A New Class of Exactly Solvable Interacting Fermion Models in One Dimension

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We investigate a model containing two species of one-dimensional fermions interacting via a gauge field determined by the positions of all particles of the opposite species. The model can be solved exactly via a simple unitary transformation. Nevertheless, correlation functions exhibit nontrivial interaction-dependent exponents. A similar model defined on a lattice is introduced and solved. Various generalizations, e.g., to the case of internal symmetries of the fermions, are discussed. The present treatment also clarifies certain aspects of Luttinger's original solution of the "Luttinger model." [S0031-9007(98)05437-4]

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Exactly solvable models [1–4] have played an important role in the current understanding of one-dimensional interacting many-particle systems. These, together with the idea of dominant low-energy bosonic excitations of Fermi systems [5], gave rise to the emergence of the "Luttinger liquid" [6] as a unifying concept. Nevertheless, the technicalities of these exact solutions (bosonization, Bethe ansatz) are often rather complex. In the present paper we wish to introduce a class of interacting models which can be diagonalized by a simple (pseudo-)unitary transformation, yet exhibit nontrivial Luttinger-liquid behavior. The models can be defined both in the continuum and on a lattice, and can have rather arbitrary single-particle band structure. Only the interactions are constrained to be of a particular "gauge form." The long-distance asymptotics of correlation functions can then be determined exactly. Our investigation was inspired by Luttinger's original treatment of the "Luttinger model," and we will comment on this connection further below.

We start by considering the simplest model in our class, a one-dimensional fermion model with two species of particles, designated by a pseudospin index $\sigma = \pm$, having coordinates $x_{\sigma i}$ and momenta $p_{\sigma i} = -i\partial_{x_{\sigma i}}$. The Hamiltonian of our model then is

$$H = \frac{1}{2}\sum_{\sigma i} \Pi_{\sigma i}^2 , \tag{1}$$

where we have introduced a "covariant momentum" $\Pi_{\sigma i} = p_{\sigma i} + \sigma A_\sigma(x_{\sigma i})$, i.e., in this model, *particles interact via a gauge potential, given for a particle at* x *by* $A_\sigma(x) = \sum_j V(x - x_{-\sigma j})$. The potential V is an even function. On a ring of length L, we will assume that V is periodic. Clearly, the Hamiltonian is not time-reversal invariant, but it is invariant under simultaneous time reversal and charge (σ) conjugation.

The model can now be straightforwardly diagonalized by a (in general pseudo-)unitary transformation: noting that

$$e^{iS} p_{+i} e^{-iS} = p_{+i} - \partial_{x_{+i}} S , \tag{2}$$

one can chose $S \equiv S(\{x_{+i}\}, \{x_{-j}\})$ so as to eliminate the interaction in Eq. (1) by

$$S(\{x_{+i}\}, \{x_{-j}\}) = \sum_{i,j} E(x_{+i} - x_{-j}) , \tag{3}$$

where E is the indefinite integral of the interaction potential:

$$E(x) = \int_0^x dx' \, V(x') . \tag{4}$$

The transformed Hamiltonian then takes the form

$$\tilde{H} = e^{iS} H e^{-iS} = \frac{1}{2}\sum_{\sigma i} p_{\sigma i}^2 . \tag{5}$$

The eigenfunctions of $\tilde{H}$ clearly are Slater determinants of plane wave states $|\{k_{-,i}\}, \{k_{+,j}\}\rangle$, characterized by the sets of wave numbers $\{k_{-,i}\}$ and $\{k_{+,j}\}$ for the $-$ and $+$ particles, respectively. Consequently, the eigenfunctions and eigenvalues of the original Hamiltonian are obtained straightforwardly:

$$H e^{-iS} |\{k_{-,i}\}, \{k_{+,j}\}\rangle = \frac{1}{2}\sum_{\sigma i} k_{\sigma i}^2 e^{-iS} |\{k_{-,i}\}, \{k_{+,j}\}\rangle . \tag{6}$$

At first sight it thus appears that the spectrum of the interacting Hamiltonian is independent of the interaction. Conformal field theory, or equivalently Luttinger liquid theory, then would imply that the asymptotic form of correlation functions (which is directly determined by the eigenvalue spectrum) is also interaction-independent. This conclusion is, however, incorrect: periodic boundary conditions have to be treated carefully. In fact, keeping all other coordinates fixed, one easily finds $S(x_{-,i} = L) - S(x_{-,i} = 0) = -N_+\delta$, where L is the length over

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which periodic boundary conditions are applied, N_+ is the total number of + particles, and the *phase shift* δ is given by

$$\delta = \int_0^L dx\, V(x)\,. \tag{7}$$

An analogous result, with N_+ and δ replaced by N_- and $-\delta$, holds for the phase shift of the + particles. Consequently, the quantization condition on the wave numbers is given by

$$Lk_{\pm,i} \mp N_\mp \delta = 2\pi n_{\pm,i}\,, \tag{8}$$

where the $n_{\pm,i}$ are integer quantum numbers analogous to those used in the noninteracting case. Clearly, particles of one given "spin" orientation give rise to an effective Aharonov-Bohm flux acting on the other species, the value of the flux depending on the number of particles present. It should now also be clear why we refer to the transformation Eq. (2) as *pseudo*unitary: unless δ is "accidentally" an integer multiple of 2π, the plane wave states of the interacting and noninteracting problems obey different boundary conditions and therefore define different Hilbert spaces.

The ground state energy E_0 can be found in any sector with $N_\pm$ particles, as follows. In order to minimize E_0 we must choose $n_{\pm,i} = n^0_{\pm,i} \pm [[\frac{\delta}{2\pi} N_\mp]]_{\rm int}$, where $n^0_{\pm,i}$ are the quantum numbers in the absence of the interaction [7]. The change in energy due to the interaction, $\delta E = \frac{2\pi^2}{L^2}\{N_+[[\frac{N_-\delta}{2\pi}]]^2_{\rm rem} + N_-[[\frac{N_+\delta}{2\pi}]]^2_{\rm rem}\}$, is not extensive, but on the scale expected from a magnetic field applied to the ring.

An effective Luttinger liquid description in terms of a bosonic field theory for the low-energy properties can be obtained from the low-energy excited states [6]. To be precise, we start from a ground state with $N_{\pm 0} = 2n_0 + 1$ and assume that $N_{\pm 0}\delta$ is an integer multiple of 2π. We now add $n_{\pm R}$ ($n_{\pm L}$) particles at the right (left) Fermi points of the $\pm$ particles. Introducing particle number and current quantum numbers $N_\pm = n_{\pm R} + n_{\pm L}$ and $J_\pm = n_{\pm R} - n_{\pm L}$ the second order variation of the ground state energy is

$$\begin{aligned} E^{(2)} = \frac{1}{2L^2}(2n_0 + 1)[&(\pi^2 + \delta^2)(N_+^2 + N_-^2) \\ &+ \pi^2(J_+^2 + J_-^2) \\ &+ 2\pi\delta(J_+N_- - J_-N_+)]\,. \end{aligned} \tag{9}$$

Up to quantum fluctuations, $N_\pm$ and $J_\pm$ are related to bosonic fields and their conjugate momentum density via $N_\pm = -(L/\pi)\partial_x\phi_\pm$ and $J_\pm = L\Pi_\pm$. The effective Hamiltonian including the low-energy quantum fluctuations then takes the form

$$\begin{aligned} H = \frac{n}{4}\int dx\,\{&[1 + (\delta/\pi)^2][(\partial_x\phi_+)^2 + (\partial_x\phi_-)^2] \\ &+ \pi^2[\Pi_+^2 + \Pi_-^2] \\ &+ 2\delta[\Pi_-\partial_x\phi_+ - \Pi_+\partial_x\phi_-]\}, \end{aligned} \tag{10}$$

where $n = 4n_0/L$ is the particle density. Introducing new variables

$$\tilde\phi_\pm = \phi_\pm, \qquad \tilde\Pi_\pm = \Pi_\pm \mp \frac{\delta}{\pi^2}\,\partial_x\phi_\mp \tag{11}$$

the Hamiltonian takes an apparently noninteracting form [Eq. (10) with $\delta = 0$]. However, the expression of single-fermion operators [6] is changed and therefore the asymptotic decay law of the single particle Green's function is obtained as

$$G_{R\pm}(x) = \langle\psi_{R\pm}(x)\psi^\dagger_{R\pm}(0)\rangle \approx e^{i(k_F \pm N_{\mp 0}\delta)x}x^{-1-\alpha}, \tag{12}$$

with $\alpha = \delta^2/(2\pi^2)$. Thus, for any nonvanishing δ the decay is faster than $1/x$, leading, among other things, to the well-known power-law singularity of the momentum distribution function at k_F. The correctness of Eq. (12) can be checked independently using the eigenfunctions of Eq. (6): One obtains a Töplitz determinant of the form previously considered by Luttinger [2], and which has the same asymptotic power law as obtained by the bosonization approach.

Similarly, correlations of two-particle operators decay as $x^{-\eta}$, with interaction dependent exponent η. Specifically:

$$\psi^\dagger_{R\pm}\psi_{L\pm} \Rightarrow \eta_1 = 2\,, \tag{13}$$

$$\psi^\dagger_{R\pm}\psi_{L\mp} \Rightarrow \eta_2 = 1 + (1 \mp \delta/\pi)^2\,, \tag{14}$$

$$\psi_{R\pm}\psi_{L\pm} \Rightarrow \eta_3 = 2 + 2(\delta/\pi)^2\,, \tag{15}$$

$$\psi_{R\pm}\psi_{L\mp} \Rightarrow \eta_4 = 1 + (1 \pm \delta/\pi)^2\,, \tag{16}$$

The most slowly decaying correlations identify the dominant incipient instabilities. In the spin language, for positive δ then *spiral* spin-density wave correlations and opposite-spin Cooper pairing correlations with one fixed spin orientation ($\uparrow\downarrow$ and $\downarrow\uparrow$ are *not* degenerate) are favored, whereas for negative δ correlations with reversed spin orientations dominate. Adding a density-density interaction between the two spin orientations, the degeneracy between pairing and spin-density wave correlations is lifted. The density correlations, Eq. (13), are not affected by the interactions because they are diagonal elements of the density matrix which themselves are unchanged by the unitary transformation, Eq. (2). We notice that the exponent for pairing correlations with equal pseudospin, Eq. (15), is just twice the exponent of the single-particle Green's function, i.e., there are no singular vertex corrections in this particular two-particle correlation function. Finally, from Eq. (8) it is clear that the value of δ is relevant only modulo 2π. Consequently, the results (12) to (16) are valid only for $|\delta| \le \pi$. Outside this interval δ has to be taken modulo 2π. We note that *the scaling relations between the different exponents in Eqs. (12)–(16) are different from those of standard fermionic Luttinger liquids* because of the presence of time-reversal breaking terms in the Hamiltonian.

We can note here that the pseudounitary transformation translates into a "Jastrow"-like phase factor, of which the

"projection" counterpart, i.e., a Jastrow factor without the "i", is well known in the literature as leading to non-Fermi liquid behavior if used as a variational (not exact) wave function [8].

We can now comment on Luttinger's original solution of his model [2]. In first-quantized form his Hamiltonian contains only first derivatives:

$$H_{\rm Lutt} = \sum_{\sigma i} \sigma \Pi_{\sigma i} = \sum_i p_{+i} - \sum_j p_{-j} + 2 \sum_{ij} V(x_{-j} - x_{+i}) . \quad (17)$$

We first remark that this Hamiltonian is a conserved quantity as far as the Hamiltonian (1) is conserved, and shares all (nondegenerate) eigenfunctions. It is unbounded from below, though, unlike H in Eq. (1). Hence the issue of finding its *ground state* is replete with difficulties familiar from relativistic field theories. The second-quantized version of the model $H_{\rm Lutt}$ can be solved consistently and exactly by filling the Dirac sea and using bosonization [3]. This leads, among other things, to an asymptotic decay exponent of the single particle Green's function $\alpha = 1/\sqrt{1 - (\delta/\pi)^2} - 1$. In a first-quantized framework, a consistent but different solution can be obtained if one is willing to consider quasi-ground states where single-particle states below a certain very negative cutoff energy $E_{\rm cutoff}$ are left empty (evidently, the model does not have a conventional ground state). This "rapidity cutoff" in fact is frequently used in the Bethe ansatz solution of field theoretical models [9,10]. Such a state becomes natural if one is interested in finding the ground state of H in Eq. (1), and examines the eigenvalue of $H_{\rm Lutt}$, a commuting operator, in this state. The transformations used for Eq. (1) also can be used here, and the solution found as earlier and lead to a shift in its eigenvalue due to interactions $\delta E_{\rm Lutt} = \frac{2\pi}{L}\{N_+[[\frac{N_-\delta}{2\pi}]]_{\rm rem} + N_-[[\frac{N_+\delta}{2\pi}]]_{\rm rem}\}$, a number of the $O(1)$. The correlation function can be found using Luttinger's original paper and lead to the same asymptotic decay exponent α of the Green's function as in Eq. (12). This result was in fact obtained in Luttinger's paper [2], i.e., *Luttinger's result in fact applies to the first-quantized solution of the model described here* [11]. The same result for correlation exponents can also be obtained by considering variations of the energy with particle number, similar to what we described above. We note that the Mattis-Lieb and Luttinger results for α, though different in general, agree to the lowest nontrivial order in δ. The differences at higher order clearly have to be attributed to the different cutoff procedures used in the two calculations.

We now turn to similar models defined on a one-dimensional lattice. Specifically, we will consider the Hamiltonian with $\phi_m = \sum_l \alpha_{m-l} n_{l,-\sigma}$,

$$H = -\sum_\sigma \sum_{m=1}^{L} [\exp(i\sigma\phi_m)\, c^\dagger_{m\sigma} c_{m+1\sigma} + \text{h.c.}] + V , \quad (18)$$

where periodic boundary conditions are implied, $\alpha_{m+L} = \alpha_m$ and $c_{L+1,\sigma} = c_{1\sigma}$, and the number operator $n_m = c^\dagger_m c_m$.

The interaction term V can be either zero, or one of two nontrivial functions which retain the exact solvability. We will consider the XXZ model and the Hubbard model, i.e.,

$$V_{XXZ} = V \sum_{j,\sigma} n_{j\sigma} n_{j+1,\sigma} \quad \text{and} \quad V_{\rm Hub} = U \sum_j n_{j\uparrow} n_{j\downarrow} , \quad (19)$$

and will show below that these are exactly solvable. The XXZ model corresponds to two copies of the usual model, wherein the two species of particles (spin up and down) talk to each other only via the phase factors. The Hubbard model corresponds to the usual two body interaction.

We now perform a unitary transformation induced by $U = \exp(iS)$ where $S = \sum_{1\le l,m\le L} \beta_{l,m} n_{l\uparrow} n_{m\downarrow}$. It is easy to see that $\beta_{i,j} = -\beta_{j,i}$, i.e., an odd function is appropriate, and we will assume it to be so. Thus we find $e^{iS} c_{m\sigma} e^{-iS} = c_{m\sigma} \exp(-i\sigma \sum_l \beta_{m,l} n_{l,-\sigma})$. The transformed Hamiltonian takes the form

$$H' = -\sum_\sigma \left[\sum_{m=1}^{L-1} \exp(i\tilde\phi_m) c^\dagger_{m\sigma} c_{m+1\sigma} \right] - [\exp(i\tilde\phi_L) c^\dagger_{L\sigma} c_{1\sigma} + \text{H.c.}] + V , \quad (20)$$

with $\tilde\phi_m = \sigma \sum_l \{\beta_{m,l} - \beta_{m+1,l} + \alpha_{m-l}\} n_{l,-\sigma}$ and $\tilde\phi_L = \sigma \sum_l \{\beta_{L,l} - \beta_{1,l} + \alpha_{L-l}\} n_{l,-\sigma}$. We now use the freedom in defining β to cancel the interior terms in the phase factor by choosing $\beta_{m+1,l} - \beta_{m,l} = \alpha_{m-l}$. The hop across the $L \leftrightarrow 1$ bond has a total phase

$$\chi_\sigma = \sigma \sum_l [\beta_{L,l} - \beta_{1,l} + \alpha_{L-l}] n_{l,-\sigma} . \quad (21)$$

It is in fact not necessary to solve explicitly for β, although it is easy enough to do so for simple choices of α. By adding the $L-1$ difference equations we get $\beta_{L,l} - \beta_{1,l} + \alpha_{L-l} = \sum_{n=1}^{L} \alpha_{n-l} = \delta$. Thus $\chi_\sigma = \hat N_{-\sigma} \sigma \delta$. The number operator $\hat N_\sigma \to N_\sigma$ in any sector, and hence we see that the problem collapses to one with lattice fermions having twisted boundary conditions. If $V = 0$, we can follow the logic used for the continuum model to determine asymptotics of correlation functions. It turns out that up to the trivial replacement $v_F = \pi N/2L \to 2\sin(\pi N/2L)$ one obtains the same expression for $E^{(2)}$ as in the continuum limit, and, consequently, the same low-energy effective Hamiltonian, Eq. (10), and the same expressions for correlation exponents [Eqs. (12) to (16)] apply.

In the presence of a nonzero extra interaction V, previous work [12] can be used where the Bethe Ansatz has been adapted to the case of a "spin twist," which is precisely the case needed here. We write the solution

immediately: for the XXZ model the energy is the usual sum over cosines and

$$Lk_n^\sigma = 2\pi I_n^\sigma + \sigma\delta N_{-\sigma} + \sum_m \theta(k_n^\sigma - k_m^\sigma), \quad (22)$$

with the phase shift and the usual integers I_n. For the Hubbard model the analogous equations are available in Ref. [12]. In these models, one has non-Fermi-liquid behavior even in the absence of α_r, and adding this changes the exponents, and indeed even the symmetries of the model. The detailed behavior of the lattice models and the resulting exponents will be reported elsewhere.

There are a number of further possible generalizations of the present model [13] or with its counterpart with further internal symmetries [14]. A striking case is that of *unequal masses of particles*, i.e., in Eq. (1) we could allow a m_σ dividing the Π_σ^2, which would be unaffected by the pseudounitary transformation. This can also be generalized to lattice models.

In conclusion, we have presented a class of lattice and continuum fermion models which are exactly solvable by a pseudounitary transformation, leading to nontrivial and non-Fermi-liquid behavior, with exponents depending on the interaction. The models do not have an unbounded spectrum, eliminate the problem of the negative energy Dirac sea and consequent Schwinger terms, and thus help us to focus on the physics of the interactions in one dimension in a bounded, and even a finite dimensional Hilbert space (for the lattice models). Note that the momenta of *each of the* $N_\pm$ particles has to be readjusted by the addition of even one particle of the opposite species. This basic fact results in an infrared catastrophe that underlies the non-Fermi-liquid nature of the resulting solution, as captured in our model at a *minimal level*. The method used embeds the original problem considered by Luttinger in a family of commuting Hamiltonians which contains both bounded as well as unbounded operators. By focusing on the problem of finding the *ground state* of the bounded operators one comes up with eigenfunctions which are of the type considered by Luttinger, enabling us to make a connection between the methods used by him (Töplitz determinants and the Szegö formula for asymptotics) with more recent conformal/Luttinger liquid methods.

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[14] Another generalization is obtained by giving additional internal degrees of freedom to the $\sigma = +$ and $\sigma = -$ particles. For example, assuming that they both occur in m different "flavors" one obtains a model with an internal $\mathrm{SU}(m) \times \mathrm{SU}(m)$ symmetry. By a calculation analogous to that leading to the exponents in Eqs. (13) to (16) one finds $\eta_1 = 2, \eta_2 = 2 \mp 2\delta/\pi + m\delta^2/\pi^2$, $\eta_3 = 2 + 2m\delta^2/\pi^2$, and $\eta_4 = 2 \pm 2\delta/\pi + m\delta^2/\pi^2$. As expected from symmetry, these exponents are independent of the flavor indices appearing in the corresponding operators. One can further solve the case where the number of flavors for the $+$ and $-$ particles is different.