II. CLOCKS, CLOCK RATES AND TIME DILATION

The notion of a *clock* is basic to our discussion of SRT. In its simplest form, we take it to be a single particle that either undergoes a periodic motion, like a vibrating molecule, or else goes "on" at one time and "off" at a later time, like a meson that is formed and later decays.[1] We shall assume that such clocks "tick" uniformly, that is, that they neither slow down nor speed up, that the mean lifetime of a pi meson is the same this year as it was last year, or that the vibrational frequency of an ammonia molecule is the same now as it was a thousand years ago. We also assume that clocks tick homogeneously, that is, that a clock's ticking rate does not depend on its spatial coordinates. Hence if we bring any two clocks together their relative rates will not depend upon where or when we bring them together.

We can use the language of a spacetime diagram and the concept of a clock to describe the behavior of moving clocks. This description will provide us with an essential key to the understanding of SRT.

First we imagine the following idealized experiment: Take a number of identical clocks, say a number of hypothetical mesons, each of which exists for exactly τ seconds. (We ignore the uncertain nature of the decay process). Suppose they are all born simultaneously at the origin of a horizontal ruler, or x-axis, and then move along the axis away from the origin, some to the left and some to the right, at various different constant velocities. One of the mesons could even remain right at the origin; it would have therefore zero velocity. In general, a meson moving at velocity v will move a distance $x = vt$ away from the origin after a time t, where t is the time measured by a clock at the origin of the x-axis. Now each meson will subsequently die (or decay); the death of each meson constitutes a distinct event with spacetime coordinates $(x/c, t)$. Here, we have arranged our experiment so that all the mesons are born at the event whose spacetime coordinates are $(0, 0)$.

Now we ask a question: How is the spacetime coordinate t for each of the decay events related to the meson lifetime τ? "Huh?" we say, "What do you mean? Since each meson lives for τ seconds, $t = \tau$ for each decay event." This is what Newton would have said too: This is the Newtonian prediction for clock rates. The entire experiment may be illustrated

[1] It may seem at first glance that a real particle that is formed and later decays does not constitute an accurate clock, because of the uncertain nature of the decay process. Given a number of particles, some will decay at times less than the mean life, some will decay at times greater than the mean life, and in general it is impossible to predict exactly when any given particle will decay. However, it is possible to determine the *mean* lifetime of a number of particles to any desired accuracy simply by observing a sufficient number of such particles, and in this sense, decaying particles are just as good clocks as vibrating molecules. Indeed, for a vibrating molecule it is necessary to observe it for a large number of cycles in order to determine its frequency precisely; this is analogous to observing a large number of decays in an exponentially decaying system.

on a spacetime diagram, shown in Fig. 6. Here we have drawn the world line for each clock, and have indicated each supposed decay event by a little black dot on the diagram. A line drawn through all such events we call the line of "τ-second ticks".

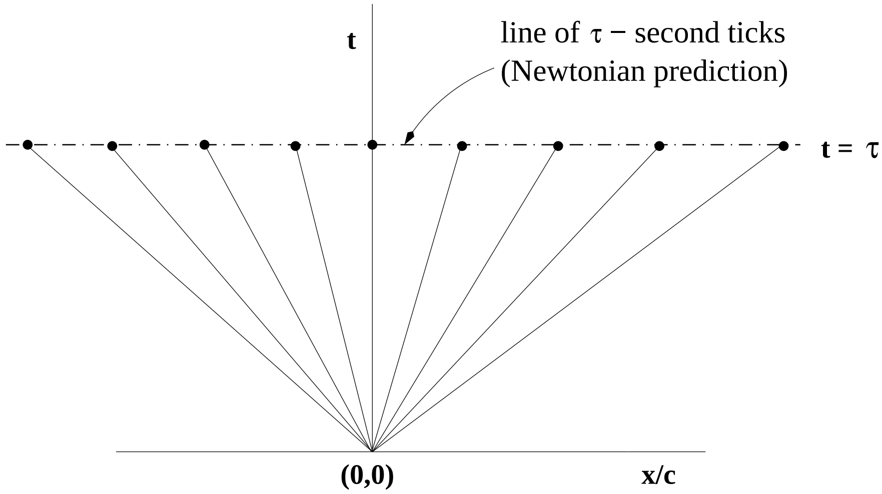

Figure 6—Newtonian spacetime

The surprising result is that this prediction is *wrong*. It is approximately true *only* for clocks that move at speeds much less than the light velocity. When we examine the behavior of real clocks moving at speeds comparable with the light velocity, we find that the line of "τ-second ticks" is not a straight line, but is curved. The form of the curve, while originally predicted by Einstein, is also based on experiments using high speed mesons, experiments we discuss in the next section. The result is that the curve is described by the hyperbola:

$$t^2 - \frac{x^2}{c^2} = \tau^2$$

This curve is shown on a spacetime diagram in Fig. 7 on the next page.

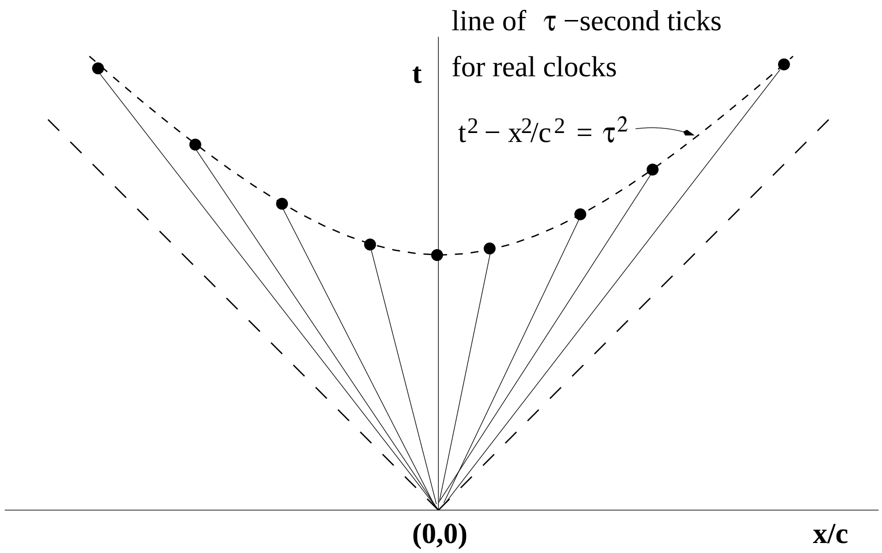

Figure 7—Real spacetime

Note that for $x/c \ll t$, that is, for $v \ll c$, we may neglect the term x^2/c^2 in the equation for the hyperbola, and $t^2 \approx \tau^2$, or $t \approx \tau$. This is the region of spacetime where the Newtonian concept of absolute time is approximately correct.

Exercise

The Newtonian prediction for clock rates is sufficiently accurate if the clock velocity is small enough. What is the maximum clock velocity allowed such that clock rates be accurately predicted by Newtonian theory to within 1 per cent? To within 0.01 percent? (Hint: Make use of the binomial theorem. Answer: .14c, .014c).

The clocks used in the experiments leading to Fig. 7 are typically mu mesons (muons), for which $\tau = 2.2$ microseconds. The experiment may also be done with pi mesons (pions), for which $\tau = 0.027$ microseconds. In both cases, the locus of decay events is given by the hyperbola. The hyperbola thus represents a description of the experimental data for real clocks, and is not subject to dispute. On the basis of these experimental data (see the following section for details), we are led to believe that *any* clock moving from the origin of our spacetime diagram to an event with spacetime coordinates $(x/c, t)$ will record a time interval τ where

$$\tau^2 = t^2 - \frac{x^2}{c^2}$$

One must distinguish between t and τ. Here t is called the *coordinate* time, or the time recorded by a clock at a fixed value of x, say at $x = 0$. τ, on the other hand, is the elapsed time recorded by an observer traveling with the clock. It is called the *proper* time. The proper time is characteristic of the clock, and does not depend on how the clock moves.

Note that the parameter c, the light velocity, enters into the equation for the hyperbola. This is an indication that the light velocity is a fundamental quantity that must enter into the theory of high-velocity motion of particles. This also provides us with the motivation for scaling our axes as we have discussed earlier in connection with Fig. 3.

In actual experiments, not all clocks will pass through the same event. Our assumptions that all clocks tick uniformly and homogeneously will allow us to compare clocks ticking at different times and at different places.

We may generalize the above relation: If a clock is carried at constant velocity from the event $(x_1/c, t_1)$ to the event $(x_2/c, t_2)$, it will record a time interval given by $\Delta\tau$, where

$$(\Delta\tau)^2 = (t_2 - t_1)^2 - \frac{1}{c^2}(x_2 - x_1)^2 \equiv (\Delta t)^2 - \frac{1}{c^2}(\Delta x)^2$$

This forms the basic rule for thinking about spacetime geometry, and forms a concrete basis for SRT. It is an experimental basis, a way of summarizing the description of real observations. The quantity $\Delta\tau$ has a name. It is called the *metric*, or *spacetime interval*, separating the spacetime events $(x_1/c, t_1)$ and $(x_2/c, t_2)$.[2]

This expression for the spacetime interval looks a lot like the Pythagorean theorem, *except for the minus sign.* That minus sign is *crucial.* It is the single feature leading to the surprising differences between spacetime and ordinary Euclidean geometry.

[2] Note that $(\Delta\tau)^2$ can be positive, negative or zero, depending on the relative values of the spacetime coordinates of the two events. The ramifications of this will become clear in subsequent sections. By $\Delta\tau$ we mean $(\pm(\Delta\tau)^2)^{1/2}$, where the sign is chosen to make $\Delta\tau$ real.

Example

Consider a clock, starting from the event E_1: $x/c = 0$, $t = 0$, that travels at uniform velocity to the event E_2: $x/c = 6$ sec, $t = 10$ sec. What is the elapsed time on the clock? It is given by

$$\tau = [(t_2 - t_1)^2 - \frac{1}{c^2}(x_2 - x_1)^2]^{1/2} = (10^2 - 6^2)^{1/2} = 8\,\text{seconds}$$

Note the obvious: 8 seconds is less than 10 seconds. We can illustrate with a spacetime diagram:

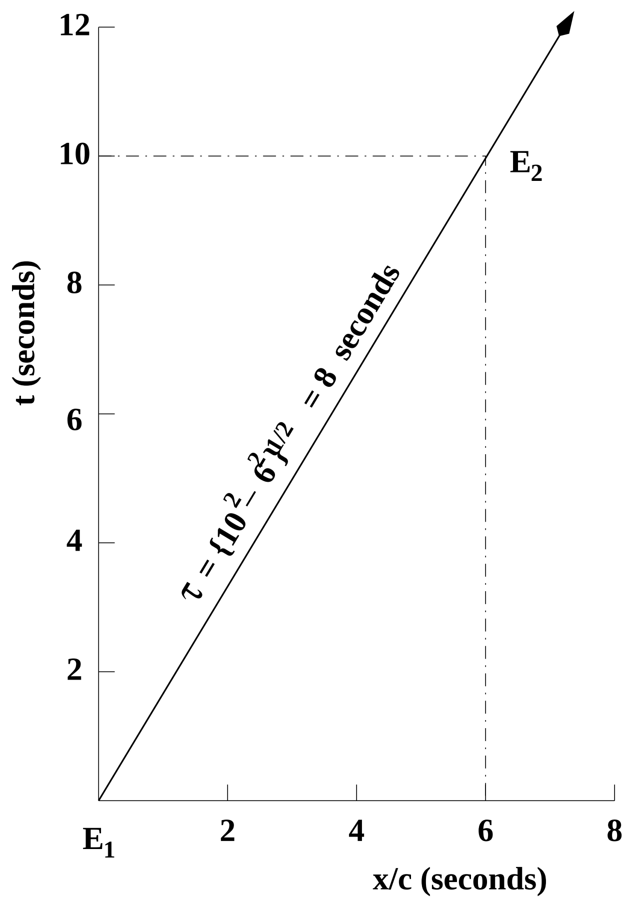

Figure 8—World line of a clock moving at $v = 0.6c$

If our clock were to travel instead from the origin to the event $x/c = 8$ sec, $t = 10$ sec, the elapsed time on the clock would be even less, only 6 seconds. And if our clock were somehow able to travel to the spacetime point $x/c = 10$ sec, $t = 10$ sec, again at uniform velocity, it would show no elapsed time at all.

This is strange stuff indeed. It can be made a bit less strange, however, if we realize two things: First, note that the distances of 6, 8 and 10 light-seconds are extremely large—several times as far away as the moon—so that for a clock to travel that far in only a few seconds it must travel at a very high velocity, much higher than we are accustomed to, even with the highest speed spaceship we could make. (Note that for the case illustrated in Fig. 8, our clock must move at $v = 0.6c$ in order to pass through the two events). So perhaps we shouldn't be too astonished if we encounter strange results.

Second, although we represent our spacetime by a drawing on a piece of paper, so that it *looks* like ordinary Euclidean space, it is *not* a representation of Euclidean space, and the rule for measuring distance in Euclidean space (the Pythagorean theorem) does not apply. We must use the spacetime interval rule instead.

The behavior of moving clocks is called the relativistic *time dilation* for the following reason: Consider Fig. 9 (a generalization of Fig. 8), in which a clock moves with constant velocity from the event $(0, 0)$ to the event $(x/c, t)$.

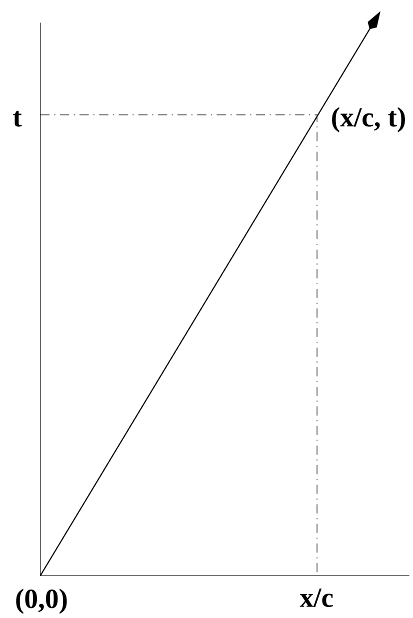

Figure 9—World line of a clock moving at v

What will this moving clock read when it passes the second event? That is, what will be the elapsed time as recorded by this clock? Since the clock moves at speed v, $x = vt$, and we may calculate:

$$\tau^2 = t^2 - \frac{x^2}{c^2} = t^2 - \frac{v^2}{c^2}\,t^2 = t^2(1 - \frac{v^2}{c^2})$$

The coordinate time t, which may be measured by a pair of synchronized stationary clocks situated at $x = 0$ and $x = x$, is therefore related to the proper time interval τ

recorded by the moving clock:

$$t = \frac{\tau}{\sqrt{1 - v^2/c^2}}$$

The ratio of the clock readings depends only on the relative velocity v. This particular function of v appears so often that it is given a special symbol, γ:

$$\gamma \equiv \frac{1}{\sqrt{1 - v^2/c^2}}$$

Since v/c is always observed to be less than 1, we see that γ is always greater than 1. Hence, as measured by the stationary observer's clock, the time intervals of the moving clock are lengthened, or *dilated*, by the quantity γ, hence the expression *time dilation*.

An additional implication resulting from this behavior of clocks is that moving unstable particles will travel farther than we might think before decaying. Newtonian theory would predict that a moving particle whose lifetime was τ would move a distance of $v\tau$ before decaying. The SRT rule for clocks says that the moving clock appears to run slowly, and so the particle can actually cover a distance $\gamma v\tau$ before decaying. This effect may also be illustrated with a spacetime diagram as shown in Fig. 10.

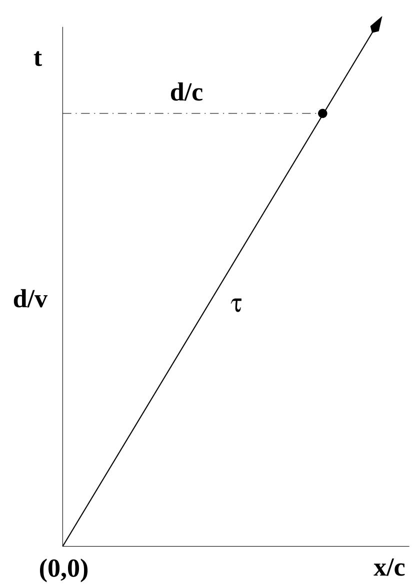

Figure 10—Distance traveled by a moving clock

Again, suppose that the particle decays after a time interval τ during which time it travels a distance d. We have

$$\tau^2 = \frac{d^2}{v^2} - \frac{d^2}{c^2} = \frac{d^2}{v^2}(1 - \frac{v^2}{c^2}), \quad \text{or} \quad d = \gamma v\tau$$

This result plays an important role in the experiments on real clocks, discussed in the following section.

So far, we have considered only clocks moving at constant speeds relative to each other. How do we treat a clock that is moving on a *curved* world line, that is, a clock that is undergoing *acceleration*? Contrary to the opinions of some effervescent mythologists, General Relativity need not be invoked. SRT gives the complete description. We merely consider an infinitesimal interval of elapsed time along the world line of the clock:

$$d\tau = (dt^2 - \frac{1}{c^2}\,dx^2)^{1/2}$$

To calculate a total elapsed time for any given clock between any two specified events, we simply add up all the infinitesimal spacetime intervals, that is, we perform a line integral along the world line:

$$\tau = \int_{E_1}^{E_2} d\tau = \int_{E_1}^{E_2} \left(dt^2 - \frac{1}{c^2}\,dx^2\right)^{1/2} = \int_{t_1}^{t_2} \left[1 - \frac{1}{c^2}\,(\frac{dx}{dt})^2\right]^{1/2} dt$$

Of course, to perform this integral, we need to know the equation of the world line, that is, we must know how x depends on t, so that we can determine how dx/dt depends on t and do the integral. The result we get for τ depends not only upon the spacetime coordinates of E_1 and E_2, but also upon the equation of the world line connecting E_1 with E_2.

Now for a surprise: The *greatest* elapsed time recorded by a clock carried between any two events E_1 and E_2 occurs when the world line connecting the two events is *straight*. All other world lines yield *shorter* elapsed times. Contrast this with Euclidean geometry, where the *shortest* distance between two points is along the straight line connecting the points. We compare the two geometries in Fig. 11.

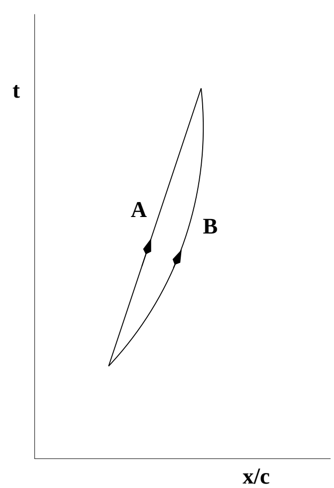

Figure 11(a)
Spacetime: $\tau_A > \tau_B$

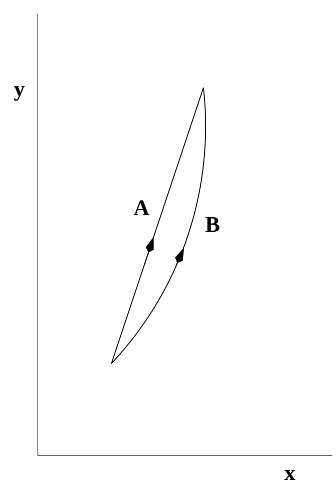

Figure 11(b)
Euclidean space: $d_A < d_B$

Problem

For this exercise, assume that the light velocity is 5 miles per hour. Dave starts from home at 6 AM and walks down a long straight road at 1 mile per hour. His friend Erin starts (from the same home) 9 hours later (at 3 PM) and follows Dave, walking at 2 miles per hour.

Draw their world lines (to scale) on a suitable spacetime diagram, and determine graphically the coordinates of the event E: Dave and Erin meet. Their dog Fido leaves home just when Erin does, pursuing Dave at 4 miles per hour, meets Dave, reverses direction and returns to Erin (also at 4 miles per hour), reverses to Dave, etc., until the event E. How far does Fido walk? Now Dave, Erin and Fido each carry ordinary clocks, all of which have been synchronized at 6 AM, the moment when Dave leaves. What are the readings of each of the 3 clocks at the event E, when they are all back together again? (Partial answer: Erin's clock reads 11:15 PM)