are the dominant ones), which can get very easily pinned by impurities. On such a ground state, the disorder will act very efficiently and drastically reduce the stiffness compared to the pure value, therefore making the persistent currents smaller. Such an argument is in agreement with higher dimensions.[26]

This is to be contrasted with a previously studied spinless model.[12–14] In that case, both the attractive and repulsive ground states have density fluctuations, and both can be equally well pinned by disorder. Since in the attractive case the superconducting fluctuations tend to screen the disorder, the stiffness increases with attractive interactions. For the interactions to have a beneficent effect on the persistent currents, one must necessarily take a realistic model in which the main effect of the interactions will be to homogenize the density as is the case for the Hubbard model. That there are these two competing effects of interactions can also be seen from (23) and (11). Repulsive interactions would tend to reduce K_ρ, and therefore decrease the persistent currents. This corresponds to the fact that superconducting fluctuations are killed by repulsive interactions (and simultaneously asymptotic density fluctuations are enhanced). This is

the only effect present for a spinless model. But for a model with spin, repulsive interactions make K_σ and $g_{1\perp}$ larger, traducing the fact that due to the repulsion, spin up and spin down do not want to be at the same position in space, therefore making the short-distance density fluctuations smaller and the system harder to pin by disorder, which increases the persistent currents. These two competing effects should be present in higher dimensions as well.

If the interactions are infinitely repulsive, the system becomes equivalent to a model of spinless fermion with a Fermi momentum of $2k_F$. In that case, although the $2k_F$ component of the disorder is inefficient (as is also obvious from the fact that it will no longer correspond to a process on the new Fermi surface), one should worry about the $4k_F$ component of the disorder. Such a Fourier component acts on the free spinless fermion, so that one recovers the stiffness of free electrons in the presence of disorder. The crossover between the two regimes would need a detailed analysis of the coupling of the $4k_F$ component of the charge density to disorder, which is beyond the scope of this paper, but one could naively expect a maximum of the persistent currents for an intermediate value of the interactions.

If the size of the system becomes large enough, the disorder will renormalize to large values and the system will be localized. This is always the case for repulsive interactions.[27-29,15,16] For attractive interactions, a localized-delocalized transition is, in principle, possible[27-29,15,16] (for $K_\rho > 3$), and the stiffness could saturate to a finite value. We will show in the following section that for the particular case of the attractive Hubbard model, where one has only an on-site attraction, this transition does not occur and the system remains always localized.

V. NEGATIVE U

Let us consider the case of a large negative U. In that case, one would naively imagine that the system should delocalize. In fact a very large on-site attraction cannot delocalize, and *increases* the localization (and, therefore, decreases the persistent currents). In the $U \to -\infty$ limit, one can perform a large $|U|$ expansion. Only pairs of particles can hop and if one introduces the operators $b_i = c_{i,\uparrow} c_{i,\downarrow}$, the attractive Hubbard model then becomes a model of hard-core bosons with a hopping $t' = t^2/|U|$ and a disorder $\Delta' = \Delta$. The residual interaction between the bosons is also in the scale of t^2/U. In fact, using the superexchange formulation or degenerate perturbation theory, the model maps on precisely to the one-dimensional Heisenberg antiferromagnet at a fixed magnetization (related to the density of particles), with an exchange energy $4t^2/U$. By a Jordan Wigner transformation this model corresponds to spinless fermion with a narrow bandwidth and with nearest-neighbors interaction, in the presence of the old disorder. Such a system is obviously localized, and since the kinetic energy reduces with $|U|$, one expects length to diminish when the attraction is increased.

One can make the statements more quantitative for finite U by studying the RG equations in the attractive regime. In that case, it is well known that there is a gap in the spin excitation spectrum and that only the charge sector remains ungapped. Keeping only the charge excitations into account, the RG equations become[15,16]

$$\frac{dK_\rho(l)}{dl} = -\frac{1}{2}K_\rho^2 \Delta(l) , \tag{26}$$

$$\frac{du_\rho(l)}{dl} = -\frac{u_\rho^2 K_\rho}{2}\Delta(l) , \tag{27}$$

$$\frac{\Delta(l)}{dl} = [3 - K_\rho(l)]\Delta(l) , \tag{28}$$

with

$$\Delta(l) = (2C_\sigma W_\xi \alpha)/(\pi u_\rho^2) , \tag{29}$$

and C_σ is a constant of order unity coming from the ϕ_σ correlations in the perturbation expansion. Equation (16) can be used at scales above the size ξ of a Cooper pair. This approach will, therefore, be adapted for reasonably large U. For small U, it will be better to use Eqs. (16)–(21), the crossover between the two regimes occurring when $y \sim 1$.

In order to get the stiffness of the disordered system, one needs the initial values of K_ρ and u_ρ in the absence of disorder as a function of the attraction U. As for the repulsive Hubbard model, they can be deduced from the Bethe-ansatz solution.[23] Here, we will give a derivation based on an appealing formulation introduced by Sutherland (in the absence of ϕ), involving the formation of Cooper pairs which scatter without diffraction. Let the number of particles be $2M$, for which case we note the Bethe equations for the attractive U Hubbard model, with energy $E = -4\sum_{j=1,M} \cos(P_j/2)\cosh(Q_j)$, where $(P_j, Q_j) = (Re, Im)\arcsin[\psi_j + i(U/4)]$, and ψ_j satisfy the Bethe equations,[30]

$$LP(\psi_j)/2 = 2\pi J_j + 2\phi + \sum_{k=1,M} \arctan\left[\frac{2}{U}(\psi_i - \psi_j)\right] , \tag{30}$$

where J_j are integers (half odd integers) for M odd (even). The flux ϕ comes in with a factor of 2, due to the charge of the Cooper pair. These equations can also be obtained from the repulsive case[31,18] by using a particle-hole transformation on the up electrons of a half filled model, the spin excitations then map on to the above equations. This can be checked explicitly, using the idea of complementary solutions due to Woynarovich,[32] which essentially rests on the recognition that the equations for real k's in the Bethe equations of the repulsive Hubbard model are the L real zeros of a polynomial of degree $L + 2M$. Hence, the residue theorem of Cauchy helps in transforming equations involving the real k's to those over complex k's. The complex k's come in pairs, and are essentially pinned to be $\psi_j \pm i(U/4)$, in order to satisfy the growth conditions. The error involved in writing down the above Cooper pair representation is of $O(\exp(-L/\xi(U)))$, with $\xi(U)$ the Cooper pair radius.

The parameters u_ρ and K_ρ can be obtained by computing the compressibility $\chi = u_\rho/K_\rho$ and the charge
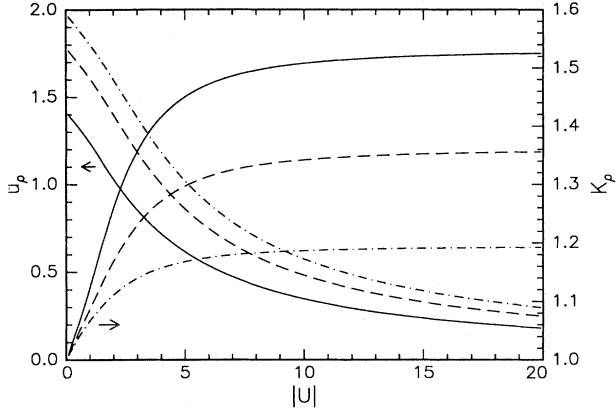
FIG. 2. Values of u_ρ and K_ρ for the attractive Hubbard model as a function of the strength of the interaction $|U|$. These values are obtained by numerical integration of the Bethe-ansatz equations for systems of $L = 200$ sites with, respectively, 90, 70, 50 particles per spin for the full, dotted, and dash-dotted lines. This corresponds to density of $n = 0.9, 0.7$, and 0.5 particles per sites, respectively.

stiffness $D = 2u_\rho K_\rho$ of the pure system from the Bethe-ansatz ground-state energy. Various values of u_ρ and K_ρ are plotted in Fig. 2, together with the stiffness D (for the pure system). As can be guessed from the large U expansion, $u_\rho \to 0$ at large U. The fact that the parameter K_ρ remains finite shows that the system remains interacting. One can check that the limiting value of K_ρ is in agreement with the one obtained for the XXZ chain[33] on which this system maps in the large U limit. Due to the reduction of the velocity, the stiffness of the clean system itself
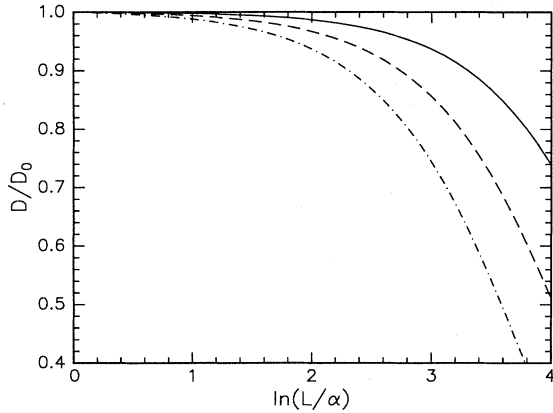

FIG. 3. Normalized stiffness D/D_0 for the attractive Hubbard model as a function of the size of the system (in units of the lattice spacing α) obtained by numerically integrating the RG equations (26)–(28). D_0 is the stiffness in the absence of disorder. The effective disorder $C_\sigma W_\xi$ is fixed to $C_\sigma W_\xi = 5 \times 10^{-4}$ and the density is $n = 0.5$ particles per site. The full, dotted, and dash-dotted lines correspond, respectively, to $|U| = 5, 10$, and 15. The corresponding bare stiffness are, respectively, $D_0 = 1.78, 1.04$, and 0.72. Here again, for a given size L and fixed disorder, the stiffness decreases with increasing attraction.

goes to zero at large U. Conversely to what happened for repulsive interactions where the stiffness of the pure system was nearly interactions independent, here, there is a drastic *reduction* of the stiffness when the attraction is increased.[23]

Using equation (16), one computes the stiffness in the presence of disorder. Here, the main effect is the decrease of the velocity u_ρ, which increases the relative strength of the disorder given by the dimensionless parameter (29). Some results are shown in Fig. 3, where we have normalized the stiffness to its value in the absence of disorder to avoid the trivial effect of renormalization of the bare stiffness by attractive interactions. In agreement with the previous section, the reduction of the stiffness due to disorder becomes more and more important as the attraction U is increased.

VI. EXTENDED HUBBARD MODEL

In order to check the validity of the arguments presented here for a slightly more general model than the Hubbard model, we also look at an extended Hubbard model with a nearest-neighbor interaction V, defined by

$$V\sum_i n_i n_{i+1} \ . \tag{31}$$

In that case, and for small U and V, the various parameters entering the equations are

$$K_\rho \simeq 1 - \frac{U}{2\pi v_F} - \frac{V}{\pi v_F}[2 - \cos(2k_F a)] \ , \tag{32}$$

$$K_\sigma \simeq 1 + \frac{U}{2\pi u_F} + \frac{V}{\pi v_F}\cos(2k_F a) \ , \tag{33}$$

$$y_\perp \simeq \frac{U}{\pi v_F} + \frac{V}{\pi v_F}\cos(2k_F a) \ . \tag{34}$$

For small U and V, the renormalization of velocities is of the second order in U, V and can be neglected. When replaced in Eqs. (21), one gets

$$(3 - K_\rho - K_\sigma - y) = 1 - \frac{U}{2\pi v_F} + \frac{2V}{\pi v_F}[1 - 2\cos(2k_F a)] \ . \tag{35}$$

If $U \gg V$, the results are unchanged compared to the case of the pure Hubbard model. In order to check whether the physical ideas introduced here on the increase of the persistent current due to repulsive interactions are correct, or whether they are an artifact of the purely local Hubbard model, one can consider the artificial limit where $U = 0$ and V remains finite. Note that the model does not boil down in that case to the spinless fermion model since the V term still introduces interactions among opposite spins.

In that case, the effect of the interaction depends on the filling. For low filling, a repulsive V will tend to favor a spin-density wave ground state again, whereas an attractive one would tend to pair particles on neighboring sites, giving a modulation of the density. For large fillings, the situation changes: a positive V will now tend to favor two particles on the same site, to avoid paying the repulsion, and, therefore, to give a charge-density

wave. An attractive V favoring two particles on a neighboring site will this time give a spin-density wave. The change between a spin-density wave towards charge-density wave ground state occurs when $K_\sigma = 1$, and as can be seen from (33), this will occur when $k_F = \pi/4$. Added to this is the competing effect that the more attractive V we have, the more there are superconducting fluctuations in the system which tend to reduce the disorder. Above quarter filling, the two effects go hand in hand and repulsive interactions are detrimental to the stiffness, whereas below quarter filling, the two effects will compete. From (35) one can see that the point where a repulsive V again becomes favorable to the stiffness is $k_F = \pi/3$. Below this filling, the fluctuations of the density generated by a repulsive V are too strong to be balanced by the superconducting fluctuations, and a positive V will increase the persistent currents.

VII. CONCLUSIONS

We have looked, in this paper, at the stiffness constant of a Hubbard model as a function of the size of the system. The stiffness constant is directly related to the persistent currents in the presence of an external flux by $J = D\phi$ for small flux. We have shown that both the attractive and repulsive Hubbard model are always localized for a macroscopic system regardless of the strength of the interactions. In fact, using the Bethe-ansatz solution or for large $-U$ simple perturbation theory, one shows that the localization length decreases for attractive interactions due to the reduction of charge velocity.

For a mesoscopic system, the stiffness in the repulsive Hubbard model is much less sensitive to disorder than for the attractive one. Therefore, the persistent currents are *enhanced* by repulsive interactions. This surprising result is related to the fact that for the attractive Hubbard model, the ground state contains strong charge-density wave fluctuations that pins easily on the impurities, whereas repulsion favors a uniform density and makes the pinning harder. This property remains valid for a model with longer-range interactions. In general, the effects of the interactions on the persistent currents is controlled by two competing effects. One is the presence of density fluctuations in the ground state. The more there will be, the more easily the system will be pinned by disorder and the more the persistent currents will be reduced compared to the pure value. In general, repulsive interactions will tend to favor a homogeneous density (local fluctuations in density will cost an energy increase with the repulsion), and, therefore, will tend to increase the persistent currents. On the other hand, attractive interactions promote superconducting fluctuations in the system that tend to screen the disorder and therefore tend to increase the persistent currents.

Previous studies of one-dimensional systems, leading to the conclusion that repulsive interactions reduced persistent currents, were performed on a spinless model. In such a rather artificial model, the first effect does not occur, since density fluctuations are always present both for attractive and repulsive interactions and, therefore, repulsive interactions are detrimental to persistent currents. In a more realistic model where the local interactions are the dominant ones (interactions in a real system do decrease with distance), the density effect will dominate and the persistent current are increased.

This study is, strictly speaking, restricted to one-dimensional systems, and a direct comparison of our results with experimental, three-dimensional, rings is not feasible. It nevertheless suggests that in mesoscopic systems, the presence of repulsive interactions can considerably enhance the persistent currents, and confirms, in the exactly solvable one-dimensional case, that the increase of the persistent currents is linked to a reduction of the local-density fluctuations by the repulsive interactions. It is, therefore, tempting to ascribe the discrepancies observed between the measured and the computed (with a free-electron theory) values of the persistent currents to such an interaction effect.

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