Physics 224 **Origin and Evolution of the Universe** **Spring 2010**

Homework Set 3
DUE: Thursday June 3

1. **Power Spectrum of Perturbations.** Suppose that the power spectrum of density perturbations $\delta \equiv (\delta\rho/\rho)$, with index n, is $P_k \equiv |\delta_k|^2 \propto k^n$.

a) Derive this expression for the rms density fluctuations on a comoving scale λ:

$$\left(\frac{\delta\rho}{\rho}\right)_\lambda \sim k^{3/2}|\delta_k| \propto \lambda^p$$

and find the index p in terms of n.

b) Find the index m for the rms mass fluctuations on a comoving scale λ:

$$(\delta M/M)_\lambda \propto M^m.$$

What is the criterion on n such that structures grow in a top-down manner (large masses collapse first)? In a bottom-up manner? If the primordial spectrum has $n = 1$ and the CDM transfer function goes as $T(k) \sim k^{-2}$ due to the horizon entry effect, what m does this correspond to and what is the physical interpretation of the mass fluctuation spectrum?

c) The dynamics of the perturbations are driven by the fluctuations in the gravitational potential ϕ. Find the index f for the rms potential fluctuations

$$\delta\phi_\lambda \propto \lambda^f.$$

What is the physical significance of the scale invariant power spectrum index $n = 1$? Also explain the difference between a process being scale free and being scale invariant. Which, if either, is Newtonian gravity?

2. **Growth of Density Perturbations.** For matter density perturbations with wavelengths much greater than the Jeans length, the time evolution is given by

$$\ddot{\delta} + 2H\dot{\delta} - (3/2)\Omega_m(t)\, H^2\, \delta = 0.$$

a) Rewrite this equation with the dependent variable being the scale factor a. Write any derivatives of a or H in terms of H and the deceleration parameter q. **b)** Consider the case of a flat matter universe where, on the scales considered, only a constant fraction Ω_{cl} clumps to form structure. (Possible realizations are a cold + hot dark matter universe or a dark + baryonic matter universe.) In this case the Ω_m in the source term of the evolution equation is replaced by Ω_{cl}. Solve the equation to find the behavior of the growing mode: $\delta \propto a^m$. Interpret. Check the limits $\Omega_{cl} = 0$ and 1. **c)** Consider the case of an open universe at a time dominated by the curvature. Write the evolution equation, substituting in for q and $\Omega_m(t)$, keeping only the leading order for the coefficient of each term as a gets large. What happens to the source term for the growth as the universe expands? Try a solution $\delta \propto a^m$. Find the dominant mode and give the physical interpretation (include explanation of the roles of both the drag and source terms).

3. **Spherical Collapse.** The evolution of a spherically symmetric, overdense perturbation in an $\Omega = 1$ universe can be solved analytically up to the point of singular collapse. As a consequence of Birkhoff's theorem (in a spherically symmetric universe, only the interior mass matters), the perturbation follows the equations of a $k = +1$ Friedmann universe, for which we have a parametric solution. **a) Perturbation overdensity** The solutions – unperturbed for the background universe (barred quantities) and perturbed for the overdense region – for the evolution of the size of a sphere and the time are given by

$$\bar{r} = r_0\,(a/a_0) = r_0\,(\eta/\eta_0)^2 \qquad \bar{t} = t_0\,(a/a_0)^{3/2} = t_0\,(\eta/\eta_0)^3$$
$$r = A\,(1 - \cos\theta), \qquad t = B\,(\theta - \sin\theta),$$

where η is the conformal time and θ is the development angle.

At early times the density perturbation must be small ($\rho \to \bar{\rho}$) so the Friedmann equations for the universe and the perturbation region look the same. Enforce this by matching x, $\dot{x}$, $\ddot{x}$ for $x = r, \bar{r}$ with the respective time variables in order to find $r(\theta)$ and $t(\theta)$, i.e. A and B. *Hint:* Remember the definition of the conformal time parameters η and θ.

The age of the universe is unique so t and $\bar{t}$ must be equal. Use this to derive $\eta(\theta)$.

Use mass conservation to express the overdensity $\rho/\bar{\rho}$ first in terms of $r/\bar{r}$ and then as a function of θ.

Verify that turnaround occurs at $\theta = \pi$ and $\rho/\bar{\rho} = 5.55$ and that virialization occurs at $\rho/\bar{\rho} = 178$. Use that the radius is half the turnaround radius (implying $V = -2K$), but use the time corresponding to $\theta = 2\pi$. (Although $V = -2K$ at $\theta = 3\pi/2$, virialization requires $\langle V \rangle = -2\langle K \rangle$, which obtains roughly at $\theta = 2\pi$).

b) Linear regime Show that the density contrast

$$\frac{\delta\rho}{\rho} \equiv \frac{\rho - \bar{\rho}}{\bar{\rho}} \propto a$$

when $\theta \ll 1$. Show that the dimensionless velocity perturbation for $\theta \ll 1$ is

$$\delta_v \equiv \frac{v - Hr}{Hr} = -\frac{1}{3}\left(\frac{\delta\rho}{\rho}\right),$$

where $v = dr/dt$ is the perturbation's expansion velocity and H is the Hubble parameter of the background universe.

c) Astrophysical application Suppose that we observe a galaxy with rotation speed σ at radius R. If we attribute this rotation speed to the mass of a (spherical) dark halo and assume the spherical collapse model in an $\Omega = 1$ universe gives an accurate description of the formation of this halo, what is the expression for the redshift of virialization z_v? What is the value of z_v if $\sigma = 180$ km s^{-1}, $R = 30$ kpc and $H_0 = 60$ km s^{-1} Mpc^{-1}?