

VII. THE EINSTEIN FORMULA FOR VELOCITY ADDITION

We are familiar with the old “rowboat-on-the-river” problem: Suppose a boat is rowed down a river such that its velocity with respect to the river bank is $v_1 = 8$ miles per hour, while the water in the river flows with velocity $v_2 = 3$ miles per hour, also with respect to the river bank. With what velocity v_{12} does the boat move with respect to the water? Following Newton (and our natural intuition), we simply subtract the two velocities: $v_{12} = v_1 - v_2 = 5$ miles per hour. While this calculation is sufficiently accurate for these low velocities, it becomes increasingly incorrect for velocities approaching the velocity of light. The following exercise will serve to illustrate the problem:

Exercise

Let one particle move to the right while another moves to the left, each with speed $c/2$. How fast does the first particle move with respect to the second? Newton would have said “ c ”. Show that this prediction is wrong by constructing a spacetime diagram with the world lines of the two particles. You can do this by using light signals as in Fig. 15, choosing t_1 and t_2 such that $(t_2 - t_1)/(t_2 + t_1) = 1/2$. Do it carefully, so the slopes are accurate. Now from your diagram, estimate graphically how fast the first particle is moving with respect to the second. You should get $(4/5)c$.

This is an example of the Einstein law for velocity addition. If one particle moves with a velocity v_1 , while another moves with velocity v_2 , the first particle moves with respect to the second with a speed v_{12} , where

$$v_{12} = \frac{v_1 - v_2}{1 - v_1 v_2 / c^2}$$

At low velocities, the extra term in the denominator may be neglected, and we get the Newtonian formula:

$$v_{12} \approx v_1 - v_2$$

Exercise

Sometimes the Einstein formula is stated in a slightly different form: Suppose a particle moves with velocity v_{12} with respect to a reference frame (a railroad car, say) that itself is moving with velocity v_2 . The velocity of the particle with respect to a stationary frame (the ground) is then

$$v_1 = \frac{v_2 + v_{12}}{1 + v_2 v_{12} / c^2}$$

Show that this formula is equivalent to the preceding one.

Another exercise

We can use the form of the Einstein formula stated in the previous exercise to show that no particle can move faster than the light velocity. Let $v_2 = v_{12} = \alpha^c$, where α is some constant number, possibly greater than 1. Show that the maximum value for v_1 is obtained for $\alpha = 1$.

It is not too difficult to derive the Einstein formula. Here is a problem that will illustrate how it can be done:

Problem

Derive the Einstein formula using our spacetime diagram method for determining particle velocities, as discussed in the previous section. Start by sketching a spacetime diagram on which are drawn the world lines of two particles, one moving at speed v_1 , and the other at speed v_2 , where $v_2 < v_1$. Our job is to determine v_{12} , the velocity of the first particle with respect to the second, in terms of v_1 and v_2 . Each velocity can be expressed in terms of clock readings using appropriate light signals. Invocation of the spacetime interval rule will then lead, with some algebra, to the desired result.

We shall discuss the Einstein formula further in Section IX, in the context of the Lorentz Transformation equations.